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E. G. Begle,
Secretary

TOPOLOGY OF LIE GROUPS

HANS SAMELSON

1. **Introduction.** The purpose of this address is to sketch some aspects of the subject known as topology of Lie groups, i.e., of the study of the topological characteristics (mainly those of homology and homotopy theory) of the underlying spaces of the Lie groups, and of the connections between topological and group theoretical properties. The interest in this field seems to stem from the fact that a variety of disciplines from algebra, analysis, and topology have found a very natural domain of application here; on the other hand, the topological study of Lie groups has resulted in contributions to other fields such as the theory of fibre bundles [116],[1] (generalized) affine connections [20], metric geometry [128], and topology (one might mention that the famous theorems of de Rham [102] were first formulated in this context).

In [13] Cartan has given a beautiful account of everything that was known about topology of Lie groups at the time; in the present paper we shall try to fill in some of the subsequent developments; some overlap is of course unavoidable (see also [115; 141]). It should be said that we shall be concerned only incidentally with the general theory of topological groups; in particular we are not concerned with the developments centering around Hilbert's fifth problem, i.e., the problem of when topological groups can be proved to be Lie groups.

2. **Definitions.** We begin by recalling briefly basic definitions and a series of classical facts: To describe a Lie group G, we have first of all a manifold of some dimension n, i.e. a (separable) Hausdorff space, usually assumed connected, in which every point has a neighborhood which is homeomorphic with Euclidean n-space E^n; such a homeomorphism sets up a coordinate system in the neighborhood of point. Secondly, the manifold carries an analytic structure: A of coordinate systems, covering the manifold, is specified, suc wherever two of the systems overlap, one has a (real-) analytic formation of coordinates, with nonvanishing Jacobian. It becom then possible to introduce the concepts of (real-) analytic functio

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An address delivered before the Chicago meeting of the Society, April 28, 1950 by invitation of the Committee to Select Hour Speakers for Western Sectional Meetings; received by the editors June 18, 1951.

[1] Numbers in brackets refer to the bibliography at the end of the paper.

nalytic mapping, analytic (or only differentiable) curves, vectors, usually thought of as tangent vectors to curves, the space of all vectors at a point, which is an n-dimensional linear vector space (over the reals), called the tangent space at the point in question, and more generally tensors of an arbitrary index constellation. Thirdly, the manifold carries a group structure, a multiplication is defined between the points of the manifold (the product of a and b denoted by $a\cdot b$ or ab) satisfying the usual group axioms. Fourth, the analytic structure and the group structure are related through the requirements that the product ab and the inverse a^{-1} depend analytically on a and b, i.e., the coordinates of ab and a^{-1} are analytic functions of the coordinates of a and b; or expressed in terms of a mapping: the map of $G\times G$ (endowed with the natural analytic structure) into G defined by $(a, b)\rightarrow ab^{-1}$ is analytic.

An important example is the group $GL(n, R)$ of all real non-singular $n\times n$ matrices, with matrix multiplication as group operation, and analyticity defined by considering the set of all these matrices in the obvious fashion as subset of n^2-dimensional Euclidean space, and thus covering it with a single coordinate system. A second example is the group $O(n)$ of all orthogonal matrices; this is a submanifold of $GL(n, R)$, defined by the equations between the matrix elements which express orthogonality ($M\cdot M'=I$). Here one needs several coordinate systems to cover the group, even to cover the subgroup $O^+(n)$ of orthogonal matrices of determinant $+1$ (this is an open-and-closed connected normal subgroup of $O(n)$, of index 2); one can represent, e.g., the points in the neighborhood of the unit element I by the formula e^S, with the skew symmetric matrix S anging through a neighborhood of the 0-matrix; the elements s_{ij} of S with $i<j$ serve then as coordinates [21; 96].

3. **Covering groups.** Homomorphisms and isomorphisms of Lie roups are always assumed to be continuous, and can then be proved o be analytic. A local homomorphism is an (analytic) map of a [illegible]borhood of the unit element e in one group into a similar set in [illegible]nd group, which preserves product relations between elements [illegible] neighborhood; local isomorphism is defined analogously. A [illegible]vering group of a group G is one that is mapped onto G by a [illegible]omorphism, called projection, which is also a local isomorphism the two groups; such a group is a covering space of G in the usual pological sense [108; 116]. Every group G possesses a unique simply onnected (i.e. all closed paths are contractible to a point) "universal" covering group $\tilde{G}$; the—discrete—kernel of the projection is omorphic with the fundamental group (cf. §6) of G. The simplest

example is the map of the real line R into the circle S^1 (complex numbers of norm 1), defined by $x \to \exp(ix)$; the kernel consists of the multiples of 2π. All groups locally isomorphic with a given group are obtained by factoring the simply connected covering group by various discrete normal (automatically central) subgroups. A local homomorphism of a simply connected group can always be extended to a homomorphism of the whole group [107; 108; 21; 96].

4. **The Lie algebra.** To every Lie group G there is attached the Lie algebra: let L be the tangent space at e; let ξ, η be two vectors of L, thought of as tangent vectors to two curves $x(t)$, $y(t)$; form the commutator $x(t) \cdot y(t) \cdot x(t)^{-1} \cdot y(t)^{-1}$, divide by t^2; the limit, as $t \to 0$, is again a tangent vector at e, depending on ξ and η, and called the commutator or Lie product $[\xi, \eta]$. This multiplication is bilinear, antisymmetric, and satisfies the Jacobi identity $[\xi, [\eta, \zeta]] + [\eta, [\zeta, \xi]] + [\zeta, [\xi, \eta]] = 0$, which reflects the associative law in G. With this multiplication L is called the Lie algebra of G. As an example, the Lie algebra of the orthogonal group $O(n)$ consists of the skew-symmetric matrices, and $[\xi, \eta]$ is the matrix combination $\xi\eta - \eta\xi$. The principal fact concerning the Lie algebra is that two groups with isomorphic Lie algebras are locally isomorphic, and that therefore almost all properties of a group must be contained in the Lie algebra. Every subalgebra is the Lie algebra of a subgroup (which is a Lie group itself, but may or may not be closed in G; on the other hand, a closed subgroup is automatically a Lie group, not necessarily connected [13 (16)]); ideals correspond to normal subgroups; homomorphism of the Lie algebra and local homomorphism of the group correspond to each other. In particular, every vector in L gives rise to a 1-parameter subgroup, i.e., a subgroup which is homomorphic image of the real line R; such a subgroup is either homeomorphic and isomorphic with R (in which case it "diverges to ∞") or its closure is a torus group T^m, i.e., a product of a certain number of circle groups S^1.

The inner automorphism $x \to axa^{-1}$ by an element a leaves the e fixed, and so induces a linear transformation A_a (actually an a morphism) of L. The map $a \to A_a$ is called the adjoint representat and the A_a's form the adjoint group; the corresponding "infin tesimal" homomorphism maps ξ into adξ, where adξ is the linear ma of L into itself defined by ad$\xi(\eta) = [\xi, \eta]$; it follows easily that "idea in L" is identical with "subspace of L, invariant under the adjoint group."

The commutator ideal L' is the ideal generated by all elements o the form $[\xi, \eta]$; it corresponds to the usual commutator subgroup. A

Lie algebra is solvable if the sequence L, L', $L''=(L')'$, $\cdots$ ends with 0; every Lie algebra has a unique maximal solvable ideal, the radical; a Lie algebra is semi-simple if its radical is 0—it is then direct sum of nonabelian simple Lie algebras. A Lie algebra is simple if it has no nontrivial ideals; frequently the (one-dimensional, abelian, non-semi-simple) Lie algebra of the real line is considered not simple ex definitione. The important Levi-Whitehead theorem [135] says that any Lie algebra has a semi-simple subalgebra which maps isomorphically on the factor algebra modulo the radical, so that one has a split extension. There are of course always the corresponding definitions and facts for the groups [21; 96].

5. **Classification.** A fact of very great importance is that all simple Lie algebras and groups are known (the construction of a general Lie group is reduced thereby to an—in general nontrivial—extension problem). One knows since Killing and Cartan [13 (2)] that of connected compact simple groups there are exactly the following (up to local isomorphism) [28; 124; 138]: The classical groups, namely the orthogonal groups $O^+(n)$ for all $n>1$ (except that $O^+(4)$ is not simple), the groups $SU(n)$ of unitary unimodular matrices ($M^*=M^{-1}$, det $M=1$) of degree n, and the symplectic groups $Sp(n)$ consisting of the unitary unimodular matrices of degree $2n$, which leave a given nondegenerate skew-symmetric bilinear form invariant; and five so-called exceptional groups G_2, F_4, E_6, E_7, E_8 (of resp. dimension 14, 52, 78, 133, 248). (One often uses the notation $A_n=SU(n+1)$, $B_n=O^+(2n+1)$, $C_n=Sp(n)$, $D_n=O^+(2n)$.) They are all semi-simple except for D_1.

For noncompact semi-simple groups there is a theorem of Cartan [13 (3, 14); 39; 130], which associates each such group uniquely with a compact group in such a way that the Lie algebras of the two groups become identical if one admits complex scalars; as an example one might cite the Lorentz group and $O^+(4)$—with complex coordinates they become identical, since then the distinction between a definite and an indefinite quadratic form disappears.

We note here a remark of Cartan's on the development of our subject (and the theory of Lie groups in general): The early discovery of the classification has had the effect that frequently general facts about Lie groups were first so to speak "verified" for the individual simple groups, and that only later a general reason was found.

6. **First topological facts.** At this point topology of Lie groups proper begins, namely the study of the topological structure of the spaces which appear as manifolds of the various Lie groups. The

interest in this question is due, besides to H. Weyl, above all to É. Cartan, who in a long series of papers [13; 14] came back to the subject again and again, pointed out its importance, made a thorough study of many special cases and went on from there to prove or predict many general results.

We begin with some elementary statements about the space of our Lie group G (cf. [13]):

The manifold is orientable. To show this we note that, for any fixed $a \in G$, the transformation sending x into ax, the left translation l_a by a, is an analytic homeomorphism of G with itself, mapping the unit element e into a; we choose now an orientation at e, and transplant it to a by the left translation l_a; this orients G coherently. The proof really shows more, namely that there exist vector fields without singularities on G, and that in fact G can be "parallelized," i.e., there exist n everywhere independent vector fields on G; we simply take n independent vectors at e, and transplant them to the arbitrary point a by l_a. It follows from topological principles [116] that all the Stiefel-Whitney characteristic classes vanish, and in particular that the Euler-Poincaré characteristic $\chi(G)$ (the alternating sum of the Betti numbers, which equals the alternating sum of the numbers of cells of the various dimensions if G is divided up into a complex (cf. [2])) is 0.

The fundamental or Poincaré group $\pi_1(G)$ [109; 116], whose elements are the homotopy or deformation classes of closed paths, starting and ending at e, is abelian: If two such paths, α and β, are represented by maps f and g of the circle S^1 into G, we form the torus $T^2 = S^1 \times S^1$, and map it into G by sending the point (x, y) into $f(x) \cdot g(y)$. Under the induced homomorphism the generators of the fundamental group of the torus, represented by $S^1 \times p$ and $p \times S^1$ (p is a suitable point of S^1), map into α and β respectively; from the elementary fact that these two generators commute one concludes that also $\alpha\beta = \beta\alpha$.

7. **Noncompact groups.** We turn next to noncompact groups. Examples indicate that the noncompactness is of a trivial character: In $GL(n)$, e.g., every matrix can be expressed (through the Gram-Schmidt orthogonalization process) as product of an orthogonal matrix and a triangular matrix ($t_{ij} = 0$ for $i < j$) with positive diagonal elements. This means that $GL(n)$ is homeomorphic with the Cartesian product of the compact group $O(n)$ and a Euclidean space of dimension $n(n+1)/2$ (the elements of the triangular matrices below the diagonal range through the whole real axis, those on the diagonal through the positive half of the real axis). Similarly, for the Lorentz

group: any matrix can be written as product of an orthogonal and a positive definite symmetric matrix; for a matrix in the Lorentz group one shows that the two factors separately belong to the Lorentz group. That means that the Lorentz group is homeomorphic to the Cartesian product of its subgroup of orthogonal matrices and the subset of positive definite symmetric matrices; and the latter set is easily shown to be homeomorphic with Euclidean E^3. The general theorem behind this (known for simply-connected groups to Cartan [13]) is due to Malcev [80] and Iwasawa [52] (cf. also Mostow [92]): Any connected group is homeomorphic to the Cartesian product of a maximal compact subgroup K (which is connected) and a subset H, which is homeomorphic with a Euclidean space; the theorem says more, namely, that there exist 1-parameter subgroups $h_1, \cdots, h_r$, isomorphic to the real line R, such that H can be written uniquely as $h_1 \cdot \cdots \cdot h_r$ (i.e., each element h of H has a unique representation $h = g_1 \cdot \cdots \cdot g_r$ with $g_i \in h_i$) and that G can be written uniquely as $K \cdot H$. Moreover all maximal compact subgroups are conjugate. The main step in the proof is to establish the theorem for the adjoint group of a semi-simple group; this is done by full use of the Lie algebra: one extends the scalars of the Lie algebra L to complex numbers; in this complex Lie algebra one finds the Lie algebra L_c of the corresponding compact Lie group (a theorem of Cartan-Weyl-Mostow [13 (14); 130; 92]); the intersection $L_c \cap L$ determines a compact subgroup K of G, which turns out to be maximal, and by a careful analysis one finds a complementary subalgebra, which generates a subgroup H made up of triangular matrices; K and H have then the required properties. For the general result, one makes use of the theorem that the decomposition is valid for a group if it is valid for a normal subgroup and the corresponding factor group; this makes possible an induction over the dimension. Practically all significant topological properties of G are therefore those of a maximal compact subgroup.

8. **Solvable groups. Some properties of subgroups.** The special case of solvable groups is worth considering directly. Solvability is equivalent to the existence of a composition series with each factor group isomorphic with either R or S^1. One proves now [52]: whenever one has a factor group G/N, with N solvable and $G/N = R$ or S^1, then G has a 1-parameter subgroup projecting isomorphically on G/N. Going through the composition series, one gets n 1-parameter subgroups $h_1, \cdots, h_n$, isomorphic with either R or S^1, such that G is uniquely represented as $h_1 \cdot \cdots \cdot h_n$. It follows that G is homeomorphic with the Cartesian product of a torus and a Euclidean space;

this is a theorem of Chevalley [**22**; **23**]. In particular, if G is simply-connected, it is homeomorphic with Euclidean n-space; this was known to Cartan long ago.

A question of Pontryagin's can be solved on the basis of the Malcev-Iwasawa theorem: If G is simply-connected, and N is a closed normal subgroup, is N simply connected? An affirmative answer is obtained by reduction to compact groups where the result was known [**52**; **74**]. Let $G=K\cdot H$, then N and G/N (the factor group) decompose according to $K\cap N\cdot H'$ and $KN/N\cdot H''$ ($K\cap N$ and KN/N are maximal compact in N and G/N). G is simply connected if and only if K is; this in turn is so (as can be proved easily on the basis of the developments in §10) if and only if $K\cap N$ and $K/K\cap N$ ($=KN/N$) are simply connected; and this in turn is equivalent to N and G/N being simply connected.

A last question, which goes somewhat in this direction, is the following: as mentioned, every subalgebra of L determines a subgroup of G. When is this subgroup closed? Malcev has given invariants of the algebra, which decide the answer [**75**; **77**; **79**; **80**; **81**]; the invariants involve subspaces of L with rational coefficients—the rationals enter the problem through the fact that one has to consider closed 1-parameter subgroups, and such groups on a torus correspond to straight lines through rational points in the Euclidean space which is the simply connected covering group of the torus. A useful result is that a subgroup is closed exactly if it contains the closure in G of each one of its 1-parameter subgroups. Similar results have been obtained by Gotô [**40**]. The problem is really somewhat more complicated, because the character of the subgroup generated by a subalgebra of the Lie algebra depends also on which one in its class of locally isomorphic groups the group G is. A consequence of these considerations, as shown by Mostow [**93**], is the fact that a semi-simple subalgebra of L generates a closed subgroup if G is simply connected. He also showed, under the same hypothesis on G, that the subgroup, generated by the subalgebra M, is closed if $\dim L - \dim M < 5$; for larger differences this is not always so.

9. **Compact groups.** We come now to compact groups. Compactness makes it possible to find in L a positive definite quadratic form which is invariant under all transformations of the adjoint group: One takes any positive definite form and averages or integrates (with an easily defined invariant integration process [**13** (16); **21**; **131**]) its transforms under the adjoint transformations over the group. In other words, the adjoint group can be assumed to consist of orthogonal matrices. It follows that the Lie algebra of a compact group is com-

pletely reducible in the sense that to every ideal (= subspace invariant under the adjoint group) there exists a complementary ideal. As a consequence, if N is a closed normal subgroup, then G is locally isomorphic with the direct product of N and the factor group G/N. This means that all extensions of one compact group by another are locally isomorphic, and so they all have the same universal covering group; they are distinguished by the different discrete normal subgroups by which one has to divide. It is possible to give a complete description of the situation [**110**; **11**; **12**]; one can make the set of extensions of a group N by a group H into a group, and one has the theorem of Shapiro [**110**] that this group of extensions is isomorphic to the group of homomorphisms of the fundamental group of H (considered as a discrete subgroup of the simply connected covering group $\tilde{H}$ of H) into the center of N, modulo those homomorphisms which can be extended to all of $\tilde{H}$.

If one pursues the above reasoning somewhat further, one arrives at the fact that any compact group is locally isomorphic with the direct product of a finite number of simple groups, and more precisely is obtained from the direct product of the simply connected covering groups of these simple groups by dividing by a discrete normal subgroup.

10. **Weyl's theorem.** The last statement is given more precision by an important theorem of H. Weyl, saying that the fundamental group of a compact semi-simple group is finite (or that the first Betti number is zero), in other words, that the universal covering group is still compact. There are several proofs for this theorem [**129**; **13**; **25**; **43**; **105**]; a geometrical proof, which gives us occasion to introduce some useful concepts, goes along the following lines [**105**]: In a compact group there exists always an invariant Riemannian metric—invariant under left and right translations. The invariant quadratic form in L(cf. §9) defines the line element, the ds^2, at e; for any other point one gets the ds^2 by left translation; the formula $x\cdot a = a\cdot a^{-1}xa$ shows that invariance under left translations and under the adjoint group implies invariance under right translations [13 (16); 1]. It turns out that the geodesics are the 1-parameter subgroups and their cosets. Now if the universal covering group (into which the metric can be lifted) is not compact, then one can find in it a straight line, i.e., a geodesic which furnishes the shortest connection between any two of its points, through e. By considering the relative position of this line to the kernel of the projection one can show that it must lie in the center, so that one gets a center of positive dimension which makes the group not semi-simple. A noteworthy fact about Weyl's

theorem is that compactness for semi-simple groups appears as a property of the Lie algebra; this is of course not so for other types of groups, as the groups S^1 and R show. Using Weyl's theorem it follows then easily that any compact group can actually be obtained as product of (a finite number of) compact, simply connected, simple, semi-simple groups and a torus group, modulo a finite normal subgroup [13; 96].

11. **The Betti numbers.** The most obvious topological problem that arises now is the determination of the topological invariants, in particular of the Betti numbers. The results indicated in §§9 and 10 mean that it is sufficient to consider the simple groups of the classification given in §5 (cf. a remark in §16). (We shall use the language of homology or cohomology, whichever is convenient; we assume some familiarity with the concepts of algebraic topology [2; 50; 61; 109].)

The problem has been approached in various manners, some of which we shall describe briefly. The first method, Pontryagin's, amounts to a direct geometric construction of the cycles, for the classical groups [99]. Consider, e.g., in the unitary group A_n the subgroup A_{n-1}, obtained by putting $a_{n+1,n+1}=1$; the (left) cosets of A_{n-1} are all analytic submanifolds of A_n, homeomorphic with each other; they fill up A_n in a very regular manner, as a matter of fact, they define a fibre decomposition [116] of A_n, i.e., a neighborhood of each coset can be mapped homeomorphically onto the Cartesian product $C^{2n+1}\times A_{n-1}$ (C^{2n+1} is an open $(2n+1)$-cell) in such a fashion that the sets $p\times A_{n-1}$ correspond to cosets of A_{n-1}. Pontryagin constructed now a cycle, given by a closed surface V_n of dimension $2n+1$, which intersects each coset in exactly one point (except the coset farthest away from A_{n-1}). The smoothness and regularity with which A_{n-1} and V_n intersect imply that their intersection number, in the sense of topology, is $+1$, and that therefore, by Poincaré duality, both are not homologous to zero in A_n and so define nonzero elements of the homology groups of A_n (with integral or rational coefficients). The same construction is now repeated in A_{n-1}, giving rise to a V_{n-1} and so on down to V_1 (of dimension 3). These cycles are now combined by an operation utilizing the group property, the Pontryagin multiplication, which with two cycles C, D associates the cycle $C\circ D$, where intuitively speaking $C\circ D$ is traced out by the point $p\cdot q$, if p runs through C and q runs through D. (The strict definition: $C\circ D$ is the image of $C\times D$ under the map $(x, y)\rightarrow xy$ of $G\times G\rightarrow G$). Pontryagin showed now that, in the sense of homology, the V_i are anticommutative, and generate the whole Betti group in such a way that the products $V_{i_1} \circ \cdots \circ V_{i_k}$, $i_1 < \cdots < i_k$, and the zero-cycle, given by a point, form a basis for the group. This can be expressed geometrically by saying that A_n, as far as the cycles go, behaves as if it were the product $S^3 \times S^5 \times \cdots \times S^{2n+1}$ (where S^n denotes the n-sphere $\{x: x \in E^{n+1}, \|x\| = 1\}$), or algebraically by saying that the Pontryagin ring (homology group with Pontryagin multiplication) is a Grassmann algebra with a unit and generators of dimension 3, 5, $\cdots$, $2n+1$. The intersection behavior also follows a very simple pattern, namely that given by the analogy with the sphere-product. Similar results were obtained for the other classical groups. We describe the results by introducing the Poincaré-polynomial of a space, the polynomial in which t^k has for coefficient the kth Betti number. We have then

$$\begin{aligned}
P_{A_n} &= (1 + t^3)(1 + t^5) \cdots (1 + t^{2n+1}),\\
P_{B_n} &= (1 + t^3)(1 + t^7) \cdots (1 + t^{4n-1}),\\
P_{C_n} &= (1 + t^3)(1 + t^7) \cdots (1 + t^{4n-1}),\\
P_{D_n} &= (1 + t^3)(1 + t^7) \cdots (1 + t^{4n-5}) \cdot (1 + t^{2n-1}).
\end{aligned}$$

12. **Differential forms.** The second approach to the Betti numbers, initiated by Cartan and completed by R. Brauer, is algebraic in character [9; 13]. It starts from a different interpretation of the Betti numbers: By the famous theorem of de Rham [102] (which was inspired by just these questions) the pth Betti number is equal to the number of linearly independent p-dimensional closed exterior differential forms defined over G modulo those forms which are derivatives of $(p-1)$-forms. (The value or expression of such a form at any point is a skew-symmetric covariant tensor with p indices or, in another description, a skew-symmetric function $\omega(\xi_1, \cdots, \xi_p)$ of p arbitrary tangent vectors of the point, linear in each variable; the derivative referred to is the exterior derivative of the form or the alternating derivative of the tensor; closed means with derivative 0; cf. [21; 50] and ref. 64 in [14]; we use the symbol d for derivation.) Now one can reduce the p-forms by integrating or averaging over the group to bi-invariant forms, i.e., to forms which are invariant under left and right translations. (Incidentally, the bi-invariant forms turn out to be identical with the harmonic forms, relative to the invariant metric of the group, and can be treated from this point of view [50].) A theorem of Cartan's [13 (15); 25] (an easy consequence of the Maurer-Cartan equations, which state that for left-invariant 1-forms the relation $d\omega(\xi, \eta) = -\omega([\xi, \eta])$ holds at e) says that a bi-invariant form is automatically closed; and this means that the pth Betti

number is equal to the number of linearly independent bi-invariant forms (modulo nothing). A bi-invariant form is determined by its expression at e (the expression at any other point is obtained by left translation), and there it has to be invariant under the adjoint group (cf. the case of the metric in §10). The pth Betti number is therefore equal to the number of linearly independent skew-symmetric tensors with p indices, at e, invariant under the adjoint group; and the problem of finding the Betti numbers is translated into a problem in representation theory. (One sees by the way that the pth Betti number can be at most $C_{n,p}$, this being the maximum number of independent skew p-tensors over a linear space of dimension n.) By use of the first main theorem of invariant theory, Brauer was able to construct, for each classical group, explicitly a certain number of invariants, which from other considerations could be shown to exhaust all possible invariants (cf. also [**130**; **131**]). The result corresponded completely to Pontryagin's, i.e., certain basic forms $\omega_1, \omega_2, \cdots$ were found, such that the products $\omega_{i_1}\cdot\omega_{i_2}\cdot \cdots \cdot\omega_{i_k}$ with $i_1<i_2< \cdots <i_k$ and the scalars constitute a basis for the group of invariant forms. The whole method of invariant differential forms is due to Cartan; he had, before Brauer gave the complete result, obtained certain facts about the Betti numbers, enough to make him state as a conjecture what then turned out to be the correct result [**13** (15)].

13. **Hopf's theorem.** The next approach, due to Hopf, differs from the first two by establishing a general theorem instead of considering each group separately [**45**; **46**]. The method is topological and starts from the map ϕ of $G\times G$ into G, given by $\phi(x, y)=x\cdot y$. The existence of the unit element e means that every cycle of $G\times G$, which "lies in one of the two factors," i.e., which is of the form $z\times e$ or $e\times z$ with some cycle z of G, is mapped by ϕ into z; for the associated or adjoint map ϕ^*, which maps the cocycles (or differential forms, if one prefers) of G into those of $G\times G$, this means that $\phi^*(a)$ is of the form $a\times 1+1\times a+R$, where 1 is the unit cocycle and R consists only of "mixed terms" $c\times d$, where both factors have positive dimensions. Now one draws algebraic conclusions. To illustrate, we show that a sphere of even dimension cannot be a Lie group (of course this follows also from the characteristic). If t^n is the basic n-cocycle of S^n, then one would have $\phi^*(t^n)=t^n\times 1+1\times t^n$ (since there are no "in between" terms in the ring of S^n). $(t^n)^2$ is 0, for dimension reasons. ϕ^* being a homomorphism, the square of $t^n\times 1+1\times t^n$ has to vanish; under the sign-rules in a Cartesian product, remembering that n is even, one obtains $2t^n\times t^n$ for the square, which is not 0, and thus has a contradiction. Refining this

argument considerably one obtains the theorem: The cohomology ring of a Lie group contains certain "primitive" elements $a_1, \cdots, a_l$ (of odd dimension and therefore anticommutative), such that the products $a_{i_1} \cdot \cdots \cdot a_{i_k}$, $i_1 < \cdots < i_k$, and the unit 1 form a basis for the cohomology group; the ring is isomorphic with the Grassmann algebra over the linear space spanned by the a_i. Any such set of elements a_i is called a set of generators of the ring (strictly speaking one would have to add the unit to this set). The number l of elements a_i has a group theoretical meaning, which we discuss in §20. One can show in addition that the homology group, with Pontryagin multiplication, is also a Grassmann algebra, and that these two Grassmann algebras are duals of each other [59; 62; 104]. While thus the theorem gives a reason for the specific structure of the cohomology rings of the classical groups as found by Pontryagin and Brauer, it does not of course determine them explicitly; for the classical groups, e.g., one can couple Hopf's theorem with a theorem about transitive transformation groups of spheres (to which we shall come later) to arrive at the Betti numbers.

14. **Cohomology theory of Lie algebras.** The fourth method works with the Lie algebra. By the integration process mentioned above, one reduces the differential forms (cf. §12) to forms which are invariant under left translations (invariance under right translations, as in §12, is not required here); such forms are determined by their expression at e; moreover the process of taking the exterior derivative can be translated into a simple algebraic operation in the Lie algebra (the Maurer-Cartan equations, §12, show how this translation goes for 1-forms; the complete formula, which follows easily from this, is one of a series of algebraic "coboundary formulae," which recently have become popular, and reads $d\omega(\xi_1, \cdots, \xi_{p+1}) = \sum_{i<j} (-1)^{i+j}\omega([\xi_i, \xi_j], \xi_1, \cdots, \hat{\xi}_i, \cdots, \hat{\xi}_j, \cdots \xi_{p+1})$; the "roofs" mean that the vectors are to be omitted). The whole cohomology theory of differential forms is so expressible by simple algebraic operations in the Lie algebra, and the problem arises, to prove topological facts about groups on this basis [25]. Koszul proved Hopf's theorem and more in this manner [59]. The development naturally parallels somewhat that in Hopf's theory. Instead of the product $G \times G$ there appear algebraic constructions like $L \times L$, the direct product of the Lie algebra with itself; instead of the natural map of $G \times G$ into G there appears the natural map of L onto the diagonal of $L \times L$; instead of the forms or cochains there appears the Grassmann algebra A over L. Hopf's theorem becomes the statement that the cohomology ring of a Lie algebra is a Grassmann algebra generated by certain "primi-

tive" elements. It is formulated and proved for any field of characteristic 0 as coefficient domain, under assumptions on L which replace the condition of compactness on G.

15. **The exceptional groups. Symmetric forms.** The five exceptional groups have resisted much longer (or rather four of them; G_2 was known for some time [13 (15)]). Recently Yen determined their Betti numbers [139], using methods which involve knowledge of the representations of the groups (somewhat in the spirit of §12) and many additional pieces of information, in particular a conjectured formula of G. Hirsch about fibre bundles, when bundle and fibre have the homology structure of products of spheres (as Lie groups do, by Hopf's theorem). The dimensions of the primitive cocycles for the five groups are: (3, 11), (3, 11, 15, 23), (3, 9, 11, 15, 17, 23), (3, 11, 15, 19, 23, 27, 35), (3, 15, 23, 27, 35, 39, 47, 59). In the meantime Chevalley and Weil [26] have found a totally different approach to the whole problem, which made the computations for all groups much simpler. The leading idea is the introduction of symmetric, invariant (under the adjoint group) forms in L instead of skew-symmetric ones; by a process which involves substitution of commutators $[\xi, \eta]$ for the elements in a symmetric form, one can associate with each symmetric p-form a skew form (of degree $2p-1$) (a special case is familiar: from the symmetric bilinear form, introduced by Cartan, $\sigma(\xi, \eta)$ $(=\operatorname{tr}(\operatorname{ad} \xi \operatorname{ad} \eta))$, one obtains in $\sigma(\xi, [\eta, \zeta])$ a skew 3-form $\psi(\xi, \eta, \zeta)$). A crucial theorem states that the primitive elements of the cohomology ring can be so obtained. The invariant symmetric forms are closely related to the generalized Casimir operators of Racah [101]; in particular the degrees of these operators for the exceptional groups, as given by him, turn into the dimensions of the primitive cocycles upon replacing p by $2p-1$.

16. **Torsion.** Besides the Betti numbers there are also the torsion coefficients to determine, or in other words, the elements of finite order in the cohomology group with integral coefficients. It is well known here that the fundamental group of the rotation group of 3-space is of order two; actually this is so for all orthogonal groups $O^+(n)$, $n>2$ [116]. Pontryagin has shown in his explicit construction [99] that the unitary and symplectic group have no torsion at all, and that all the torsion elements of the orthogonal group are of order two. The Poincaré polynomial mod 2 of $O^+(n)$ is given by $(1+t)\cdot(1+t^2)\cdot\cdots\cdot(1+t^{n-1})$. It has recently been shown by A. S. Shapiro and A. Borel, using the topological theory of sphere-bundles (in particular the theory of the characteristic cocycle) and Leray's homol-

ogy theory that the simply connected covering-groups of the orthogonal groups also have torsion (from some n on). A general study of the problem of torsion in groups and also in homogeneous spaces (cf. §17), based on the theories of H. Cartan, Koszul, Leray, Weil (cf. §22), has been made by A. Borel [9]. Among other things, he has determined completely the torsion of the universal coverings of the orthogonal groups, and of some of the exceptional groups. The cohomology mod 2 for the orthogonal group and related spaces has been investigated by C. E. Miller [86]. (That the Betti numbers of a compact group and any compact covering group are the same is easily proved by interpreting the Betti numbers via bi-invariant forms: both groups have the same adjoint group. The same device shows that for a compact group G and connected normal subgroup N the "Künneth relation" $P_G(t) = P_N(t) \cdot P_{G/N}(t)$ holds between the Poincaré polynomials.) As special facts we mention that for a compact semi-simple group the second Betti number is always zero and that the third is always positive [13 (15)] (the invariant form $\psi(\xi; \eta, \zeta)$ mentioned in §15 is nonzero on such groups; incidentally this proves that of the spheres S^n only S^1 and S^3 are groups [103]). Koszul recently proved from the Lie algebra, making use of the connection between symmetric 2-forms and skew 3-forms mentioned in §15, that for a simple Lie group (except the 1-dimensional D_1) the third Betti number is always one [54; 59]; this checks of course with the explicit results. We also note an isolated fact about noncompact simple groups: as shown by Cartan, the first Betti number is 0 or 1 [13; 13 (16)]. The distinction between the two cases is related to the question of the existence of a faithful representation [13]; this question has been investigated by Gotô [40; 41] and Malcev [78] (cf. also [3; 4; 42; 43; 83]).

17. **Klein spaces.** Closely related to Lie groups is another class of spaces, the Klein spaces. There are two equivalent definitions: as coset spaces and as homogeneous spaces [141]. If H is a closed subgroup of the Lie group G, then the left cosets of H can be considered as the points of a new space, the coset space G/H, with the customary decomposition topology [2, p. 63] (if H is normal, this is the factor group); the Lie property of G and H imply that G/H is an analytic manifold, whose dimension is the difference of those of G and H; the natural map p of G into G/H, which sends g into the coset gH, the projection, is analytic. A homogeneous space is a space (which we assume to be a manifold) W, on which G acts transitively as a transformation-group, i.e. each $g \in G$ determines a homeomorphism T_g of W with itself, such that $T_{g_1} \cdot T_{g_2} = T_{g_1 g_2}$, that $T_g(x)$, or in short

$g(x)$, depends continuously on g and x, and that for any x and y in W there is a g in G with $T_g(x)=y$.

The connection between the two concepts is as follows: If G acts on W, let x_0 be a point of W, and consider the closed subgroup $H=G_{x_0}$, consisting of those elements of G which leave x_0 fixed (it is called the isotropy or stability group at x_0). Each left coset of G_{x_0} is identical with the set of elements of G which send x_0 into one and the same point x, and conversely for each point x the set of elements which map x_0 into x is a left coset of H; one has thus a 1:1 correspondence between W and G/H, which turns out to be a homeomorphism. (The concepts make sense for any topological group; but in general one can assert only that the map from G/H to W is continuous.) In particular, W can be made into an analytic manifold, and G acts analytically on it. As an example, the group $O^+(n+1)$ acts transitively on the unit sphere S^n in E^{n+1}; the isotropy group is simply $O^+(n)$. Other examples of Klein spaces are the groups themselves (with $H=\{e\}$), the projective spaces, the complex projective spaces K (one can take $SU(n+1)$ as group G), the Grassmann-manifolds $M_{n,k}$, formed by all planes of dimension k through the origin of E^n, with the orthogonal group $O(n)$ as G. There arises the problem of the relations between the topological properties of G, H, and G/H. Actually G is a fibre bundle over G/H (with G/H as base space), with fibre H; i.e., the cosets xH fill out G in such a regular arrangement that each coset has a neighborhood which is homeomorphic with $C\times H$ (where C is a cell of dimension equal to that of G/H) in such a fashion that the sets $p\times H$ correspond to the cosets of H (cf. §11)—G is "locally a Cartesian product"; one should add that the structure group of the bundle is H [116]. This fact is quite important; a good part of our knowledge of coset spaces was obtained by the application of fibre bundle methods. (Conversely the homology theory of Lie groups has recently found applications to the theory of fibre bundles and the related field of differential geometry [20]; even knowledge of the explicit form of the cohomology ring of, e.g., the orthogonal group, has proved useful.)

A slight generalization of the above is worthwhile: if H and K are two subgroups with $H\subset K$, then G/H is a fibre bundle over G/K with fibre K/H; the projection or fibre map is the natural one, which sends each coset of H into the coset of K containing it. If, e.g., H is K_0, the component of e in K, then K/K_0 is discrete, and G/K_0 is simply a covering space of G/K, with the number of sheets equal to the number of components of K. This has some consequences for the fundamental group of a Klein space: If K is connected, then $\pi_1(G/K)$

is easily seen to be a factor group of $\pi_1(G)$ and therefore abelian; in the general case it has $\pi_1(G/K_0)$ as abelian normal subgroup, with index equal to the cardinality of K/K_0 (cf. [109]).

As to orientability, one can show by an argument similar to one used for groups in §6 that a Klein space is orientable if and only if all the transformations of the isotropy group at a point x are orientation preserving at x (i.e., of index $+1$ or of positive Jacobian at x).

18. **Compact Klein spaces. Characteristic.** A first result, only recently established by Montgomery [91], is the following: If a Klein space, with connected isotropy group, is compact, then it is coset space of a compact group. One begins by representing G and H as $E\cdot K$ and $E'\cdot K'$ from the Malcev-Iwasawa theorem (§7), where E and E' are homeomorphic with Euclidean spaces, and K and K' are maximal compact subgroups (the interchange of the factors against §7 is of course possible; we use this order now, because we use left cosets). One can assume $K'\subset K$. It is possible then to write $G/H=(G/K')/(H/K')=E\cdot(K/K')/E'$ with an (it is hoped) obvious symbolism (cf. §17). Now if the fibre is contractible to a point (as E' is), it is known [116] that there exists a cross section, i.e., a map f: $W\rightarrow E\cdot K/K'$, where f followed by the projection is the identity. Since E is also contractible, one can deform f into f', where f' maps W into K/K' (into the "slice" of $E\cdot K/K'$ at e, parallel to the second factor). The identity of W being an essential map (W is a compact manifold), it follows that the composition of f' with the projection still maps onto W, and that therefore the projection maps K/K' onto W, which means that K is transitive over W, q.e.d.

We consider now compact Klein spaces, with compact group. As noted in §6, the characteristic of the group is zero; this came from the—trivial—existence of a vector field, and the—nontrivial—relation between vector fields and characteristic. For Klein spaces there is the theorem (with a vaguely parallel proof) that the characteristic is non-negative [49; 129]. The distinction between 0 and >0 is group theoretical, as follows: an abelian subgroup of maximal dimension of the group is known to be a torus of a certain dimension $l(G)$, called the rank of G (all maximal toral subgroups are conjugate to each other); the characteristic of G/H is positive exactly if $l(G)=l(H)$. The basis of the proof is the consideration of G/T, where T is a maximal toral subgroup (all other cases can be reduced to this). It is relatively easy to show (the trivial part) that there are elements of G which transform G/T with only a finite number of fixed points, each of index $(-1)^m$ (where $m=\dim G/T$); and then one reads off from the Lefschetz fixed point formula (the nontrivial part) [2] that the characteristic equals the number of fixed points and is therefore non-negative. As an example, consider the sphere S^2; any rotation with a finite number of fixed points has exactly two of them, and the characteristic is well known to be two. Actually, all the transformations of G/H are isometries with respect to a suitable Riemannian geometry; and the theorem should have a connection with the generalized Gauss-Bonnet formula, which expresses the characteristic as integral of an n-dimensional curvature (cf. [**18**] and the references there), in other words, one should be able to show that for Klein spaces this curvature is non-negative. To illustrate the meaning of the theorem, consider Riemann surfaces of genus $p>1$, i.e., of negative characteristic. It is known that they do not admit 1-parameter groups of conformal mappings, and also that the periods of their periodic homeomorphisms are bounded [**115**], so that they are about as far from being Klein spaces as one can wish.

19. **The diagram.** We digress at this point to describe a construction, which goes back to Cartan and Weyl, which is of fundamental importance for the problem of classifying the simple Lie groups and without which no report on Lie groups would be complete. If, as above, T^l denotes a maximal toral subgroup of the compact Lie group G, then one sees easily that the normalizer N_T consists of a finite number of cosets of T^l (the reason for the finiteness is the fact that the automorphisms of a torus form a discrete group, and that therefore any element of N_T, which is close enough to e, commutes elementwise with T^l, and so belongs to it). Conjugation of T^l with elements of N_T determines then a finite group Φ of automorphisms of T^l. A basic fact, which we shall naturally not prove here, is that this group, as transformation group of the torus T, determines the Lie algebra of G [**121**]. The group Φ is easier to see if one considers the universal covering Euclidean space E^l of T^l; there Φ turns into a discontinuous group Γ of orthogonal transformations (rigid motions) by adding to the rotations given by Φ the covering transformations (deck transformations) of E^l over T^l (which are translations of E^l).

The group Γ is generated by reflections in certain (infinitely many of course) planes; the planes form the so-called diagram of the group [**121**]. The fundamental domains of Γ are convex polyhedra, which are all congruent to each other, and cover E^l without overlapping; the angles between two faces of a polyhedron are of the type π/k, k integral and not less than 2. If the group is simple, the polyhedron is a simplex, and the determination of the simple groups is reduced to the discussion of simplices satisfying the arithmetical condition on the angles. The relation of this polyhedron to the group might be better understood if one remembers that all maximal toral subgroups are conjugate to each other, that therefore each element of G has a conjugate on T^l, and that two elements of T^l are conjugate in G if an operation of Φ maps one into the other; finally, therefore, the polyhedron in E^l plays the role of a fundamental domain of G with respect to conjugation. The elements corresponding to points on the faces of the polyhedron, and their conjugates, have normalizers of dimension greater than l, actually even greater than or equal to $l+2$; such elements are called singular. The singular elements form a set of dimension at most $n-3$ (transforms of an $(l-1)$-dimensional set with at most $(n-(l+2))$-dimensional orbits). One can conclude from this construction that, if G is simply connected, the set of nonsingular (regular) elements can be contracted over itself into the point e, along 1-parameter subgroups [13 (10, 11); 121; 122; 130; 131]. A new application of these ideas has been given by Stiefel: He has devised an approach, based on the diagram of G, to the problem of finding the Betti numbers, which yields, e.g., once more the Betti numbers of the classical groups [123].

20. **The rank.** The maximal toral group T also plays a role for the group-theoretical interpretation of the number l of generators of the cohomology-ring of G (cf. §13); the simple result is that l equals the dimension of T (= the rank of G). The proof employs the "power" map p_k, which sends $x \in G$ into its kth power x^k for integral $k \neq 1$. From the homology properties of G, as described in §13, one can show that the (homology-) degree of p_k is k^l (if we denote by G also the basic n-cycle on the n-manifold G, then we have $p_k(G) = k^l \cdot G$). On the other hand, if one takes a "general" element a of T (i.e., one whose coordinates are independent mod 1, so that its powers are dense in T), one can see by an elementary consideration that there are exactly $k^{l(G)}$ elements on T which map into a under p_k, and that no other elements of G can map into a. In addition, one can show that the functional-determinant of p_k (with respect to a given orientation of G) is always non-negative, and for the elements under consideration even positive; this means that the index of p_k of all the elements which map into a is $+1$, and that the degree of p_k at a is $k^{l(G)}$. Applying now the theorem that the homology degree of a map equals its degree at any point [2], one finds $k^l = k^{l(G)}$, and so $l = l(G)$ [47].

21. **The subgroup as a cycle.** We return now to the discussion of a Klein space G/H. The submanifold H of G can be considered as a cycle, in the sense of topology, in G. As such it either bounds or it doesn't bound; this turns out to be an important distinction. The

case of H not bounding is by far the simpler (incidentally, no cycle of H can then bound in G, unless it bounds already in H—a very special behavior [104]), and one has there a very precise result on G/H: G behaves homologically (over the rationals) as if it were the Cartesian product of H and G/H (in general it does not actually factor); the cohomology ring of G/H is also a Grassmann algebra, with generators of odd dimensions, and the ring of G is isomorphic with the tensor product of the rings of G/H and H. These two facts completely determine the ring of G/H in terms of those of G and H: if from the generators of the ring of G one drops those which correspond to generators of H, then the remaining ones generate a ring isomorphic with that of G/H. This has been proved by the topological and algebraic methods discussed in §§13 and 14, suitably extended to the case of Klein spaces [59; 104]. For the topological proof one considers the natural map of G onto G/H. The principal fact in the proof is that the assumption on H implies that the Betti groups of G are mapped onto those of G/H; the remainder of the proof consists in a discussion of the behavior of the projection with respect to Pontryagin multiplication and the connection with the induced cohomology map. A particular case where this applies is the case where G/H is a sphere of odd dimension; this is one of the theorems needed for the determination of the Betti numbers of the classical groups as mentioned in §13 (the classical groups appear in a natural way as transitive transformation groups of spheres; the isotropy groups are again classical, and an induction becomes possible). Of course, one has to prove first that in this case H is not homologous to zero; but that is easy, with a construction to be described in a moment. The remaining case of a subgroup H, which is ~ 0 in G, is not completely understood, and much remains to be done. A special case of interest in connection with the problem just mentioned is known, namely when G/H is a sphere S^m of even dimension; the result is that one of the primitive generators of the ring of H has dimension $m-1$, and that the ring of G is obtained by replacing this generator by one of dimension $2m-1$. The topological proof employs again the map p from G to $G/H = S^m$. The principal fact (for odd and even m) is that S^m can be lifted back into G "with one singularity"; more precisely one can find an m-cell C^m in G, such that the boundary of C^m lies in H and the projection p maps the interior of C^m topologically on the sphere S^m minus the point which is represented by the coset H itself (p maps the whole boundary of C^m into this point). One can show now that the boundary of C^m, which is an S^{m-1} lying in H, is ~ 0 in H for odd m, from which it follows easily that then H is not ~ 0 in G (by a consideration involving Poincaré duality as in §11; one starts by completing C^m to a cycle by means of a chain in H); for even m this S^{m-1} is not ~ 0 in H, and as a matter of fact it is a primitive $(m-1)$-cycle in H (which is ~ 0 in G, since it bounds the cell C^m) [104].

22. **Algebraic methods.** A recent result of Koszul generalizing a result of §21 can be described roughly as follows: For every primitive p-cycle of H which bounds in G, there exists a $(p+1)$-dimensional generator in the cohomology ring of G/H. It gives thus a first indication of what the structure of the ring of G/H might be in the general case. Koszul's methods [59], which operate with the Lie algebra of G and a subalgebra corresponding to H (the "relative case" of §14), are inspired by the concepts of ring of a map and of spectral ring of a filtered differential algebra, which Leray has recently introduced into topology and which have found an increasing number of applications in the recent past; it is impossible to indicate definitions or proofs here. We only mention a concept which also plays a role elsewhere, that of transgression; in terms of differential forms, it is defined as follows: If ω is a closed form in G/H whose image $p^*\omega$ (under the map p^* associated with the map p of G into G/H) is homologous to zero in G, i.e., $p^*\omega = d\alpha$, then α induces on the submanifold H of G a closed form, which is said to be transgressive. This process can be considered as the dual of the process of finding cycles in G/H which are p-images of chains of G, whose boundaries are not necessarily 0, but are collapsed into 0 by p (a special case occurred in §21). A main result of the theory is that just the primitive elements are transgressive. Leray himself has announced a number of applications of his ideas [62–73]. He has determined, for the classical groups, the ring of G/T, where T is a maximal toral subgroup of G, and more generally the ring of G/H, when H has the same rank as G, i.e., when it contains T. The results [69] are best stated in terms of the Poincaré polynomials: If G and H have respectively the polynomials $\Pi(1+t^{2m_\lambda-1})$, $\Pi(1+t^{2n_\lambda-1})$, then G/H has the polynomial $\Pi(1-t^{2m_\lambda})/(1-t^{2n_\lambda})$. That this formula holds if, instead of the group-subgroup structure, one has only a fibre bundle is Hirsch's conjecture (§15); the case where G and H are spheres, so that the products contain only one factor, is one of Gysin's results (cf. ref. 38 in [116]).

Another result in this direction was given by Koszul [56] for the case where H is a circle group S^1, where therefore G is fibred into circles; it states that the Poincaré polynomial of G/H is obtained from that of G by replacing a factor $1+t^3$ by a factor $1+t^2$. A trivial

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example for this is the group $SU(2)$, which is homeomorphic with S^3, and for which G/S^1 is homeomorphic with S^2; the interest of the result comes of course from all cases but this one.

Considerable advances along these lines have been made recently by H. Cartan [16; 17], Koszul [60], and Leray [71]. The introduction of symmetric forms (= symmetric multilinear functions on the Lie algebra, cf. §15) has proved very fruitful. To mention one of the main results, let $I_S(G)$ ($I_A(G)$) denote the algebra of invariant (under the adjoint group) symmetric (skew-symmetric) forms of the group G. For a subgroup H of G the Lie algebra of H is contained in that of G; one has therefore a natural map of $I_S(G)$ into $I_S(H)$, by restriction of the arguments of the forms. This mapping completely determines the cohomology ring of G/H; the ring is constructed by suitable definition of a boundary operator in the tensor product $I_A(G)\otimes I_S(H)$. As a more specific result it has been found that a formula of the Hirsch type holds not only in the case where G and H have equal rank (cf. above), but also for symmetric spaces G/H (cf. §24); in this case the denominator polynomial can contain fewer factors than the nominator. A concept that has evolved in these developments is that of the Weil algebra $W(L)$ of a group G (or rather of the Lie algebra L of G); it is the tensor product of the Grassmann algebra $A(L)$ of skew forms on L and the algebra $S(L)$ of symmetric forms on L. The Weil algebra turns out to be the algebraic equivalent of a classifying space (for arbitrary dimension) for principal fibre bundles with fibre G (cf. §30). What corresponds to the cohomology ring of the base space is just $I_s(L)$ (with all degrees doubled). The notion of infinitesimal connection, after being given a very algebraic formulation, makes it possible to define, for a principal fibre bundle with fibre G and base space B, a homomorphism of $I_s(L)$ into the cohomology ring of B [the image is the so-called characteristic ring of the bundle (cf. §30)]; and one can give an algebraic construction for the full ring of the bundle from the connection.

Many of these and many other results, in particular for fields of arbitrary characteristic as coefficients (i.e., roughly speaking for the torsion group, cf. §16), have been obtained by A. Borel [9; 9a–9e], utilizing Leray's theory.

23. **Grassmann manifolds.** We come now to what can be called special results, namely results on explicit spaces or classes of spaces. The first results here are due to Ehresmann [32–37], on the Grassmann manifolds (cf. §17) and related spaces. The procedure is very direct; the space is decomposed into cells, and from the cell structure the invariants are calculated. The cells in question are of course not

arbitrary, but related to the algebraic-geometric properties of the space. For the Grassmann manifolds, e.g., the cells are defined by the so-called Schubert varieties [34; 35]: In Euclidean n-space E^n we consider a fixed increasing sequence $E^1\subset E^2\subset \cdots \subset E^{n-1}\subset E^n$ of subspaces (through the origin of E^n). Given now a sequence of integers a_i, $i=1, \cdots, k$, such that $0\leqq a_1<\cdots<a_k\leqq n-k$, we denote by $[a_1, a_2, \cdots, a_k]$ the set consisting of all k-dimensional subspaces L (through the origin) of E^n which have the property that $\dim L\cap E^{a_i+i}\geqq i$, for $i=1, \cdots, k$. This set is a (pseudo-) manifold of dimension $\sum_1^k a_i$; it is a generalized cell in the sense that it is obtainable from a Euclidean cell of the same dimension by making certain identifications on the boundary. All the cells so defined form a (generalized) cell decomposition of the Grassmann manifold $M_{n,k}$. The boundary relations (in the sense of combinatorial topology) have been determined by Ehresmann; the Betti numbers and torsion coefficients can then be computed. Going further in this direction, Chern [19] has determined the cohomology ring structure of $M_{n,k}$ (with coefficients mod 2); the interest in this ring is due to the fact that it is a universal antecedent of the so-called characteristic ring of any sphere bundle (cf. §30). (See also [**100a**; **100b**] in this connection.)

Similar methods have been used to determine the homology invariants of the quadrics (real, complex, quaternion, hermitian) and similar spaces [32–37; 95; 118].

A different cell decomposition, related to one used by J. H. C. Whitehead [136], has been employed by C. E. Miller [86] to investigate in detail the cohomology ring mod 2 of the orthogonal group and of several of its coset spaces, e.g. the Stiefel manifolds (cf. §30). The cells here are essentially products of projective spaces. It turns out that, while the Pontryagin ring mod 2 is still a Grassmann algebra (in particular the squares of the generators vanish) this is not so for the cohomology ring mod 2; the defining relations state that certain higher powers of the generators vanish. (In the simplest case, $O^+(3)$, which is projective 3-space, one has one generator x and the relation $x^4=0$.) In addition, the Steenrod squares (operations in the cohomology ring, recently introduced by Steenrod) are determined for these spaces, with an application to the problem of vector fields on the n-sphere (cf. §30 and [119]).

24. **Symmetric spaces.** A second method, also initiated by Ehresmann [32] and perfected by Iwamoto [51] applies to the symmetric Riemannian spaces; these are spaces which have the property that for each point the symmetry, i.e. the transformation which reverses

all geodesics through the point, is an isometry. Obvious examples are Euclidean spaces and spheres. Cartan has shown [13 (16)] that the symmetric spaces are exactly the coset spaces G/H, where G has an automorphism of order two, which leaves just the elements of H (or rather of the component of e in H) fixed, and where H is compact. As a nontrivial example let $G=SU(n)$ and let the automorphism be complex conjugation; $H=O^+(n)$ consists then of the orthogonal matrices, and G/H is the space of symmetric unimodular unitary matrices. We consider now the cohomology of an arbitrary Klein space of a compact group in terms of differential forms; by an averaging process (as in §§12 and 14) one can reduce all forms to invariant forms, where invariant means invariant under the action of the group. An invariant form is determined by its expression at any point x_0 (this leads at once to an inequality for the Betti numbers analogous to that of §12). Let us call adjoint group of the space the group of linear transformations in the tangent space at x_0, induced by the elements of G, which leave x_0 fixed (i.e. G_{x_0}, cf. §17); the value at x_0 of an invariant form is then invariant under the adjoint group. The symmetric spaces now have the special property that, by a theorem of Cartan [13 (15); 25], invariant forms are automatically closed. This means that the pth Betti number is equal to the number of independent invariant p-forms (modulo nothing), and, upon considering the values of the forms at x_0, that it equals the number of independent skew-symmetric tensors with p indices, invariant under the adjoint group; in more sophisticated language, one has to determine how often the trivial representation occurs in the pth exterior power of the adjoint group (cf. §12). As an example for the procedure, consider again the Grassmann manifold $M_{n,k}$, the manifold of k-planes in n-space, under the orthogonal group $O(n)$. The isotropy group is the direct product $O(k)\times O(n-k)$ (to keep a k-plane in itself, one can rotate in the plane, and also in the perpendicular $(n-k)$-plane). The adjoint group can be considered as operating on the vector space of all $k\times(n-k)$ matrices M; the element (T_k, T_{n-k}) (where $(T_k\in O(k)$, $T_{n-k}\in O(n-k))$ maps M into $T_kMT'_{n-k}$. One considers now the representation of $GL(k)\times GL(n-k)$, given by the same formula. The complete reduction of the pth exterior power of this representation has been given by Ehresmann [34]; one has to make use of the known theory of the representations of $GL(n)$, in particular the Young diagrams and the weights. For each irreducible constituent obtained, Iwamoto [51], using the known theory of representations of the orthogonal groups, determined what happens when one now restricts oneself to the orthogonal groups. One arrives at a fairly simple rule for the appearance of trivial representations or linear invariants, expressing the pth Betti number in terms of partitions of p. The procedure and the results are similar for all other symmetric spaces; Iwamoto treated also the classical groups themselves in this way. That the algebraic methods of §22 can be applied successfully to symmetric spaces has been mentioned already. Among other things this has led to an explicit formula for the Poincaré polynomials of the Grassmann manifolds.

25. **Subgroups of maximum rank.** Another class of (compact) coset spaces which was already mentioned in §18, and §22, consists of those spaces for which the isotropy groups are of maximum rank; according to §18 they are just the coset spaces of positive characteristic. It is easy to see that any such space is topological product of coset spaces of the same type of simple Lie groups, thus reducing the problem of classifying these spaces considerably. Wang [**125**; **126**] and Borel and de Siebenthal [**5**; **6**] have given a complete list of the subgroups of maximal rank of the classical groups, and (Borel and de Siebenthal) of the maximal subgroups of maximal rank of the exceptional groups. Naturally this analysis makes very explicit use of what one knows about the Lie algebras of the simple groups. (In the treatment of Borel and de Siebenthal a characterization of connected subgroups of maximal rank is taken as the starting point of the classification: such groups are the components of e of the normalizers of their own center.) As an example of the results: for the unitary unimodular group $SU(n)$ the subgroups in question are obtained by stringing groups of the type $U(k)$ (=group of unitary matrices of degree k) with properly chosen k-values along the diagonal and requiring the matrices so obtained to be unimodular. Out of the classification of these Klein spaces, with the help of the theory of a group modulo a maximal abelian subgroup (§18), one gets the characteristic; the Betti numbers and the ring structure are not obtained here. By the way, subgroups of maximum rank are automatically closed, due to a theorem of Malcev of §8.

Along these lines it has been shown: If a (simply connected) Klein space has characeristic 1, it is a single point [**127**]; if it has characteristic 2, it is a sphere; if it has characteristic a prime>2, it is one of a small list of spaces [**7**; **127**].

26. **Two-dimensional Klein spaces. Local groups of transformations.** A very natural problem arising here is that of determining all two-dimensional Klein spaces. Completing earlier results in which the Klein bottle had been overlooked, Mostow [**93**] showed that

exactly the following surfaces appear (noncompact groups allowed): plane, cylinder, torus, sphere, (open) Möbius strip, projective plane, Klein bottle. He arrived at this result by showing that for a coset space G/H with $\dim G - \dim H = 2$ the fundamental group has to be of a certain restricted type, which, by the classification of surfaces, admitted only the surfaces listed, and then showing explicitly that they actually are Klein spaces. His results include more, namely a complete classification (up to isomorphism) of all transitive Lie groups of transformations of surfaces.

The results mentioned in §8 on the closedness of subgroups also have a significance for Klein spaces: they relate to the problem of extending a local transformation group to a (global) transformation group. "Local transformation group" means roughly that in the definition of homogeneous space of §17, the T_g are defined only for the elements g in a neighborhood of e, and that they act only on some open subset of W; transitivity has to be defined properly. In an extension G and W may have to be replaced by locally equivalent objects. Starting from the simple observation that for a Klein space the isotropy group is always closed, it is shown [38; 93] that the extension problem amounts to determining which subalgebras of the Lie algebra of G (or which "local subgroups") give rise to closed subgroups in the simply connected form $\bar{G}$. In particular, all transitive local groups on an open set in E^k, $k<5$, can be extended to a group, acting transitively on some manifold M^k.

27. **A converse problem.** A problem which goes so to speak in the reverse direction is the following: Given a Klein space W, which groups (besides the one which defines W) can act transitively on W? (We assume the group to be effective, i.e., only the unit e gives the identity transformation of W, and compact.) It is possible to answer this question completely when W is a sphere [7; 8; 89]. To state the result: Except for three cases only those groups can act transitively on a sphere which are well known to do so, namely for even dimensions only the orthogonal group (plus G_2, which has S^6 as Klein space), for dimension $4n+1$ the orthogonal, the unimodular unitary, and the unitary group, for dimension $4n-1$ also the symplectic group and the full quaternion-unitary group (plus $S^{15} = \bar{B}4/\bar{B}3$ and $S^7 = \bar{B}_3/G_2$). The proofs depend on the general theorems on groups acting transitively on spheres (cf. §21), plus the knowledge of the Betti numbers of the classical groups. To indicate a simple case: A symplectic group can not act transitively on S^{4n+1} since otherwise, by the theorem of §21. it would have a primitive cocycle of dimension $4n+1$, which it doesn't by the results of §11. The same problem has

been treated by Matsushima [84] under the weaker assumption that W is a homology sphere, i.e., that its Betti numbers equal those of a sphere, of odd dimension. The results can be described by saying that, up to a small, finite number of cases, which remain undecided, any such space is either actually a sphere or—and this is the only new case—the space of unit tangent vectors to a sphere of even dimension ($=O(2n+1)/O(2n-1)$; with the notation of §30 we can write it as $V_{2n+1,2}$), and that the groups act in standard fashion. The methods are similar to those indicated above; much stronger use is made of the Lie algebras of the groups involved. An important role is played by a theorem of Kudo which says that under the stated assumption on W the isotropy group H is not ~ 0 in G; this is also contained in Koszul's results (§22).

For spheres of even dimension another approach has been given by A. Borel [7]: the sphere has characteristic two, and two is a prime number; this implies easily that the isotropy group must be a maximal subgroup of maximal rank, and now one can make use of the complete list of such groups in §25.

A similar result is that any group acting transitively and effectively on the torus T^n is actually T^n itself [7; 90]. For one of the proofs one shows that because of the size of the Betti numbers of T^n, i.e., because of the large number of invariant differential forms, the isotropy group has to reduce to the unit element (or at any rate be finite); this implies that the group is homeomorphic with T^n, and therefore, using the known structure of compact groups (§10), is isomorphic with T^n.

28. **Two-point homogeneous spaces.** A recent result of H. C. Wang [128] belongs to this group of problems. The question was to determine all compact (connected) metric spaces, which are two-point homogeneous in the sense that, for any four points a, b, c, d with $\rho(a, b)=\rho(c, d)$, there exists an isometry sending a into c and b into d. Wang showed by a simple ingenious argument that the group of isometries cannot have arbitrarily small normal subgroups, and therefore must be a (compact) Lie group, and the space is therefore a Klein space (of finite dimension, say n). The two-point condition implies that the isotropy group must be transitive on the directions issuing from a point, or in other words, that it is transitive over the $(n-1)$-sphere, and one can then apply the results described in §27. One obtains a complete determination of all possibilities; the spaces occurring are: for odd n only the spheres and projective spaces, for even n there are in addition complex projective spaces, quaternion

projective spaces, and the (16-dimensional) projective plane over the Cayley numbers.

29. **Noncompact Klein spaces.** Not too much is to be said about noncompact Klein spaces. Cartan showed [13 (15)] that the symmetric Riemannian spaces, whose groups of isometries are noncompact simple Lie groups, are all homeomorphic to E^n. He showed actually that in the group there exists a set of elements, called transvections (which behave somewhat like translations in Euclidean space), which set is homeomorphic with E^n, and whose elements are in 1:1 correspondence with the points of the symmetric space (under the natural projection). Incidentally, all these spaces have negative or zero Riemannian curvature (whereas closed symmetric spaces have non-negative curvature).

The analogue of the Malcev-Iwasawa theorem does not hold in general; there are spaces which are not Cartesian product of a compact space and an E^n. An example is given by one of the two sets complementary to a hyperquadric in (real or complex) projective space. Factorization does hold in the special case of a Klein space of a nilpotent Lie group [82; 85] (a Lie algebra L and its group are nilpotent if the sequence L, $L_1 = [L, L]$, $L_2 = [L, L_1]$, $\cdots$ ends with 0). For the proof one can start from the fact that the (simply connected) group is homeomorphic with E^n (§8), and that it is possible to give a description of the discrete subgroups (each such subgroup has the geometric structure of a lattice in E^n, but is in general noncommutative). Chevalley has shown (not yet published) that factorization (into a torus and a Euclidean space) holds for Klein spaces of solvable groups, provided the isotropy group is connected. Some new results have recently been announced by Mostow [94].

Nilpotent groups also provide some examples of compact Klein spaces, whose group is noncompact (the isotropy group, which by §18 must be nonconnected, is here even discrete) [85]. The results obtained concern the first and second rational cohomology group; in particular the first Betti number is found to equal dim $L -$dim L', generalizing the fact that for the torus T^n the first Betti number equals n.

30. **Homotopy properties.** As a last class of topological properties we should like to mention briefly the homotopy properties; this means principally the question of the homotopy groups [the kth homotopy group $\pi_k(G)$ is the group of continuous maps of a k-sphere S^k into G, or rather of classes of maps which can be deformed continuously into each other, under a certain addition, generalizing the

elements and the operation of the fundamental group $\pi_1(G)$] (cf. [116] for this whole section).

That the fundamental group of a group is always abelian and that it is finite for compact semi-simple groups was mentioned earlier. It was also known long ago [13] that the second homotopy group $\pi_2(G)$ of a Lie group G is always zero, which means that every continuous map of an S^2 into the group can be deformed into a single point (this follows from the construction at the end of §19, since the image of a 2-sphere can always be so deformed as to not meet the $(n-3)$-dimensional set of singular elements). For coset spaces the situation is different, as the space S^2 itself shows. This sheds some light on the theorem of Malcev and Iwasawa (§8) on the simple connectedness of a normal subgroup N of a simply connected group G: We consider the exact homotopy sequence of G and N and the fibre space isomorphism of $\pi_2(G/N)$ with the relative group $\pi_2(G; N)$ [116]. Since $\pi_1(G)$ and $\pi_2(G)$ vanish, it follows that $\pi_1(N)$ is isomorphic with $\pi_2(G/N)$, even if N is not normal; if N is normal, then $\pi_2(G/N)$ is zero, and so is then $\pi_1(N)$. The results on higher homotopy groups are far from complete; what is known is in very close connection to what is known about the homotopy groups of spheres; the reason for this is that most of the existing results are obtained for groups acting transitively on spheres, by fibre space relations [29; 30; 126; 133; 134; 136; 116]. A special result of Pontryagin's [100] says that $\pi_4(A_n)=0$, for $n \geq 2$; this is of interest because it shows that A_n is not Cartesian product of S^3 with another space, whereas the homology properties would allow it.

Homotopy properties of Lie groups also play a role for the theory of fibre bundles. One particular problem relates to the so-called classifying or universal bundles. For a group G, a (principal) fibre bundle B with fibre G is called n-classifying or n-universal if it is arcwise connected and the homotopy groups $\pi_i(B)$ vanish for $1 \leq i < n$. (The bundles, which have G as transformation group of their fibre, and which have a polyhedron X of dimension $<n$ as base space, are then in 1:1 correspondence with the homotopy classes of maps of X into the base space of B, whence the term classifying [116; 137].) Now, if G is a compact Lie group, it is easy to construct such a bundle for any n: G can be considered as subgroup of some orthogonal group $O(k)$ (since it has a faithful representation). We form now $O(n+k)/O(n) \times I_k$ (where $O(n) \times I_k$ means the elements of $O(n+k)$ whose last k diagonal elements are 1); this is a fibre bundle over $O(n+k)/O(n) \times G$, with fibre $O(n) \times G/O(n) \times I_k = G$ (cf. §17). Moreover its homotopy groups π_i vanish for $1 \leq i < n$; for $k=1$ this

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follows from the fact that $O(n+1)/O(n)\times I_1$ is nothing else but the sphere S^n; for larger k it follows similarly, by considering the chain $O(n+k)$, $O(n+k-1)$, $\cdots$, $O(n)$. An interesting example is obtained for $G=O(k)$, the case of sphere bundles, where the fibre is a $(k-1)$-sphere; the base space $O(n+k)/O(n)\times O(k)$ of the universal bundle can then be identified with the space of $(k-1)$-spheres on a $(n+k-1)$-sphere considered by Whitney [137], or equivalently with the Grassmann manifold $M_{n+k,k}$ of k-planes in $(n+k)$-space. Knowledge of the homology properties of $M_{n+k,k}$ is then of obvious importance, since for sphere bundles, by the classification theorem above, one has to consider maps of a space X into $M_{n+k,k}$; such a map induces a homomorphism of the cohomology ring of $M_{n+k,k}$ into that of X, which depends only on the bundle corresponding to f; the image ring is called the characteristic ring of the bundle. (An algebraic counterpart has been mentioned in §22.)

The manifold $O(m)/O(m-k)\times I_k$, which appeared here, can also be described as the space, each point of which is a system of k mutually orthogonal unit vectors at the origin in Euclidean m-space (a k-frame); it is called the Stiefel manifold $V_{m,k}$. The $V_{m,k}$ were introduced by Stiefel [120] in connection with the problem of constructing over a manifold M^m a system of k vector fields, mutually orthogonal at each point of M^m. One is led, in obstruction theory [116], to a characteristic cocycle, which measures the degree of impossibility of the construction of the vector fields. The dimension of the cocycle is the first dimension for which the homotopy groups of $V_{m,k}$ are not zero plus one and the coefficients are taken exactly from this group. As indicated above, the critical dimension for $V_{m,k}$ is $m-k$; the corresponding groups $\pi_{m-k}(V_{m,k})$ are known [116]. To quote the result: they are infinite cyclic if $m-k$ is even or $k=1$, of order two if $m-k$ is odd and $k>1$.

One special, but important, case is the following: We have seen that the sphere S^n is the coset space of the orthogonal group $O(n+1)$ by the subgroup $O(n)$. Does there exist a map f of S^n into $O(n+1)$, such that f followed by the projection of $O(n+1)$ onto $O(n)$ is the identity? Phrased differently: does there exist a continuous family of rotations, simply transitive over the sphere; or again, does the decomposition of $O(n+1)$ into cosets of $O(n)$ admit a cross section, i.e., a surface, which meets each coset of $O(n)$ in exactly one point? If a cross section existed, then, as one sees easily, $O(n+1)$ would be the Cartesian product of $O(n)$ and S^n, and, more interesting, there would exist n everywhere independent vector fields, tangent to S^n, i.e. in the language of topology, S^n would admit a parallelism. The obvious approach to this is to construct a cross section by starting at an arbitrary point of $O(n+1)$, expanding it gradually and studying the singularity or obstruction, into which one runs after a while [116]. We only state some results: It is an old story that spheres of even dimension do not admit a single (nonzero) vector field. B. Eckmann and G. W. Whitehead have shown that spheres of dimension $4n+1$ admit only one independent vector field [29; 133]; for spheres of dimension $4n-1$ there exist more than one field, constructed e.g. with quaternions and Cayley numbers. Recently Steenrod and J. H. C. Whitehead [119] have shown, with the help of new homology invariants, that at most the spheres of dimension 2^k-1 admit a parallelism (the only cases where a parallelism is known are S^1, S^3, and S^7, corresponding to complex numbers, quaternions, and Cayley numbers (cf. also [86])).

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UNIVERSITY OF MICHIGAN

THE OCTOBER MEETING IN WASHINGTON

The four hundred seventy-third meeting of the American Mathematical Society was held at the National Bureau of Standards on Saturday, October 27, 1951, in conjunction with a meeting of the Institute of Mathematical Statistics. This meeting was a portion of the celebration of the fiftieth anniversary of the founding of the National Bureau of Standards. The registered attendance at the meeting was 225 and included the following 180 members of the Society.

J. C. Abbott, Milton Abramowitz, M. I. Aissen, T. W. Anderson, R. P. Bailey, J. H. Barrett, L. K. Barrett, E. F. Beckenbach, E. G. Begle, E. E. Betz, Gertrude Blanch, R. C. Blanchfield, J. H. Blau, Joseph Blum, H. W. Bode, H. F. Bohnenblust, J. R. Bowman, A. T. Brauer, R. S. Burington, L. J. Burton, G. H. Butcher, F. P. Callahan, W. R. Callahan, H. C. Carter, W. C. Carter, E. A. Coddington, L. W. Cohen, Natalie Coplan, E. F. Cox, G. F. Cramer, A. B. Cunningham, J. H. Curtiss, G. B. Dantzig, D. A. Darling, C. R. DePrima, J. B. Diaz, W. F. Donoghue, William H. Durfee, Samuel Eilenberg, Benjamin Epstein, J. L. Ericksen, Trevor Evans, F. D. Faulkner, N. J. Fine, F. H. Fowler, L. K. Frazer, F. N. Frenkiel, L. M. Fulton, A. S. Galbraith, Landis Gephart, H. H. Germond, H. E. Goheen, Michael Goldberg, Leon Goldstein, R. A. Good, R. D. Gordon, W. H. Gottschalk, E. C. Gras, J. W. Green, R. E. Greenwood, D. W. Hall, Marshall Hall, Jr., J. F. Hannan, L. S. Hart, Philip Hartman, H. J. Hasenfus, E. V. Haynsworth, G. A. Hedlund, Vaclav Hlavatý, A. J. Hoffman, Temple Hollcroft, L. A. Hostinsky, E. A. Hoy, D. R. Hughes, Witold Hurewicz, S. B. Jackson, F. E. Johnston, Joseph Kampé de Fériet, R. E. Keirstead, M. S. Klamkin, S. H. Lachenbruch, Jack Laderman, O. E. Lancaster, Lamar Layton, Patrick Leehey, Marguerite Lehr, B. A. Lengyel, J. H. Levin, D. C. Lewis, T. P. G. Liverman, B. J. Lockhart, D. B. Lowdenslager, Eugene Lukacs, E. J. McShane, Irwin Mann, Murray Mannos, C. G. Maple, W. H. Marlow, M. H. Martin, W. T. Martin, Joseph Milkman, C. E. Miller, D. D. Miller, H. J. Miser, Don Mittleman, R. W. Moller, E. W. Montroll, C. N. Moore, T. W. Moore, W. E. Moore, C. R. Morris, W. R. Murray, P. P. Nesbeda, I. L. Novak, M. W. Oliphant, Alex Orden, J. C. Oxtoby, L. E. Payne, Anna Pell-Wheeler, I. D. Peters, G. W. Petrie, J. W. Ponds, F. M. Pulliam, O. J. Ramler, C. J. Rees, M. S. Rees, J. N. Rice, R. P. Rich, P. R. Rider, H. P. Robertson, L. V. Robinson, L. R. Sario, S. S. Saslaw, I. R. Savage, J. B. Scarborough, A. T. Schafer, R. D. Schafer, Henry Scheffé, G. E. Schweigert, I. E. Segal, W. H. Sellers, J. A. Silva, R. C. Simpson, A. D. Solem, J. J. Sopka, G. L. Spencer, I. A. Stegun, W. J. Strange, E. G. Swafford, Olga Taussky, J. H. Taylor, B. J. Tepping, Feodor Theilheimer, W. R. Thickstun, J. A. Tierney, John Todd, C. B. Tompkins, A. W. Tucker, J. W. Tukey, S. M. Ulam, J. L. Vanderslice, A. H. Van Tuyl, M. C. Waddell, D. H. Wagner, G. C. Webber, H. F. Weinberger, Harry Weingarten, Alexander Weinstein, M. E. White, P. M. Whitman, G. T. Whyburn, L. S. Whyburn, R. F. Williams, N. Z. Wolfsohn, M. A. Woodbury, D. M. Young, F. H. Young, E. H. Zarantonello, Neal Zierler, J. A. Zilber.

Professor J. C. Oxtoby, of Bryn Mawr College, addressed the Society at 2:00 P.M. on *Ergodic sets* by invitation of the Committee to Select Hour Speakers for Eastern Sectional Meetings. This session was presided over by Dr. S. M. Ulam.

There were sessions for the presentation of contributed papers at 10:00 A.M. and 3:15 P.M. presided over by Professor H. P. Robertson and Professor C. N. Moore respectively.

At 1:30 P.M. members of the Society and of the Institute were welcomed by Dr. A. V. Astin, Acting Director of the National Bureau of Standards, at a joint session presided over by Dr. J. H. Curtiss, Chief of the Applied Mathematics Division.

Abstracts of the papers presented at the meeting are listed below, those with the letter "*t*" after their numbers having been read by title. Papers numbered 9 and 27 were read by Dr. Taussky and Dr. Carter respectively. Mr. Boone was introduced by Professor K. L. Chung, Mr. Leslie and Dr. Love by Professor L. C. Young.

ALGEBRA AND THEORY OF NUMBERS

1*t*. A. A. Aucoin: *Systems of Diophantine equations.*

Solutions of the following Diophantine systems are given: (1) $\prod_{j=1}^{m}\sum_{k=1}^{s} a_{ijk}x_k = f_i(y_i)$ $(i=1, \cdots, n)$, where $f_i(y_i)=f_i(y_{i1}, \cdots, y_{iv})$ are homogeneous polynomials of degree m with integral coefficients and m and n are relatively prime. (2) $f_1(x) = g_1(u)$, $f_2(x)=g_2(v)$, where $f_1(x)=f_1(x_1, \cdots, x_p)$, $f_2(x)=f_2(x_1, \cdots, x_q)$ are homogeneous polynomials of degree n and are such that integers $x_i=a_i$ exist for which all the partials of f_1 as well as those of f_2, of all orders less than $n-1$, vanish. g_1 and g_2 are homogeneous polynomials with integral coefficients of degree m where m and n are relatively prime. (3) $f_1(x_i, y_i, z_i)=g_1(x_i, y_i, z_i)$, $f_2(x_i, y_i, z_i)=g_2(x_i, y_i, z_i)$, where the functions involved are polynomials, homogeneous in each of the sets of variables x_i, y_i, z_i. The solution depends upon the degree of homogeneity. The method applies to more than two equations. (Received August 15, 1951.)

2*t*. H. W. Becker: *Planar rhyme schemes.*

Among forty odd other interpretations, $M_n=(2n, n)/(n+1)$ is the number of planar rhyme schemes of n letters. (Nonplanar rhyme schemes, such as *abab*, have crossovers in their Puttenham diagrams.) A lexicon theorem ranks them individually. Known and new breakdowns of M_n have planar rhyme significations: ${}_{\mathrm{I}}M_{n,m}=M_{n-m}\cdot M_{m-1}$ (Euler), last "*a*" in mth position; ${}_{\mathrm{II}}M_{n,m}=(2n-m-1, n-m)m/n$ (reverse Delannoy), m *a*'s, initial ascending run of length m, $m+1$ progeny in M_{n+1}; ${}_{\mathrm{III}}M_{n,m}=2^{n-1-2m}(n-1, 2m)M_m$ (Touchard), m inversions; ${}_{\mathrm{IV}}M_{n,m}=(n, m-1)(n-1, m-1)/m$ (W. H. Wise), m different letters, m ascending runs, $m-1$ rising pairs; ${}_{\mathrm{V}}M_{n,m}=(n, m)\Delta^{n-m}M_0=(n, m)N_{n-m}$, m singletons (unrhymed letters); ${}_{\mathrm{VI}}M_{n,m}=(n-1, m)\Delta^{n-m}M_1$, m couplets (consecutive-rhyme pairs); ${}_{\mathrm{VII}}M_{n,m}=N^{n-m}(N+1)^{m-1}$, last singleton in mth position. These lead to variations of the classic generating function for M_n. Thus $N_n=(M-1)^n$, the subset of M_n without singletons, has the generating function $1/(1-tN)=[1-\{(1-3t)/(1+t)\}^{1/2}]/2t$. (Received September 12, 1951.)

3*t*. H. W. Becker: *Network severances.*

Given a net $m\times n$ subject to simultaneous failure of one knot in each row, what are the possibilities ${}_mS_n$ of severances of the net? The difference equations through $m=6$ are set up, solution of the last one of the second degree being ${}_4S_n=2F_{2n}$ (F the Fibonacci series). A more interesting approach is ${}_mS_n=m\cdot 3^{n-1}-2V_n$, the subtrahend being an edge effect correction. $V_n=3V_{n-1}+W_{n-1}$, where W_n is the number of rhyme schemes of n letters, such that consecutive letters are of the form k, k or k, $k\pm 1$. In turn, $W_n=3W_{n-1}-X_{n-1}$, where $X_n=\Delta^n M_1={}^aW_n$, the membership of W_n terminating in "a". There are numerous relations between V_n, W_n, X_n, and $M_n=(2n, n)/(n+1)$. For example, if ${}^m\Delta U_n=U_{n+1}-m\cdot U_n$ (J. Touchard, International Mathematical Congress, Toronto, pp. 465–472), the case $m=3$ inverts to $W_{n+1}=(3-M)^n$, $V_{n+1}=M^{-1}(3-M)^n(-)^{n-1}$, and the elegant dual $X_{n+1}=M(3-M)^n$, $M_{n+1}=X(3-X)^n$. Typical of their generating functions is $1/(1-tX)=[3t-1-\{1-t(2+3t)\}^{1/2}]/2t$. There are also numerous subset isomorphisms with planar rhyme schemes. (Received September 12, 1951.)

4. W. W. Boone: *An extension of a result of Post.* II.

Magnus (J. Reine Angew. Math. vol. 103) terms a certain alteration in the form of a word of a group the *elimination of a generator*. The problem of determining when—for a given word—a similar elimination, described below, can be carried out is unsolvable. One can explicitly exhibit: a particular group, $G_{T'}$, with a *finite* number of defining relations R_i on a *finite* number of generators, $g_1, g_2, \cdots, g_n$, such that it is recursively unsolvable to determine for an arbitrary positive word W of $G_{T'}$ (i.e., a word in which all exponents are positive) whether or not W is expressible as a positive word W' not containing g_1 and made up only of generators occurring in W. Alternatively, regarding exponentiation as an abbreviation, words as made up of the distinct *symbols*, g_i^{+1}, g_i^{-1}, and $g_i^{+1}g_i^{-1}=g_i^{-1}g_i^{+1}=$ THE VOID WORD as among the R_i, the foregoing obviously may be formulated as an (unsolvable) problem regarding the *elimination of a symbol*. The result follows by a comparison of proofs of (cf. abstract, *An extension of a result of Post*, Journal of Symbolic Logic vol. 16, no. 3) $\sum\vdash {}_sW_1$ and $\sum Bq_0B\Omega\dashv\vdash {}_sW_2$, T_s amended to represent $G_{T'}$, B a free generator, W_1 and W_2 arbitrary words, Ω such that every T_s-symbol not a q-symbol occurs therein. (Received September 13, 1951.)

5. A. T. Brauer: *Limits for the characteristic roots of a matrix.* V.

It was shown in part 2 of this paper (Duke Math. J. vol. 14 (1947) pp. 21–26) that each characteristic root of an arbitrary square matrix of order n must lie in the interior or on the boundary of at least one of certain $n(n-1)/2$ ovals of Cassini which are determined by the elements of the matrix. It is shown in this paper that this result remains correct if each of these ovals is replaced by another oval which in general is smaller. From this result an improvement of the well known theorem of Minkowski on positive determinants is obtained. (Received September 13, 1951.)

6t. Jesse Douglas: *On finite groups with two independent generators.*

Although the theory of finite groups is a well worked branch of mathematics, the study of the present topic in its full generality seems not to have been previously undertaken. A detailed account will appear in a series of notes to be published in Proc. Nat. Acad. Sci. U.S.A. vol. 37 (1951). (Received July 12, 1951.)

7t. Jesse Douglas: *On the invariants of finite abelian groups.*

This paper offers a new proof, of a simple and illuminating nature, of the classic theorem which characterizes any finite abelian group to within isomorphism by its invariants—certain prime-power factors of its order. Some new interpretations of these invariants are brought to light in terms of functions $I(x)$, $J(x)$, $K(x)$ determined by the structure of the group—e.g., $p^{I(x)}$ is the number of elements whose period is a divisor of p^x. (The group is presumed to be of prime-power order p^a, which entails no essential loss of generality.) Graphical considerations are adduced: e.g., the graph of $I(x)$ is always rising and concave downward (both in the wide sense), the slope of its initial segment is equal to the number of elements in any basis, elements of order p^x are present in a basis if and only if the graph has a corner at the point x, and the number of such elements is equal to the "curvature" of the graph at that point, etc. The detailed account will appear in Proc. Nat. Acad. Sci. U.S.A. vol. 37 (1951). (Received July 11, 1951.)

8. Marshall Hall, Jr.: *A combinatorial problem on abelian groups.*

If $\begin{pmatrix} a_1, \cdots, a_n \\ c_1, \cdots, c_n \end{pmatrix}$ is a permutation of the elements of a finite abelian group A, then the sum $b_1 + \cdots + b_n$ of the differences $c_i - a_i = b_i$, $i = 1, \cdots, n$, is zero (the group operation being addition). It is shown conversely that if we are given elements b_i with $b_1 + b_2 + \cdots + b_n = 0$, then there exists a permutation with these b's as its differences in some order. (Received August 20, 1951.)

9. T. S. Motzkin and Olga Taussky: *Matrices with property* L.

Two $n \times n$ matrices A and B are said to have property P—considered by Frobenius, McCoy and other—(or property L, considered by Kac) if there exists an arrangement $\lambda_1, \cdots, \lambda_n$ and $\mu_1, \cdots, \mu_n$ of their characteristic roots such that any function (or linear function) $f(A, B)$ has $f(\lambda_i, \mu_i)$ as its characteristic roots. Pairs of matrices with property L are investigated, and it is shown in particular that hermitian matrices with property L have property P, are commutative, and can be transformed into diagonal matrices by the same similarity. Matrices with property L and $n > 2$ do not in general have property P. (Received August 23, 1951.)

10. D. H. Wagner: *On free products of groups.* Preliminary report.

It is proved that the homomorphisms on a free group F onto a free product are characterized as those homomorphisms (onto) which map some free basis of F into the union of the free factors. This result was, in effect, obtained by I. Gruschko (Rec. Math. (Mat. Sbornik) N. S. vol. 8 (1940) pp. 169–182) for the case where F is finitely generated. The present proof uses a transfinite convergence procedure developed by H. Federer and B. Jónsson (Trans. Amer. Math. Soc. vol. 68 (1950) pp. 1–27) in conjunction with a procedure developed by the writer for reducing a finite subset of a free product. These reductions are motivated by J. Nielsen's reductions for free groups (Matematisk Tidsskrift (1921) pp. 77–94) and possess similar properties. An example is given of a free product which is a homomorphic image of a group which cannot be decomposed into a free product. (Received April 28, 1951.)

ANALYSIS

11*t*. A. A. Aucoin: *A generalization of Abel's transformation.*

Abel's transformation is generalized to $\sum_{j=n+1}^{n+k} a_j^p b_j = \sum_{j=n+1}^{n+k} A_j^p (b_j - b_{j+1}) - A_n^p b_{n+1} + A_{n+k}^p b_{n+k+1} - \sum_{i=0}^{p-2} C_{p,i+1} \sum_{j=n}^{n+k-1} A_j^{p-1-i} a_{j+1}^{i+1} b_{j+1}$, where $A_k = \sum_{r=0}^{k} a_r$ and p is an integer > 1. Among the theorems on infinite series proved with this transformation are two which follow when $b_j = 1$. The first is: If $\sum_{j=0}^{\infty} a_j^p$ converges to S and if $\sum_{j=0}^{\infty} A_j^{p-1-i} a_{j+1}^{i+1}$ converge to S_i $(i = 0, \cdots, p-2)$, then A_n^p converges to $S +$

$\sum_{i=0}^{p-s} C_{p,i+1} S_i$. The second is a corresponding theorem when the a's are functions of a variable x. (Received August 15, 1951.)

12*t*. E. A. Coddington: *A characterization of ordinary self-adjoint differential systems.*

If D denotes the set of all complex-valued functions of class C^n, $n \geqq 1$, on a bounded interval $a \leqq t \leqq b$, the formal linear differential operator L of order n is defined for all $x \in D$ by $L(x) = p_0 x^{(n)} + p_1 x^{(n-1)} + \cdots + p_n x$, where the p_m are complex-valued functions of class C^{n-m}, and $|p_0(t)| \neq 0$, on $a \leqq t \leqq b$. Suppose L coincides with its Lagrange adjoint L^*, defined for all $x \in D$ by $L^*(x) = (-1)^n(\bar{p}_0 x)^{(n)} + (-1)^{n-1}(\bar{p}_1 x)^{(n-1)} + \cdots + \bar{p}_n x$. Let $l_i(x) = \sum_{j=1}^{n} M_{ij} x^{(j-1)}(a) + \sum_{j=1}^{n} N_{ij} x^{(j-1)}(b) = 0$, $i = 1, \cdots, n$, be n linearly independent boundary conditions with complex constants M_{ij}, N_{ij}. It is shown that the differential system consisting of L and the conditions $l_i(x) = 0$, $i = 1, \cdots, n$, is self-adjoint if and only if $MB^{-1}(a)M^* = NB^{-1}(b)N^*$, where M and N are the matrices with elements M_{ij}, N_{ij} respectively, and $B(t)$ is the nonsingular skew-hermitian matrix (depending only on the coefficients p_m in L) associated with the semi-bilinear form in Green's formula. This result can be used to characterize self-adjoint operators in the Hilbert space $L^2(a, b)$ which arise in a natural way from the formal operator L. (Received September 11, 1951.)

13*t*. E. A. Coddington and Norman Levinson: *A boundary value problem for a nonlinear differential equation with a small parameter.*

The purpose of this paper is to relate the existence, uniqueness, and general behavior of the solution $y = y(x, \epsilon)$, for small $\epsilon > 0$, of the two-point boundary value problem (B): $\epsilon y'' + f(x, y)y' + g(x, y) = 0$, $(' = d/dx)$, $y(0) = y_0$, $y(1) = y_1$, with the solution $u = u(x)$ of the corresponding "degenerate" initial value problem (I): $f(x, u)u' + g(x, u) = 0$, $u(1) = y_1$. Theorem 1 (Existence). Let $(0, y_0)$, $(1, y_1)$ be two points in the real (x, y)-plane, and assume: (1) f, g are real functions such that the problem (I) has a solution $u = u(x)$ on $0 \leqq x \leqq 1$, with $u(0) \geqq y_0$; (2) f, g have continuous first derivatives in a region $R: 0 \leqq x \leqq 1$, $|y - u(x)| \leqq a$, $a > 0$, which includes the point $(0, y_0)$; (3) there exists a constant $k > 0$ such that $f > k$ for (x, y) in R. Then, for all sufficiently small $\epsilon > 0$, there exists in R a solution $y = y(x, \epsilon)$ of (B). Further, $y(x, \epsilon) \to u(x)$, $y'(x, \epsilon) \to u'(x)$, as $\epsilon \to 0$, uniformly on any subinterval $0 < \delta \leqq x \leqq 1$. Theorem 2 (Uniqueness). Under the assumptions of Theorem 1, for sufficiently small $\epsilon > 0$, there exists at most one solution $y = y(x, \epsilon)$ of (B) in any region $R_0: 0 \leqq x \leqq 1$, $|y - u(x)| \leqq a_0 < a$, $a_0 > 0$. The theorems hold under slightly weaker, but more complicated, assumptions. If $\epsilon < 0$, the role of the left and right boundaries $x = 0$ and $x = 1$ are interchanged. (Received August 15, 1951.)

14. D. A. Darling: *On the distribution of the roots of certain almost periodic functions.*

Let $\alpha_1, \alpha_2, \cdots, \alpha_n$ be rationally independent real numbers and $f(t) = \sum_{i=1}^{n} a_i \sin \alpha_i t$. Let the positive roots of $f(t)$ be $t_1 < t_2 < \cdots < t_i < \cdots$ and define $N_k(x)$ as the number of those intervals (t_{i+1}, t_i) whose length is $\leqq x$ for $i = 1, 2, \cdots, k$. Then $\lim_{k \to \infty} N_k(x)/k = F(x)$ exists and it is a matter of some theoretical and practical interest to find this distribution function. The following result is proven: Let $X_1, X_2, \cdots, X_n$ be independent random variables, each having a uniform distribution over $(0, 2\pi)$; let ξ^* be the smallest positive and ξ_* be the largest negative root of the equation $\sum a_i \sin (X_i + \alpha_i \xi) = 0$, and let $G(x)$ be the distribution function of the random variable $U=(\xi^*-\xi_*)(\sum\alpha_i^2)^{1/2}$. Then $F(x)=(E(U^{-1}))^{-1}\int_x^\infty(1/t)dG(t)$. Under certain restrictions on the $\{a_i\}$ and $\{\alpha_i\}$, in the n-dimensional space the $(n-1)$-dimensional surface $\sum_{i=1}^n a_i \sin x_i=0$ and a system of lines having direction numbers $\alpha_1, \alpha_2, \cdots, \alpha_n$ are considered. Then $F(x)$ gives the distribution of lengths of the secants intercepted by two adjacent sheets of the surface. The mean and variance have a correspondingly simple geometrical interpretation. In general, if $p_1(t), p_2(t), \cdots, p_n(t)$ are arbitrary functions having a common period, the roots of $\sum a_i p_i(\alpha_i t)$ can be treated in the above manner. (Received September 4, 1951.)

15*t*. H. Margaret Elliott: *On approximation to analytic functions by rational functions.*

Let C consist of a finite number of mutually exterior analytic Jordan curves. Let $w=\phi(z)$ map the exterior of C conformally onto $|w|>1$ so that the points at infinity in the two planes correspond; denote by C_R the locus $|\phi(z)|=R>1$. Let $f(z)$ be analytic in the interior of C and continuous on the corresponding closed region $\overline{C}$. Suppose $f^{(k)}(z)$ exists and is continuous on C with modulus of continuity $\omega(\delta)$, $\omega(\delta)\neq 0$. Let points α_{nj}, $j=1, \cdots, n$; $n=1, 2, \cdots$, be given on or exterior to C_A, $A>1$. It is proved that there exist rational functions $r_n(z)=(a_{n0}z^n+a_{n1}z^{n-1}+\cdots+a_{nn})/(z-\alpha_{n1})\cdots(z-\alpha_{nn})$ such that $|f(z)-r_n(z)|\leq M\omega(1/n)/n^k$ for z on $\overline{C}$. Conversely, under a mild hypothesis on C and $\Omega(x)$, it is proved that if rational functions $r_n(z)$ all of whose poles lie on or exterior to C_A, $A>1$, exist such that $|f(z)-r_n(z)|\leq\Omega(n)/n^k$ for z in $\overline{C}$, then $f(z)$ is analytic interior to C and continuous on $\overline{C}$; $f^{(k)}(z)$ exists and is continuous on $\overline{C}$ with modulus of continuity $\omega(\delta)\leq L[\delta\int_a^{a/\delta}\Omega(x)dx+\int_{1/\delta}^\infty[\Omega(x)/x]dx]$, where $a>1$ is a constant independent of δ. (Received August 21, 1951.)

16*t*. Abolghassem Ghaffari: *Laguerre polynomials satisfying a certain functional equation.*

This paper endeavors to find the most general solution of the functional equation (1) $f(x, s; y, t)=\int_V f(x, s; z, u)f(z, u; y, t)dz$, $s<u<t$, where f is Lebesgue-measurable in V. By applying Fréchet's method (Proc. London Math. Soc. vol. 39 (1935) pp. 515–540) one is led to take for the solution of (1) the series (2) $f(x, s; y, t)=\sum_{n=0}^\infty A_n(x, s)B_n(y, t)$, where A_n, B_n form a complete biorthonormal set of functions over V. Using this method and taking for the region V the interval $(0, +\infty)$, one finds that (3) $f(x, s; y, t)=\sum_{n=0}^\infty\psi_n(x)\psi_n(y)\theta^n(s, t)$, where the infinite sequence of functions $\psi_n(x)=\Gamma^{1/2}(n+1)\Gamma^{-1/2}(n+\alpha+1)x^{\alpha/2}\exp(-x/2)L_n^{(\alpha)}(x)$ form a complete orthonormal set over $(0, +\infty)$, $\theta(s, t)=a(s)/a(t)$ where $a(s)$ is a positive increasing function $\neq 0$, and $L_n^{(\alpha)}(x)$ is the nth Laguerre polynomial. It is shown that for s, t fixed such that $s<t$, and x, y varying arbitrarily, the series (3) is absolutely convergent in $(0, +\infty)$ and uniformly convergent over $(\epsilon, +\infty)$, ϵ being an arbitrary small but fixed positive number. It is proved that for $s=t$ and $x\neq y$ the series (3), being divergent, is summable uniformly in $(0, +\infty)$ by the method of arithmetic means with *zero sum* everywhere; and that for $s=t$, $x=y$, the series (3) diverges essentially and becomes *infinite*. The author has shown that the solution (3) is different from those obtained by A. Kolmogoroff (Math. Ann. vol. 104 (1931) pp. 415–458). (Received September 4, 1951.)

17*t*. R. V. Kadison: *Infinite unitary groups.*

The uniformly closed invariant subgroups of the unitary groups of the various factors are determined. It is shown that the unitary groups of factors of type II_1

and III are topologically simple. The uniform closure $G_f^{(1)}$ of the set of unitary operators which act as the identity on the complement of some subspace of finite relative dimension is a closed, proper, invariant subgroup of the unitary groups of factors of type I_∞ (all bounded operators) and factors of type II_∞. All such subgroups are obtained as direct products of $G_f^{(1)}$ with closed subgroups of the scalars $\{\lambda I; |\lambda| = 1\}$. (The group $G_f^{(1)}$ is itself topologically simple but is not algebraically simple.) It follows, for example, that finite products of self-adjoint unitary operators, or unitary operators with three points in their spectrum, etc., contained in a factor, lie uniformly dense in the group of all unitary operators in the factor. (Received September 4, 1951.)

18. M. S. Klamkin: *Asymptotic sums of series of the form* $\sum_{(b_r \neq b_s)=1}^{N} (\prod_{r=1}^{p} a_{b_r})^{-1}$.

Asymptotic sums of the series $\sum_{(a_r \neq a_s)=1}^{N} (a_r a_s)^{-1}$, $\sum_{(a_r \neq a_s)=1}^{N} (a_r a_s a_t)^{-1}$, and $\sum_{(a_r \neq a_s)=1}^{N} (a_r a_s a_t a_u)^{-1}$ are derived, assuming that $\sum_1^{\infty} (a_r)^{-1}$ diverges, by elementary means. Each of the given series is rearranged and expressed in terms of various sums which have already been derived in the author's previous paper, *Sums and asymptotic sums of series related to the harmonic series*. The results are then specialized for the case of $a_r = 1/r$. (Received September 13, 1951.)

19*t*. R. T. Leslie and E. R. Love: *An extension of Mercer's theorem.*

Mercer's summability theorm is generalised to the extent of replacing the arithmetic means by the means of any regular, or merely convergence-preserving, method of summation. The relations between this extension and some known converses of Toeplitz's theorem, due to R. P. Agnew, are discussed in detail. The proof of the theorem rests on an extension of an inequality of W. A. Hurwitz for the oscillation of Toeplitz means of a bounded sequence. (Received September 24, 1951.)

20*t*. J. S. MacNearney: *Continued fractions in which the elements are operators. I. The J-sequence, associated linear equations, and the general continued fraction.*

This paper is designed to lay the foundations for a generalization of some of the classical studies of continued fractions. Continued fractions over an additive Abelian group A are defined so that the elements and the approximants are operators, here taken to be homomorphisms or isomorphisms among the subgroups of A. Analogues of numerators and denominators, even and odd parts, and equivalence transformations of continued fractions are discussed. Linear fractional transformations are defined among sets of operators in such a way that continued fractions over A are generated by sequences of these transformations. Since this paper is concerned primarily with algebraic relationships which are consequences of the definitions and no questions of convergence are considered, no topology is assumed in the group A. It is anticipated that subsequent reports will require further that A be, for example, a metric linear space or a normed vector space over the complex numbers. An appendix indicates how this theory includes the ordinary notion of a continued fraction and, also, how it includes a certain generalization of Jacobi-matrices (see, for examples, H. S. Wall, *J-matrices of interior order k*. Preliminary report, Bull. Amer. Math. Soc. Abstract 57-2-137). (Received August 29, 1951).

21*t*. J. L. Massera: *Conditional stability of homeomorphisms.*

Let T be a homeomorphism of the neighborhood $\mathfrak{W}$ of the origin o of a Banach space $\mathfrak{B}$ onto a neighborhood of o, $To=o$; $\mathfrak{B}$ is the product of two Banach spaces $\mathfrak{X}$, $\mathfrak{Y}$ and any point of $\mathfrak{B}$ will be denoted $z=(x, y)$, $x\in\mathfrak{X}$, $y\in\mathfrak{Y}$. Write $z_1=Tz_0$ and assume that T is given by the equations $x_1=Lx_0+\phi_1 z_0$, $y_1=My_0+\phi_2 z_0$, L, M linear, $\|\phi_i z\|/\|z\|\to 0$ as $z\to o$, $\|Lx\|\leq\lambda\|x\|$, $\|My\|\geq\mu\|y\|$, $0\leq\lambda<1$, $\lambda<\mu$. Then, if dim $\mathfrak{Y}<\infty$, a set of stability $\mathfrak{E}\subseteq\mathfrak{W}$ exists with the properties: (i) $o\in\mathfrak{E}$; (ii) $T\mathfrak{E}\subseteq\mathfrak{E}$; (iii) if $z\in\mathfrak{E}$, $T^n z\to o$ as $n\to+\infty$; (iv) $\mathfrak{E}$ is tangent at o to the space $\mathfrak{X}$; (v) the projection of $\mathfrak{E}$ in $\mathfrak{X}$ coincides with the projection of $\mathfrak{W}$ in $\mathfrak{X}$; (vi) any set of stability having properties (i)–(iv) is a part of $\mathfrak{E}$. If moreover T is differentiable at any point of $\mathfrak{W}$, the differential being continuous at o (if the differential is continuous at any point of $\mathfrak{W}$; if T is analytic), then $\mathfrak{E}$ is the graph of a transformation $y=Ex$ having a differential continuous at o (continuous in a neighborhood of o; E is analytic). In the latter cases the assumption dim $\mathfrak{Y}<\infty$ may be replaced either by dim $\mathfrak{X}<\infty$ or by the assumption that T^{-1} is differentiable at o. (Received September 10, 1951.)

22. C. N. Moore: *On the theory of patterns and its application to the prime pair problem.*

If we stop the procedure known as the sieve of Eratosthenes at the end of a finite number of steps, the successive differences of the integers remaining constitute a set of even integers in which a certain initial sequence is indefinitely repeated. Such a sequence may be designated as a pattern. By making use of the pattern 6, 4, 2, 4, 2, 4, 6, 2 resulting from stopping the sieve after the removal of multiples of five, an estimate may be obtained of the extent to which a further continuation of the sieve process eliminates the differences two in the earlier portions of later patterns. This enables one to infer the infinitude of prime pairs. (Received September 12, 1951.)

23. Leo Sario: *Existence of functions as a boundary property.*

Denote by α, β, γ, δ the classes of nonconstant harmonic functions (integrals) with following additional properties: α all, β bounded, γ half-bounded, δ Dirichlet bounded functions. Let α_0, β_0, δ_0, γ_0 be the corresponding subclasses of functions with vanishing periods along dividing cycles and a, b, g, d the subclasses of single-valued harmonic functions. Classes of analytic functions are denoted by couples of letters, corresponding to the real and imaginary part respectively. Consider an arbitrary Riemann surface R and a compact subregion D, bounded by a finite set l of simple analytic curves. In the boundary neighborhood $N=R-D$ let L_0 be the normal linear operator which furnishes the harmonic function with the minimal Dirichlet integral among functions with given harmonic boundary values on l. *There are functions* ff' (1) *for* $f=\beta_0$, γ_0, δ_0 *and* $f'=\alpha$, α_0 *on every* R *of positive (finite or infinite) genus; on a planar* R *there are these functions only if* R *is hyperbolic*; (2) *for* $f=b$, g, d *and* $f'=\alpha$, α_0 *on any* R *if and only if some* $\phi+i\bar{\phi}\in ff'$ *in* N, *subject to* $\int_\beta d\bar{\phi}=0$, *satisfies* $\phi\neq L_0\phi$; (3) *for* $f=b$, d *and* $f'=a$, b, d *on any* R *of finite genus if and only if some* $\phi+i\bar{\phi}\in ff'$ *in* N *fulfils* $\phi\neq L_0\phi$. (Received August 27, 1951.)

24. L. V. Toralballa: *Some theorems about functions in a general field with valuation.*

Some classical theorems in the theory of functions of a complex variable are proved to hold in a general field with valuation. Two of these theorems are: 1. Let G be a field with valuation which is complete in the topology induced by this valuation. Let f be a function on G to G such that its derivative is defined and is equal to zero at each point of E where E is a rectifiable arc-wise connected open sub-set of G. Then f is a constant over E. Let G be as in "1," and let C be a rectifiable curve in G. C is said to be locally arc-wise regular at P when there exists a sphere, $S(P, \epsilon)$, such that the set of all ratios arc length ab/distance ab (where a is not equal to b; a, b are elements of C as well as of the sphere, $S(P, \epsilon)$) is bounded. We then have: 2. Let the curve C be arc-wise regular at s_0. Let f be integrable over C, and let $F(s)$ be $\int_a^s f(s)\, ds$ where a and s are points of C. Then $F(s)$ is continuous at s_0. If, in addition, $f(s)$ is continuous at s_0, then $F(s)$ is differentiable at s_0 and $F'(s_0) = f(s_0)$. (Received September 13, 1951.)

25. E. H. Zarantonello: *On trigonometric interpolation.*

Let $P(t)$ be a trigonometric polynomial of degree n, $P^{(r)}(t)$ its rth derivative, and $P^{[r]}(t)$ its rth divided difference $\Delta^r P(t)/\Delta t^r$ with regard to a constant increment $\Delta t = 2\pi/m$. If $m > 2n$ and $1 < p < \infty$, then $A \leq \{\int_0^{2\pi} |P^{(r)}(t)|^p dt\}^{1/p} / \{\sum_1^m |P^{[r]}(t_k)|^p \cdot \Delta t_k\}^{1/p} \leq A_{p,r}$, where the t_k are m equidistant points modulo 2π, $A_{p,r}$ a constant depending on p and r only, and A an absolute constant. For $r = 0$ $(P^{(r)} = P^{[r]} = P)$ this reduces to a previous result due to J. Marcinkiewicz (*Sur l'interpolation*, Studia Mathematica vol. 6 (1936)). As a consequence of the above inequalities, it follows that if $f(t)$ is a periodic function with a p-integrable rth derivative and $P(t)$ a trigonometric polynomial interpolating f at equidistant points, then $\{\int_0^{2\pi} |P^{(r)}|^p dt\}^{1/p} \leq A_{p,r}\{\int_0^{2\pi} |f^{(r)}|^p dt\}^{1/p}$ for $1 < p < \infty$. Furthermore, the approximation of f by P is $O((\Delta t)^{r-1/p})$ for the uniform distance and $O((\Delta t)^r)$ for the p-norm distance. Applications are made to the discretization of some totally continuous functional-equations appearing in conformal mapping and in the theory of free boundaries in hydrodynamics. (Received September 13, 1951.)

APPLIED MATHEMATICS

26. Gertrude Blanch: *On the numerical solution of parabolic partial differential equations.*

The equation treated has the form $(\partial u/\partial t) = (\partial^2 u/\partial x^2) + f(x, t, u)$, with continuous initial and boundary conditions over a closed region of the $x-t$ plane. To solve the equation by finite difference approximations, a lattice is introduced with intervals h and k in the x and t-directions, respectively. It is known that if the finite difference approximation is of order two, then the latter converges to $u(x, t)$ as h approaches zero, provided the mesh ratio, k/h^2, is no greater than 1/2. For at least some difference approximations of order four, the mesh ratio can be as large as 2/3. The primary objective of the present study is to seek that mesh ratio, for a given difference approximation, which will lead to results that are within a given upper bound of error in the solution with the least amount of work. It is shown that the largest admissible mesh ratio is not always the most economical one, and that much depends on the function $f(x, t, u)$, the initial, and the boundary conditions. Five numerical examples are given to illustrate the analysis. In the process of computing, the technique recommended by Hartree and Womersley of extrapolating an improved solution from two difference approximations was tried with considerable success. (Received September 13, 1951.)

27. W. C. Carter and G. L. Spencer: *On the numerical solution of hyperbolic systems of partial differential equations with two characteristic directions.*

To find bounds for the truncation error in the numerical solution by difference methods of the Cauchy problem for hyperbolic systems of partial differential equations, a careful examination is made of the details of H. Lewy's proof of convergence of the difference process to the actual solution (Math. Ann. vol. 98 (1927) pp. 179–191). The conditions for the existence of a solution are relaxed, and it is shown that the sequence of numerical solutions corresponding to decreasing mesh size is uniformly convergent. By considering the properties of a subsequence of difference solutions corresponding to mesh sizes $h/2^n$, h fixed, extrapolation formulae giving improved numerical results are developed. Various approximation methods are considered, and examples are given in the case of the supersonic axisymmetric irrotational flow of a compressible perfect fluid. (Received September 10, 1951.)

28t. E. A. Coddington: *The stability of infinite differential systems associated with vortex streets.*

Consider the k-fold infinite linear system of differential equations (L): $d\rho/dt = A\rho$, where A is a matrix whose elements $A_{n,m}$, n, $m = 0, \pm 1, \pm 2, \cdots$, are $k \times k$ matrices of complex constants and $A_{n,m} = A_{n-m}$. If the elements of the matrix A_p are the Fourier constants of continuous functions on $[0, 1]$, one gives an integral representation of the unique analytic solution of (L) for any initial $\rho^0 = (\rho_n^\alpha) \in H_2$, i.e., $\|\rho^0\| = (\sum_{n=-\infty}^{\infty} \sum_{\alpha=1}^{k} |\rho_n^\alpha|^2)^{1/2} < \infty$. For $k = 1$ this was treated by Wintner. The system (L) is stable relative to $\rho^0 \in H_2$ if for every solution ρ with this initial ρ^0, $\limsup_{t\to\infty} \|\rho\| < \infty$. Necessary and sufficient conditions for stability are given. An application is made to a problem of von Kármán (Nach. Ges. Wiss. Göttingen (1912) pp. 547–556) concerning parallel rows of vortices. The linearized problem resulting from perturbing two rows of vortices at $(2nb+c, a)$, $(2nb-c, -a)$ is of type (L). If $r = a/b$, $q = c/b$, this system is shown to be unstable for $q \neq 1/2$. Contrary to previous beliefs, *instability* also occurs when $q = 1/2$ even if r satisfies the necessary condition $\cosh^2 \pi r = 2$. Further applications are considered (Received August 15, 1951.)

29t. Abolghassem Ghaffari: *Supersonic flow and simple waves.*

This paper integrates, without using the theory of characteristics, the equations of plane irrotational flow of a perfect gas in the case when the hodograph is degenerated. Taking into account the conditions of continuity, zero rotation, and Bernoulli's equation, together with the fundamental assumption, which implies that the components of fluid velocity, u and v, are each a function of fluid speed w only, it is found that $(du/dw)^2 + (dv/dw)^2 = w^2/a^2$, where a is the local speed of sound. Moreover if $u = -w \sin \phi$ and $v = w \cos \phi$, where ϕ is the angle between w and y-axis as in Meyer's notation (G. Taylor and W. Maccoll, *Aerodynamics theory*, ed. by Durand, vol. 3, div. H, chap. IV, 1935), one finds that $(d\phi/dw)^2 = (w^2 - a^2)/w^2a^2$, so that the motion is necessarily *supersonic*. It is shown that the lines of equal speed, pressure (isobar) and density are *all straight lines*, although they are not necessarily concurrent as in the original Prandtl-Meyer theory. Furthermore if θ is the angle of slope of isobar and $\psi = \theta - \phi$, it is found that $\phi = \lambda^{-1} \tan^{-1} (\lambda \tan \psi) - \psi$, where $\lambda = (\gamma - 1)^{1/2}(\gamma + 1)^{-1/2}$ and γ is the adiabatic index. The generalization of the Prandtl-Meyer *wedge-solution* for the flow past an arbitrary fixed boundary is obtained analytically. (Received September 4, 1951.)

30. L. E. Payne: *The flow about bodies of revolution and generalized electrostatics.*

The irrotational symmetric flow of an ideal incompressible fluid about certain bodies of revolution is obtained by the method of generalized electrostatics developed by Weinstein (Proceedings of the Fourth Symposium in Applied Mathematics, American Mathematical Society, *Fluid dynamics*, 1951, in print). This method makes possible the determination of the flow about a lens without requiring the multi-sheeted Riemann space employed by Shiffman and Spencer (Quarterly of Applied Mathematics vol. 5 (1947) pp. 270–288). The stream function for the lens is found to be expressible in terms of Legendre functions of the Mehler type. The method of generalized electrostatics is also used in the determination of the flow about various profiles in plane hydrodynamics. (Received September 12, 1951.)

31. H. F. Weinberger: *The intermediate problem and error estimation in the method of Weinstein.*

In a paper previously presented to the Society (Bull. Amer. Math. Soc. Abstract 57-6-514), the author gave an estimate of the error in the upper bounds for eigenvalues of an operator L in a space $\mathfrak{Q}\subset\mathfrak{H}$ obtained by the method of Weinstein. For this estimate, it was necessary to use the particular nth intermediate problem in the space $\mathfrak{H}\ominus(p_1, \cdots, p_n)$ where p_i is the projection into the subspace $\mathfrak{P}=\mathfrak{H}\ominus\mathfrak{Q}$ of the ith eigenvector of L in $\mathfrak{H}$. It is found in applications, however, that it may be possible only to approximate p_i by another vector f_i, rather than to evaluate it exactly. For such cases, an estimate of the difference between the eigenvalues of L in $\mathfrak{H}\ominus(p_1, \cdots, p_n)$ and in $\mathfrak{H}\ominus(f_1, \cdots, f_n)$ with $f_i\cong p_i$ is obtained. This, then, gives upper and lower bounds of the eigenvalues of L in $\mathfrak{Q}$ in terms of the eigenvalues of L in $\mathfrak{H}\ominus(f_1, \cdots, f_n)$. The latter can be evaluated by means of the Weinstein determinant. (Received September 12, 1951.)

32t. L. A. Zadeh and K. S. Miller: *Generalized ideal filters.*

The notation $v=Nu$ signifies that $v(t)$ is the response of a network N (linear or nonlinear) to a signal $u(t)$. N_2N_1 represents a tandem combination of N_1 and N_2, with N_1 preceding N_2. Consider two classes of signals $\mathfrak{M}$ and $\mathfrak{N}$ both of which contain the null element. A network N is called an ideal filter if $N(u+v)=u$ for all $u\in\mathfrak{M}$ and $v\in\mathfrak{N}$. ($\mathfrak{M}$ and $\mathfrak{N}$ are called, respectively, the acceptance and rejection manifolds of N.) An immediate consequence of this definition is that any ideal filter is idempotent. The converse, however, is true only for linear filters. If N is an ideal filter (linear or nonlinear) and L is a nonsingular linear network, then LNL^{-1} is an ideal filter whose acceptance and rejection manifolds are $L(\mathfrak{M})$ and $L(\mathfrak{N})$ respectively. Physical applications of ideal filters to the filtration and simultaneous transmission of signals are considered. (Received September 5, 1951.)

Topology

33. W. H. Marlow: *Limits of functions on directed sets.* I. Preliminary report.

If α is on the set of cofinal subsets of a directed set X to 2^A, A a given set, then $L_*\alpha=\bigcap\alpha(X_c)$, intersection for all cofinal subsets X_c of X, and $L^*\alpha=\bigcap\alpha(X_r)$, intersection for all residual subsets X_r, are defined to be the *lower* and *upper limits* of α, respectively. (See J. Tukey, *Convergence and uniformity in topology*, Princeton, 1940, for meaning of 2^A, cofinal and residual.) If α and β are *monotone nondecreasing* ($X_{c'}\subseteq X_c$ implies $\alpha(X_{c'})\subseteq\alpha(X_c)$), a typical result involving the Boolean function $\alpha\cup\beta$ is: (1) $L_*(\alpha\cup\beta)\subseteq(L_*\alpha)\cup(L^*\beta)$. A partition of the points in $L^*(\alpha\cup\beta)$ according to membership in the limits of α and β and $L_*(\alpha \cup \beta)$ is given and some properties of *convergence* (when $L_*\alpha = L^*\alpha$) are developed. For f on X to 2^Z, Z a space with *topological function* "t" (merely a function on 2^Z to 2^Z), the *lower* and *upper limits of f relative to t* are defined by replacing α by the function whose value at the place X_σ is $t(\{f(x_\sigma) \mid x_\sigma \in X_\sigma\})$ in the definitions of $L_*\alpha$ and $L^*\alpha$, respectively. The previous results apply if $t(A \cup B) = tA \cup tB$ or, in some cases, if t is only monotone nondecreasing. Then (1) specializes to limits of different functions on X or to limits of a single function relative to different topological functions. (Received September 13, 1951.)

34t. W. H. Marlow: *Limits of functions on directed sets*. II. Preliminary report.

This is a continuation of Part I. After E. W. Chittenden, the upper limit of f on X to 2^Z relative to a Kuratowski closure function is called the *Peano limit* of f, Lf. Since f and the function whose value at the place $x \in X$ is $\{f(x') \mid x' > x\}$ have the same Peano limits, properties of Lf, f arbitrary, follow from results for the case: $x' > x$ implies $f(x') \subseteq f(x)$. Conditions are obtained which are both necessary and sufficient for $(Lf) \cap A \neq 0$, $A \subseteq Lf$, $Lf \subseteq A$, and $Lf = A$ when A is closed in a compact Hausdorff space. A dominance relation for functions on directed sets is introduced and properties of directed families of open sets are exploited. Of central importance are functions whose domains contain residual subsets upon which the null set is not a value. Functions on directed sets to compacta are treated and an extension of the "metric convergence" of Hausdorff (*Mengenlehre*, Berlin, 1935) is made. Results of this paper are shown to be related to those of E. H. Moore and H. L. Smith, G. Birkhoff, and others. (Received September 13, 1951.)

L. W. COHEN,
Associate Secretary

THE NOVEMBER MEETING IN AUBURN

The four hundred seventy-fourth meeting of the American Mathematical Society was held at Alabama Polytechnic Institute, Auburn, Alabama, on Friday and Saturday, November 23–24, 1951. The total attendance was 130 including the following 66 members of the Society:

O. R. Ainsworth, M. G. Boyce, A. T. Brauer, R. K. Butz, H. E. Campbell, A. C. Cohen, Haskell Cohen, R. M. Conkling, J. C. Currie, V. E. Dietrich, F. G. Dressel, B. M. Drucker, W. L. Duren, Patrick Du Val, J. C. Eaves, M. W. Effron, D. O. Ellis, M. E. Estill, Tomlinson Fort, H. K. Fulmer, Wallace Givens, D. B. Goodner, E. H. Hadloch, B. F. Hadnot, E. V. Haynsworth, Nickolas Heerema, Melvin Henriksen, G. B. Huff, Ernest Ikenberry, J. L. Kelley, R. J. Koch, F. W. Kokomoor, G. B. Lang, R. J. Levit, F. A. Lewis, C. W. McArthur, Nathaniel Macon, G. W. Medlin, E. P. Miles, Knox Millsaps, Benjamin Ernest Mitchell, Benjamin Evans Mitchell, J. D. Morales, D. R. Morrison, W. V. Parker, E. H. Pearson, W. D. Peeples, A. S. Rayl, Herbert Robbins, J. H. Roberts, F. V. Rohde, W. A. Rutledge, H. D. Sprinkle, H. S. Stanley, H. E. Taylor, J. M. Thomas, H. S. Thurston, G. J. Trammell, B. O. Van Hook, T. L. Wade, J. H. Wahab, W. M. Whyburn, Ernest Williams, G. N. Wollan, C. T. Yang, J. W. Young.

By invitation of the Committee to Select Hour Speakers for Southeastern Sectional Meetings, Professor H. E. Robbins of the University of North Carolina and Professor J. L. Kelley of Tulane University addressed the Society. Professor Robbins' address, entitled *Mathematical aspects of the sequential design of experiments*, was delivered at 8:00 P.M. on Friday, with Professor W. V. Parker presiding. The address by Professor Kelley, entitled *Duality in algebraic analysis*, was given at 10:00 A.M. on Saturday, with Professor Tomlinson Fort presiding.

Four sessions for the presentation of contributed papers were held on Friday afternoon and Saturday morning. Presiding officers for these sessions were Professor J. H. Roberts, F. W. Kokomoor, T. L. Wade, and W. L. Duren.

President Ralph D. Draughon of Alabama Polytechnic Institute extended greetings at the opening of the evening session on Friday. All in attendance at the meetings were guests of the host institution and its Department of Mathematics at a reception and social hour which followed the hour address on Friday evening. A resolution expressing the appreciation of the Society to its hosts for the meeting was offered by Professor Tomlinson Fort and unanimously adopted.

Abstracts of the papers presented follow below. Abstracts whose number are followed by the letter "*t*" were presented by title. Paper number 59 was read by Professor Dressel. Professor Strother was introduced by Professor Wallace, Mr. Davis by Professor Brauer, Mr. Newton by Professor Fort, Mr. J. W. Ellis by Professor Pettis, and Professor Willoughby by Professor Whyburn.

ALGEBRA AND THEORY OF NUMBERS

35. A. T. Brauer: *Limits for the characteristic roots of a matrix.* IV. *Applications to stochastic matrices.*

It is well known that the characteristic roots of a stochastic matrix $A=(a_{\kappa\lambda})$ lie in the circle $|z|\leqq 1$. The point $z=1$ is always a characteristic root. If A is unreduced, then 1 is a simple root. Let a_{rr} and a_{ss} be the smallest elements of the main diagonal of A and $a_{rr}\leqq a_{ss}$. It was proved by Fréchet (Publications de la Faculté des Sciences de l'Université Masaryk no. 178 (1933) pp. 1–24) that the characteristic roots of A lie in the circle $|z-a_{rr}|\leqq 1-a_{rr}$. In this paper it is proved that the circle can be replaced by the oval of Cassini $|z-a_{rr}|\,|z-a_{ss}|\leqq(1-a_{rr})(1-a_{ss})$. Let $k_1, k_2, \cdots, k_n$ be arbitrary numbers and B the matrix $(a_{\kappa\lambda}-k_\lambda)$. Assume that A is unreduced. It is shown that the roots of A coincide with those of B and have the same multiplicity for A and B, except that the simple root 1 of A is replaced by $1-k_1-k_2-\cdots-k_n$. Therefore it is often possible to obtain better bounds for the nontrivial characteristic roots of A by applying the results of the first 3 parts of this paper (Duke Math. J. vol. 13 (1946) pp. 387–395; vol. 14 (1947) pp. 21–26; vol. 15 (1948) pp. 871–877) to the matrix B instead of A. (Received October 9, 1951.)

36t. Leonard Carlitz: *A theorem of Glaisher.*

Glaisher (Quarterly Journal of Mathematics vol. 31 (1900) p. 325) showed that if $(x+1)(x+2)\cdots(x+p-1)=x^{p-1}+A_1x^{p-2}+\cdots+A_{p-1}$, where p is a prime >3, then for $1\leqq t<2^{-1}(p-1)$, $A_{2t}/p\equiv -B_{2t}/2p$, $A_{2t+1}/p^2\equiv(2t+1)B_{2t}/4t \pmod{p}$. In the present paper the author (1) makes several new applications of Glaisher's formulas, (2) shows that $A_{2t+1}\equiv -((p-2t-1)/4t)p^3B_{2t}-((2t+1)/2t)p^3\sum_{k=1}^{t-1}(1/4k)B_{2k}B_{2t-2k}$ $(\text{mod } p^4)$, and (3) extends Glaisher's theorem to more general sequences. The generalization depends on the relation of the A_r to Bernoulli number of higher order. (Received October 8, 1951.)

37t. Leonard Carlitz: *Congruences for the coefficients of hyperelliptic functions.*

In a previous paper (Duke Math. J. vol. 16 (1949) pp. 297–302) the writer showed that if $f(x)=\text{sn}\,(x,\,k^2)=\sum_1^\infty a_ix^i/i!$, and $f^m(x)=\sum_{i=m}^\infty a_i^{(m)}x^i/i!$, $a_i^{(1)}=a_i$, then $\sum_{i=0}^r(-1)^iC_{r,i}a_p^{r-i}\,a^{(k)}_{m+r+i(p-1)}\equiv 0 \pmod{p^r}$ for $m\geqq 0$, $r\geqq 1$, $k\geqq 1$, and $l=k^2$ integral $(\text{mod } p)$. In the present paper similar results are obtained for the coefficients of hyperelliptic functions and more generally for functions $g(x)$ with inverse of the type $\sum_1^\infty c_mx^m/m$; however the modulus is now only p^s, where $s=[2^{-1}(r+k+1)]$, where $k=[h/p]$. Like results are also obtained for the coefficients of $x/g(x)$. (Received October 8, 1951.)

38t. Leonard Carlitz: *Some congruences for the Bernoulli numbers.*

Vandiver (Duke Math. J. vol. 5 (1939) pp. 548–551) proved that $\sum_{i=0}^r(-1)^iC_{r,i}B_{(i+k)(p-1)}\equiv 0 \pmod{p^{r-1}}$ for $p>2$, $k>0$, $r>0$, $k+r<p-1$. In the present paper it is shown by a somewhat different method that $\sum_{i=0}^r(-1)^iC_{r,i}B_{m+i(p-1)}$

$\equiv 0 \pmod{p^{r-1}}$, where $m=(hp+k)(p-1)$, $1\leq r<k+r<p$. It is also shown that $\sum_{i=0}^{r}(-1)^r C_{r,i}(B_{(i+1)(p-1)}+1/p-1)/(i+1)\equiv 0 \pmod{p^r}$ for $r<p-1$, and that $B_{m(p-1)}+1/p-1\equiv 0 \pmod{p^r}$ for $p^r | m$. (Received October 8, 1951.)

39. J. C. Currie: *Canonical forms for the Lorentz-Hermitian matrix.* Preliminary report.

A simple, purely algebraic derivation is given for the diagonal or near-diagonal canonical forms for the Lorentz-Hermitian matrix. (Received October 5, 1951.)

40. A. S. Davis: *The Euler-Fermat theorem for matrices.*

Let A be a square matrix of order n with integral rational elements whose determinant is relatively prime to the prime number p. J. B. Marshall (Proceedings of the Edinburgh Mathematical Society (2) vol. 6 (1939–1941) pp. 85–91) determined an exponent q such that $A^q\equiv I \pmod{p}$ for every A, where I is the unit matrix, and proved that for certain small values of p and n this exponent is the smallest possible. I. Niven (Duke Math. J. vol. 15 (1949) pp. 823–825) showed that Marshall's exponent is always the best possible. In this paper this problem is generalized for arbitrary moduli m. It is shown that an exponent w exists such that $A^w\equiv I \pmod{m}$ for every A and it is proved that w is the best possible exponent. (Received July 17, 1951.)

41. V. E. Dietrich: *Algebras of finite projective planes from incidence matrices.*

For each ordering of lines of each of two pencils and points of the line of centers there is associated a unique standard incidence matrix. A block submatrix is designated the *kernel*. The elements of the kernel are order n permutation matrices. Each element of the first column and first row of the kernel is I_n. Each of the remaining rows and columns, designated *row sets* and *column sets*, corresponds to a set of singly transitive permutations on n letters. For a fixed y-axis, origin, and ideal line, every coordinatization of the affine plane possesses a ternary relation reducible to two binary relations, *addition* and *multiplication*, if and only if each column set of the corresponding kernel is isomorphic to the additive set which is then a group. Multiplication is right-distributive over addition if and only if the column sets are automorphisms of a group, which further implies that the additive group is abelian of type $(p, p, \cdots, p)$. Necessary and sufficient conditions on the kernel for commutativity, for associativity, and for left-distributivity of multiplication then imply the plane is Desarguesian. Using a result of Wedderburn's allows omission of either commutativity or left-distributivity. (Received October 8, 1951.)

42*t*. J. C. Eaves: *On the skeletons and characteristic roots of sets of matrices having a delayed commutativity property.*

If $A, B, C, \cdots$ is a finite set of $n\times n$ commutative matrices having elements in a field and having characteristic roots $\alpha_i, \beta_i, \gamma_i, \cdots$ $(i=1, 2, \cdots, n)$ respectively and if $f(x, y, z, \cdots)$ is any rational function, then by a well known theorem due to Frobenius the characteristic roots of the matrix $f(A, B, C, \cdots)$ are $f(\alpha_i, \beta_i, \gamma_i, \cdots)$ $(i=1, 2, \cdots, n)$. This property of the characteristic roots is called the property "p." McCoy introduced a type of delayed commutativity in his quasi-commutative matrices which he showed also possess the "p" property. In this paper some pairs and some finite sets of matrices having a delayed type of commutativity and possessing finite chains of commutes are considered and are shown to have the "p" property. A generalization is given for a finite set of quasi-commutative matrices. (Received October 10, 1951.)

43t. D. O. Ellis: *On immediate inclusion in partially ordered sets and the construction of homology groups for metric lattices.*

In this note conditions are found which characterize the immediate inclusion relations arising from partial orderings of a set among binary relations on the set. Characterizations of modularity and distributivity are given in lattices in terms of immediate inclusion. It is shown how the chains of elements of a given norm in a normed distributive lattice may be formed and the boundary operator mapping the group of p-chains onto a subgroup of the q-chains (the face relation being immediate precession) may be defined for $q<p$ in case the lattice has the finite ascending chain condition. An additional condition is imposed under which the p-bounding q-chains are q-cycles for the r-chains for $r<q<p$ so that homology groups H_{pr}^{q} may be defined. The methods are elementary in nature. (Received September 10, 1951.)

44. D. O. Ellis: *Remarks on isotopies.*

One considers property g. A. of a groupoid $G(^*)$, namely the possession of four permutations P, Q, R, and S so that $xP^*yQ=yR^*xS$. One finds: (1) *If $G(^*)$ has unit e and is g. A., there is a permutation V so that $x^*y=yV^{-1*}xV$, the set $\{eV^n\}$, $n=0$, ± 1, ± 2, $\cdots$, is a subgroup of $G(^*)$, and if eV is idempotent, $G(^*)$ is Abelian.* (2) *A unit possessing semigroup which is g. A. is Abelian.* (3) *A quasigroup (finite groupoid with left and right nonsingular elements) is isotopic to an Abelian group (Abelian semigroup) if and only if it is g. A. and associative in the sense of Evans* (J. London Math. Soc. vol. 25 (1950) pp. 196–201). It is also shown that: (4) *Two semilattices are isotopic if and only if they are isomorphic.* (5) *A semilattice admits a second operation to form a lattice if and only if it is isotopic to a semilattice of a lattice in which case the two lattices are isomorphic.* (Received August 6, 1951.)

45. Wallace Givens: *Fields of values of a matrix.*

Given a square matrix A and a positive definite Hermitian matrix H, the field of values of A with respect to H, $F_H(A)$, is the set of all complex numbers $(x, Ax)_H/(x, x)_H$, where x is an arbitrary nonzero vector and $(x, y)=x^*Hy$ is the inner product in the metric defined by H. It is shown that: (1) $F_H(A)=F_I(B)$ where I is the unit matrix and B is a matrix similar to A; (2) if A has a nonsimple elementary divisor with root λ, then λ is an interior point of $F_H(A)$ for every H; (3) if A and H are simultaneously decomposable, $F_H(A)$ is the minimum convex set containing the component regions; (4) the intersection of the $F_H(A)$ for all H is the minimum convex polygon $P(A)$ containing all the roots of A; (5) a necessary and sufficient condition for $F_H(A)=P(A)$ for some H is that the elementary divisors corresponding to roots lying on the boundary of $P(A)$ shall all be simple. (Received October 11, 1951.)

46. Emilie V. Haynsworth: *Bounds for determinants with dominant main diagonal.*

Let D be a determinant of order n for which the absolute value of each element a_{ii} in the main diagonal is greater than or equal to the sum of the absolute values of all other elements a_{ij} in the same row. In a large number of papers bounds have been obtained for such determinants. (For references to the literature on such determinants, see the paper by O. Taussky, *A recurring theorem on determinants*, Amer. Math. Monthly vol. 56 (1949) pp. 672–676.) Recently G. B. Price, Proceedings of the American Mathematical Society vol. 2 (1951) pp. 497–502, has improved these results in the following manner: Set $d_i^{(i)} = |a_{ii}| - \sum_{j=1, j\neq i}^{n} |a_{ij}|$ and $S_i^{(i)} = |a_{ii}| + \sum_{j=1, j\neq i}^{n} |a_{ij}|$. Then $|D| \geqq a_{11} \prod_{k=2}^{n} d_k^{(k)}$, and $|D| \leqq a_{11} \prod_{k=2}^{n} S_k^{(k)}$. In this paper the following lower bound is obtained: $|D| \geqq a_{11} \prod_{k=2}^{n} d_k^{(k)} + \sum_{i=2}^{n} [(\sum_{j<i} |a_{ij}|) \cdot \prod_{k=1}^{i-1} d_k^{(k)} \prod_{k=i+1}^{n} d_k^{(k)}]$. Similarly, a better upper bound is obtained. Moreover, some related theorems on determinants with relatively large main diagonals are proved, and applications made to the characteristic roots of a matrix. (Received October 8, 1951.)

47. G. B. Huff: *On matrices A such that A^t is a matric polynomial in t.*

If A is a square matrix with elements from a field F of complex numbers, there may exist a matric polynomial $F(X)$ such that $A^t = F(t)$ for every rational integer t. This paper studies the implications of the definition $A^C = F(C)$, where A has the property described above and C is a square matrix. It is shown that under appropriate conditions the usual laws hold and applications are indicated. (Received October 1, 1951.)

48. R. J. Levit: *The minimax solutions of a system of linear equations.*

Let S denote the totality of real solutions $x = (x_1, \cdots, x_n)$ of a consistent system of linear equations, $\sum_{j=1}^{n} a_{ij}x_j = b_i$, $i = 1, \cdots, m$, with real coefficients. If $\max |x_j|$, $j = 1, \cdots, n$, has a minimum value ρ for x in S, ρ is called the *minimax of S*; and $x^* = (x_1^*, \cdots, x_n^*)$ is called a *minimax solution* of the system if $\max |x_j^*| = \rho$. In the case of a single equation, explicit expressions for ρ and x^* are known. (See H. Rademacher and I. Schoenberg, Canadian Journal of Mathematics vol. 2 (1950) pp. 251–252.) In the present paper the existence of a minimax solution is established for the general linear system and explicit expressions for ρ and x^* are obtained. The result is then used to generalize some theorems of C. de la Vallée Poussin concerning best approximation to functions on finite sets. (See his monograph, *Leçons sur approximation des fonctions d'une variable réelle*, pp. 78–86, Paris, 1919.) (Received October 11, 1951.)

49. Nathaniel Macon: *On the approximation of irrational numbers.*

Let α be irrational and t real. Denote by $x_{\min}(t)$ the smallest integer x such that there exists an integer y with $1 \leqq x \leqq t$ and $|\alpha x - y| < 1/t$. Denote the consecutive values of $x_{\min}(t)$ so obtained by $x_0 = 1 < x_1 < x_2 < \cdots$ and the corresponding values of y and t accordingly. Koksma [Mathematica vol. 6 (1938) pp. 113–143] proved that at least one of the three inequalities $x_i/t_i > (4+\lambda^2)^{1/2}$ $(i = n, n+1, n+2)$ holds for any λ such that $\lambda_{n+2} = (x_{n+2} - x_n)/x_{n+1} \geqq \lambda$. He showed that a well known theorem of Borel [J. Math. Pures Appl. (5) vol. 9 (1903) pp. 329–375] for the convergents of the expansion of α as a regular continued fraction is contained in the case $\lambda = 1$. It is proved here, without used of continued fractions, that either at least two of the inequalities $x_i/t_i > 5^{1/2}$ $(i = n-2, n-1, \cdots, n+2)$ or else $x_n/t_n > 3$. This is the analogue of a theorem of Brauer and Macon [Amer. J. Math. vol. 71 (1949) pp. 349–361] for the convergents of the continued fraction α. (Received October 12, 1951.)

50. G. W. Medlin: *Bounds for the imaginary characteristic roots of real matrices.*

A. Brauer [Duke Math. J. vol. 14 (1947) pp. 21–26] has shown that each characteristic root of the matrix $A=(a_{\kappa\lambda})$ must lie in the interior or on the boundary of at least one of the $n(n-1)/2$ ovals $|z-a_{\kappa\kappa}|\cdot|z-a_{\lambda\lambda}|\leq P_\kappa P_\lambda$, where $P_r+|a_{rr}|$ is the sum of the absolute values of the elements of the rth row. In a paper presented at the symposium of the Institute for Numerical Analysis at Los Angeles in August 1951, he obtained other ovals containing the roots which often are smaller. Moreover, he will present an improvement of this result for the real characteristic roots of real matrices at the meeting of the American Mathematical Society at Washington in October. It will be shown in this paper that one of Brauer's methods for obtaining bounds for the real roots can be modified so that new bounds are obtained for the imaginary roots of real matrices. These are often better than Brauer's bounds. (Received October 8, 1951.)

51. Benjamin Evans Mitchell: *A canonical form for nonderogatory matrices under unitary transformation.*

Techniques of the theory of groups with operators are applied to show that any complex matrix A may be unitarily transformed to a triangular matrix with unique nonderogatory diagonal blocks $A_1, \cdots, A_k$ where A_i has only one distinct characteristic root and has positive elements in the diagonal below the main diagonal. Moreover the Jordan normal form of A is $A=B_1\dotplus\cdots\dotplus B_k$, where A_i is similar to B_i for $i=1, \cdots, k$. Let A be unitarily equivalent to T_1 and also to T_2 where T_1 and T_2 are in the above triangular form and each has the same diagonal blocks $A_1, \cdots, A_k$. Then if no two A_i have the same characteristic root, any unitary matrix U such that $T_1U=UT_2$ is a diagonal matrix. Thus if A is nonderogatory, the absolute value of all elements below the diagonal blocks in the above triangular form is invariant, and hence by a simple convention a unique triangular canonical form is given for a specified ordering of the characteristic roots. (Received October 5, 1951.)

52. D. R. Morrison: *Bi-regular rings and the ideal lattice isomorphisms.*

Generalizing an observation of Foster, the set R^0 of idempotent elements in the center of a ring R is a Boolean ring under a modified addition and ring multiplication. If I is an ideal of R, then I^0 is an ideal of R^0. If J is an ideal of R^0, then RJ is an ideal of R and $(RJ)^0=J$. A necessary and sufficient condition that $RI^0=I$ for each ideal I of R is that every principal ideal of R have a unit element, or, in the terminology of Arens and Kaplansky, that R be bi-regular. Thus in the bi-regular case, the (principal) ideal lattices of R and R^0 are lattice-isomorphic. It follows that every theorem on Boolean rings which is expressible in terms of the ideal and principal-ideal lattices is extended immediately to bi-regular rings. A corollary is Arens and Kaplansky's extension to bi-regular rings of Stone's "mathematical equivalence" between Boolean rings and Boolean spaces. Relationships are established between the homomorphisms and subdirect sum representations of R and of R^0. (Received October 11, 1951.)

53. Erna H. Pearson: *On certain nonhomogeneous Diophantine equations.* Preliminary report.

The congruence $1+AX^e\equiv BY^e \pmod{p}$, where p is a prime of the form $1+ec$, and $(AB, p)=1$, is considered. If g is a primitive root of p, this congruence may be written $1+g^ix^e\equiv g^jy^e \pmod{p}$, where $A\equiv g^{i+ue}$; $B\equiv g^{j+ve}$; $x\equiv g^sX$; $y\equiv g^tY$; $0\leqq i<e$; and $0\leqq j<e$. The author determines the number of solutions of the congruence for a given i or j when $(XY, p)=1$, and the conditions on i and j for solutions when $(XY, p)=p$. If no solutions exist for a given (i, j), and $(d, p)=1$, there are no integral solutions of the Diophantine equation $(d+k_1p)+(dg^i+k_2p)x^e-(dg^j+k_3p)y^e=0$. The author also carried out computations for certain primes of the above form with $e=5, 7, 11$, and 13, and noted the number of solutions of the congruence for given (i, j). (Received October 10, 1951.)

54. J. H. Wahab: *Some new cases of irreducibility for Legendre polynomials.*

Holt (Proc. London Math. Soc. vol. 11 (1912–1913) pp. 351–356 and vol. 12 (1913) pp. 126–132) proved the Legendre polynomial $P_n(x)$ of degree n irreducible for $n=2^m$, 2^m+1, and for certain degrees near odd primes or twice odd primes. Ille (Thesis, Abstract in Jahrbuch der Dissertationen der Universitat Berlin (1924)) established irreducibility for $n=(p-1)p^k$ (p, odd prime) and completed Holt's proof that a pure quadratic factor can not exist. In this paper a theorem of Grosswald (Proceedings of the American Mathematical Society vol. 1 (1950) pp. 553–554) is used to prove: (1) If $P_n(x)$ is reducible and if n and the degree of each factor are expressed as the sums of different powers of two, then each power of two used in the expression for n will occur once, and only once, among the terms in the representations of the degrees of the factors; (2) Only polynomials in x^2 are possible factors; (3) $P_n(x)$ is irreducible for $n=t\cdot 2^m+0, 1, 2, 3$ ($t<19$) and for some new cases near an odd prime or twice an odd prime; (4) There are only nine $P_n(x)$ ($n\leqq 500$) not known to be irreducible. (Received July 5, 1951.)

55. R. A. Willoughby: *p-algebras.*

A. L. Foster has studied p-rings in connection with his general K-ality theories and has shown that each p-ring is related to a certain "p-algebra" by an inter-equational bond which generalizes the Boolean ($p=2$) case. The primitive operations of a p-algebra, realized in the (corresponding) p-ring, are $\times$, = the ring multiplication, and $^\wedge$, the unary operation defined by $a^\wedge=1+a$. The purpose of this paper is to present a set of postulates for p-algebras and to study certain of their intrinsic properties. The postulates are essentially in equational form and, in fact, the equations satisfied over a general p-algebra are shown to be just those satisfied over the simplest case (= the p-algebra corresponding to the field of residues mod p). In this study the "normal representation theorem" discovered by Foster constitutes a basic tool. The initial additive form of this theorem is here shown to have a convenient multiplicative adaptation. A second important tool is developed in the form of certain cancellation and substitution laws. (Received October 10, 1951.)

ANALYSIS

56. J. W. Ellis: *A general set-separation theorem.*

The result in this paper has as applications Stone's theorem on ideals in distributive lattices, Tukey's theorem on separation of convex sets, and a theorem of Klee's on cones in linear topological spaces. Let $\mathfrak{S}$ denote the collection of all subsets of a set S. Let ϕ be a function on $\mathfrak{S}$ to $\mathfrak{S}$ satisfying: (i) for all A, $\phi(A)\supset A$; (ii) for all A, $\phi[\phi(A)]\subset\phi(A)$; (iii) $\phi(A)=\bigcup\{\phi(F)\colon F \text{ finite}, F\subset A\}$; (iv) if $p\in S$ and F is finite,

$\phi(F\cup\{p\})\subset\bigcup[\phi(\{a\}\cup\{p\}): a\in\phi(F)]$. Let ψ be another function, also satisfying (i)–(iv), related to ϕ in such a way that, if $a\in\psi(\{b\}\cup\{c\})$ and $d\in\phi(\{e\}\cup\{c\})$, then $\phi(\{a\}\cup\{e\})\cap\psi(\{b\}\cup\{d\})\neq\Lambda$. Then if $\phi(A)=A$ and $\psi(B)=B$, and $A\cap B=\Lambda$, there are sets $A'\supset A$ and $B'\supset B$ such that $\phi(A')=A'$, $\psi(B')=B'$, $A'\cap B'=\Lambda$, and $A'\cup B'=S$. (Received October 11, 1951.)

57. Tomlinson Fort: *Reducibility of differential equations*. Preliminary report.

In this abstract consideration is given only to the third order equation. Generalization to the nth order equation will be given in the paper. The equation $y'''+p_1(x)y''+p_2(x)y'+p_3(x)y=0$ where $p_1(x)$, $p_2(x)$, $p_3(x)$ are continuous when $a\leqq x\leqq b$ is called reducible over this interval if it has two solutions y_1 and y_2 which constitute a fundamental system of solutions of an equation $y''+P_1(x)y'+P_2(x)y=0$ where $P_1(x)$, $P_2(x)$ are continuous over $a\leqq x\leqq b$. Necessary and sufficient conditions for reducibility are obtained involving differential equations. Sufficient conditions are obtained involving $p_1(x)$, $p_2(x)$, $p_3(x)$ alone. (Received October 5, 1951.)

58*t*. R. E. Fullerton: *On the rectification of contours of a Fréchet surface.*

Let S be a Fréchet surface with respresentation $T: x=x(w)$, $w\in[0,1;0,1]$ from E_2 to E_3 and let f be a real-valued Lipschitzian function on $[S]$, the set of points occupied by S. Let the contour β_t^* be the boundary of the set $\beta_t=E[w\in[0,1;01]\,|\,f(x(w))<t]$. Let $\{t_i\}$ be any countable set of real numbers with $\min_{x\in[S]}f(x)<t_i<\max_{x\in[S]}f(x)$, $i=1,2,\cdots$, for which the length of each $\beta_{t_i}^*$ is finite in the sense defined by Cesari (Bull. Amer. Math. Soc. Abstract 57-3-188). It is shown that there exists a mapping T' Fréchet equivalent to T for which each $\beta_{t_i}^*$ is a union of arcs the sum of whose lengths is finite. (Received October 8, 1951.)

59. J. J. Gergen and F. G. Dressel: *Uniqueness for p-regular mapping.*

Let Γ be a circle about the origin, and let S be the interior of Γ. Let S' be a finite domain whose boundary is a simple, closed, rectifiable curve. Let $p(x, y)$ be real, positive, and continuous on $S+\Gamma$, and have continuous bounded derivatives p_x, p_y on S. In a previous paper (Duke Math. J., vol. 18 (1951) pp. 185–210) the authors obtained a generalization of the existence part of the Riemann mapping theorem for equations $pu_x=v_y$, $pu_y=-v_x$, and domains S and S'. In the present paper uniqueness results are obtained. (Received October 8, 1951.)

60. B. F. Hadnot: *Note on a class of series.*

Consider the series (1) $\sum_{n=1}^{\infty}c_ng(s+n)\,n^{-s}\Gamma^{s}$, where for fixed s, $g(s+n)n^{-s}\Gamma^{s}$ has a determinate asymptotic representation by means of a power series in $1/n$. For a particular class of functions $g(s)$ with the mentioned property for $g(s+n)n^{-s}\Gamma^{s}$, the author has established theorems relative to convergence, uniform convergence, and the analytic character of the function represented by the series (1). (Received October 10, 1951.)

61. E. P. Miles: *Harmonic functions in 3-space generated by analytic functions of a hypervariable.* Preliminary report.

Let $t=ix+jy+kz$ where $i^2=i$, $ij=ji=j$, $ik=ki=k$, $j^2=k$, and $jk=kj=-i$. A function $f(t)=iU(x, y, z)+jV(x, y, z)+kW(x, y, z)$ is said to be an analytic function of the hypervariable t provided $U_x=V_y=W_z$, $U_y=V_z=-W_x$, and $U_z=-V_x=-W_y$. The components U, V, and W satisfy $\nabla^2U=-\nabla^2V=\nabla^2W$; hence $U+V$, $V+W$, and their difference $U-W$ are harmonic. The functions $(i+j)f(t)$, $(j+k)f(t)$, and $(k-i)f(t)$ where $f(t)$ is analytic have harmonic components. t^n where n is a positive integer is analytic with derivative nt^{n-1}. The ordinary sum, product, and quotient laws for differentiation hold, where quotients are defined with the aid of a multiplicative conjugate and a generalized absolute value. The conjugate for t, $\bar{t}=i(x^2+yz)+j(-z^2-xy)+k(y^2-xz)$, and the generalized absolute value, $|t|=i(x^3-y^3+z^3+3xyz)^{1/3}$, satisfy the equations $t\bar{t}=|t|^3$, $|\bar{t}|=|t|^2$, $(\bar{\bar{t}})=|t|^3t$ and $|t|^3=(x-y+z)\cdot(x^2+y^2+z^2+xy-xz+yz)$. The function $1/t=\bar{t}/|t|^3$ is analytic except at points of the plane $x-y+z=0$ and the normal to this plane at the origin. Analytic functions of a complex variable of the exponential and trigonometric type expandable in a McLaurin's series are readily generalized to analytic functions of t. (Received October 8, 1951.)

62. T. A. Newton: *Application of extended summation by parts.*

An extended summation by parts formula is developed by considering the expression $\sum a_i\Delta^n b_i$. A particular transformation for power series is then obtained which in many cases gives a closed expression for the sum of a power series. Applying this formula to the series $\sum_{i=0}^{\infty} i^{(m)}x^i$ yields properties of this series that are analogous to certain properties of the gamma function. Also, it is found that Euler's transformation is a particular case of this transformation for power series. (Received October 10, 1951.)

63*t*. B. J. Pettis: *Certain subalgebras of Boolean algebras.*

This paper is concerned with the relations between certain subalgebras generated by an arbitrary subset E of a Boolean algebra A. By means of a theorem involving an abstract space admitting a family of binary operations, previous results due to Sierpinski, Saks, von Neumann, and Ambrose concerning representations for the σ-ring generated by E are obtained and in some cases improved. (Received October 12, 1951.)

64. J. W. Young: *Nonclassical orthogonal polynomials.*

Investigation is made of the conditions under which the class of second order homogeneous linear differential equations $(A_1x^{r+2}+A_2x^{s+2})y_n''+(B_1x^{r+1}+B_2x^{s+1})y_n'+[L_1(n)x^r+L_2(n)x^s]y_n=0$ possesses orthogonal polynomial solutions. It is shown that the classical orthogonal polynomials, such as those of Legendre, Hermite, and Laguerre, are contained as special cases. Particular attention is given to the nonclassical orthogonal polynomial solutions obtained when $L_1(n)=K_1L(n)$, $L_2(n)=K_2L(n)$. It is found that there are two general types of such solutions, one of order and subdegree linear in n, and having an algebraic weight function and finite interval of orthogonality; the other of order quadratic in n, and subdegree linear in n, and having a transcendental weight function and semi-infinite interval of orthogonality. With suitable restrictions, derivative definitions and integrated square formulas are obtained for these solutions. Certain other properties of interest in these solutions are also considered. (Received October 11, 1951.)

APPLIED MATHEMATICS

65. O. R. Ainsworth: *Asymptotic evaluation of displacements in one*

elastic medium due to a P-wave point source in a second elastic medium.

A P-wave point source is located $(0, 0, a)$ in an isotropic, elastic, medium occupying $z>0$. A second medium occupies $z<0$, and is contiguous to the first medium along $z=0$. The displacements due to this source are computed asymptotically when the point of observation is far removed from the origin and is in the second medium at a distance of the order of a. (Received October 9, 1951.)

66. Ernest Ikenberry: *On the significance of the arbitrary parameters entering into a Bruns' series expansion of the distribution function in phase space.* Preliminary report.

The parameters $k_i(q, t)$ in the Hermite polynomials $H_n(k_i p_i)$ in a Bruns' series expansion of $f(q, p, t)$ are of no physical significance, since $\partial f/\partial k_i=0$, the distribution function being independent of any particular mode of representation (J. of Chem. Phys. vol. 19 (1951) pp. 467–470). From the properties of the operator D and the Liouville equation $Df=0$ there is readily obtained, for any suitable function $Q(q, p, t)$, $D(fQ)=fDQ+QDf=fDQ$. Likewise, $D_k(fQ)=fD_kQ$, where the operator D_k is defined by $D=D_k+\sum_i (Dk_i)\partial/\partial k_i$. The infinite set of partial differential equations which the coefficients in the Bruns' series satisfy are obtainable by choosing Q's which are products of Hermite polynomials $H_n(k_i p_i)$ and integrating the two members of the preceding relation over momentum space. These coefficients are functions of the $k_i(q, t)$, but the preceding relation shows that the partial differential equations are formally independent of the manner in which the k_i depend upon the position coordinates and the time. In solving the partial differential equations, there is no loss of generality in treating the k_i as constants. (Received October 8, 1951.)

GEOMETRY

67. Patrick Du Val: *Loci whose curve sections are hyperelliptic.*

This study is of the algebraic k-fold ${}^{\pi}H_k^n$ of order n, not generated by ∞' $[k]$s, whose general curve section is hyperelliptic and of genus π (expanding classical work of Castelnuovo, Rend. Circ. Mat. Palermo vol. 4 (1890), and Enriques, Rend. Accad. Lincei vol. 2 (1893), on the cases $k=2, 3$ respectively). ${}^{\pi}H_k^n$ is rational, and normal in $[n-\pi+k-1]$, and has a unique pencil of $(k-1)$-dimensional quadrics, tracing the canonical involution on the general curve section; the ambient $[k]$s of these generate a normal $(k+1)$-fold of order $n-\pi+1$, of which ${}^{\pi}H_k^n$ is the residual section by a quadric through $n-2\pi-2$ $[k]$s (or for $n<2\pi+2$, is coresidual to a quadric section plus $2\pi+2-n$ $[k]$s). If there is a point common to all quadrics of the pencil, this is a $(\pi+1)$-ple point of ${}^{\pi}H_k^n$. The necessary and sufficient condition for ${}^{\pi}H_k^n$ to be the projection of ${}^{\pi}H_k^{n+1}$ from a simple point of itself is that a quadric of the pencil breaks up into a pair of $[k-1]$s. If ${}^{\pi}H_k^n$ is not a cone, then either $n\leqq 2\pi+2$, or $n=2\pi+3$, $k\leqq \pi+4$, or $n+2k\leqq 4\pi+8$. (Received October 9, 1951.)

68. Benjamin Ernest Mitchell: *Realization of the quaternion.*

The quaternion, being a four-dimensional concept, transcends intuitional perception. However, four perfectly good "views" may be had of it in four mutually orthogonal planes after the manner of the descriptive geometry. Furthermore, one may consider two of these four planes Gaussian and thus achieve an analytical realization of the quaternion through a one-to-one lineo-linear transformation of either of these planes into the other. And finally the quaternion is co-dimensional with the line joining two points in the complex projective plane which in turn at the hand of

Laguerre has received intuitive representation in an ordered real point-pair. (Received October 8, 1951.)

STATISTICS AND PROBABILITY

69. A. C. Cohen: *The truncated Poisson distribution*. Preliminary report.

Consider a random sample from a Poisson frequency distribution, $f(x)=e^{-m}m^x/x!$, which is truncated at known terminal points. Using the principle of maximum likelihood, equations are derived for estimating the parameter m, both (1) when the number of unmeasured (missing) observations is unknown and (2) when the number of unmeasured observations is known. The results obtained are applicable to samples from which either one or both tails have been truncated. (Received October 10, 1951.)

TOPOLOGY

70. Haskell Cohen: *Codimension*. I.

Let X be a compact space, and let G be the coefficient group for the Alexander-Kolmogoroff cohomology groups. Cd(X), the codimension of X, is defined to be the greatest lower bound of those integers n such that every element of $H^n(A)$ is extendable to $H^n(X)$ for each closed subset A in X. It is shown that the sum theorem for closed sets holds, and also that cd$(X\times I)=1+$cd(X). The definition is extended to include the codimension of locally compact spaces and the codimension of the pair (X, X_0) where X_0 is a closed subset of X. It is shown that the theorems mentioned above hold in the latter two cases as well. (Received October 10, 1951.)

71*t*. S. T. Hu: *On finite-dimensional compact connected groups*.

Let G be a finite-dimensional compact connected group and G_0 a closed connected subgroup of G. It is well known that there is a totally disconnected closed central normal subgroup Z of G such that the quotient group $M=G/Z$ is a Lie group. Let $h: G\rightarrow M$ denote the natural projection and let $M_0=h(G_0)$. The main theorem of the paper states that the Čech cohomology ring $H(G/G_0)$ of the homogeneous space G/G_0 with real coefficients is isomorphic with $H(M/M_0)$ of the homogeneous space M/M_0 of the compact connected Lie group M over its closed connected Lie subgroup M_0. In particular, if G_0 reduces to the neutral element e of G, then $H(G)\approx H(M)$. Numerous applications are given in the paper; most of them are the generalizations of the theorems proved for compact connected Lie groups by H. Hopf and his school. (Received September 21, 1951.)

72. W. L. Strother: *Fixed point theorems for multi-valued functions*.

This paper gives a definition of continuity and two fixed-point theorems for multi-valued functions. Theorem one states that every continuous multi-valued function on the closed unit interval to itself has a fixed point. Theorem two states that every multi-valued function on a Tychonoff cube to itself which is continuous and satisfies one additional condition has a fixed point. Whether the product of spaces with the fixed point property for single-valued functions necessarily has the fixed point property is an open question. Theorem one above is used to answer this in the negative for multi-valued functions. (Received October 10, 1951.)

73t. A. D. Wallace: *An addition theorem for maps.*

Special cases of the following result are known. Let $X = X_1 \cup X_2$, $A = X_1 \cap X_2$, where X_1 and X_2 are closed. Let $f_i:(X, A) \rightarrow (Y, B)$ be maps ($i = 1, 2$) such that $f_1(X_2) \subset B \supset f_2(X_1)$ and $f_1 | A = f_2 | A$. Define $f_1 \vee f_2$ by $(f_1 \vee f_2) | X_1 = f_1 | X_1$ and $(f_1 \vee f_2) | X_2 = f_2 | X_2$. Then $f_1 \vee f_2 : (X, A) \rightarrow (Y, B)$ is a map such that $(f_1 \vee f_2)^* = f_1^* + f_2^*$ if X is fully normal or X is locally compact Hausdorff and $F(A)$ is compact. Here the * denotes the induced homomorphism of the Alexander-Kolmogoroff cohomology groups. Suppose (for example) that Y is a group and that B is the identity element. Then $(f_1 \cdot f_2)^* = f_1^* + f_2^*$. This is the analogue for cohomology of the commutativity of the homotopy groups of a group. (Received October 10, 1951.)

74. C. T. Yang: *Open maps and homomorphisms.*

Floyd [Ann. of Math. vol. 51 (1950) p. 573] characterized light open maps as follows: Let X and Y be connected, locally connected compacta and let f be a light map of X onto Y, then f is open if and only if for any connected open set V in Y and any component U in $f^{-1}(V)$, $f^*: H^r(\overline{V}, \overline{V}/V) \rightarrow H^r(\overline{U}, \overline{U}/U)$ is an isomorphism into, where the cohomology groups are the Čech groups with integral coefficients. By means of the construction of the Alexander-Kolmogoroff groups a direct proof of this characterization is given without involving the homotopy classes of maps into the unit circle and the result is slightly generalized. For instance, the cohomology groups can be taken as the Alexander-Kolmogoroff groups with coefficients in a nontrivial additive abelian group without torsion and either (i) compact Hausdorff spaces replace compacta or (ii) delete the compactness of X and Y and let f be closed. Further, the dual of a theorem of Whyburn [Duke Math. J. vol. 4 (1938) p. 8] follows as a corollary. (Received October 10, 1951.)

W. M. WHYBURN,
Associate Secretary

THE DECEMBER MEETING IN PASADENA

The four hundred seventy-sixth meeting of the American Mathematical Society was held at the California Institute of Technology, Pasadena, California, on Saturday, December 1, 1951. Approximately 140 persons attended, including the following 89 members of the Society:

O. W. Albert, H. L. Alder, T. M. Apostol, Richard Arens, M. M. Beenken, Clifford Bell, M. C. Bergen, Benjamin Bernholtz, H. F. Bohnenblust, J. V. Breakwell, R. E. Bruce, Herbert Busemann, W. D. Cairns, Paul Civin, P. H. Daus, Robert Davies, A. C. Davis, E. A. Davis, R. Y. Dean, D. B. Dekker, A. H. Diamond, R. P. Dilworth, Milton Drandell, Roy Dubisch, D. G. Duncan, W. D. Duthie, H. A. Dye, Arthur Erdélyi, Harley Flanders, G. E. Forsythe, O. A. Gross, G. J. Haltiner, H. J. Hamilton, V. C. Harris, A. R. Harvey, L. A. Henkin, A. D. Hestenes, M. R. Hestenes, P. G. Hodge, P. G. Hoel, Alfred Horn, D. H. Hyers, J. R. Jackson, C. G. Jaeger, P. B. Johnson, D. H. Lehmer, M. M. Lemme, J. W. Lindsay, Y. L. Luke, J. L. McGregor, J. C. C. McKinsey, M. W. Maxfield, C. B. Morrey, T. S. Motzkin, J. W. Odle, Barrett O'Neill, R. S. Pierce, D. H. Potts, W. T. Puckett, Edgar Reich, J. B. Robinson, R. M. Robinson, P. G. Rooney, Herman Rubin, I. J. Schoenberg, Raymond Sedney, Seymour Sherman, G. E. F. Sherwood, I. S. Sokolnikoff, R. H. Sorgenfrey, M. L. Stein, Robert Steinberg, E. G. Straus, A. C. Sugar, J. D. Swift, Alfred Tarski, A. E. Taylor, F. B. Thompson, D. L. Thomsen, F. A. Valentine, R. L. Vaught, Morgan Ward, W. R. Wasow, M. A. Weber, J. G. Wendel, P. A. White, A. L. Whiteman, František Wolf, F. H. Young.

By invitation of the Committee to Select Hour Speakers for Far Western Sectional Meetings, Professor J. C. C. McKinsey of Stanford University delivered at 11:00 A.M. an address entitled *Notions and problems of game theory*. Professor McKinsey was introduced by Professor H. F. Bohnenblust. There were sessions for contributed papers in the morning and afternoon, presided over by Professors A. E. Taylor, D. H. Lehmer, and C. B. Morrey. Following the meetings, those attending were guests at tea in the Athenaeum.

Following are abstracts of papers presented at the meeting. Papers with abstract numbers followed by "*t*" were presented by title. Paper number 96 was presented by Professor Hodge and number 85 by Mr. Thompson. Professor Kalicki was introduced by Professor Alfred Tarski, and Dr. Selmer by Professor T. M. Apostol.

ALGEBRA AND THEORY OF NUMBERS

75. T. M. Apostol: *Theorems on generalised Dedekind sums.*

The sums in question are defined by $s_p(h, k) = \sum_{\mu=1}^{k-1} (\mu/k)\overline{B}_p(h\mu/k)$, where $\overline{B}_p(x)$ is the pth Bernoulli function and h and k are relatively prime integers. For odd $p>1$, the author derives the formula (1): $s_p(h, k) = ip!(2\pi ik)^{-p}\sum_{\mu=1}^{k-1} \cot(\pi h\mu/k)\zeta(p, \mu/k)$, where $\zeta(s, a)$ is the Hurwitz zeta function, defined for $R(s)>1$ by the series

$\sum_{n=0}^{\infty} (n+a)^{-s}$. Formula (1) is then applied to give an analytic proof of the reciprocity law for $s_p(h, k)$. If p is allowed to tend to 1 in (1), the second member tends to $-(2\pi k)^{-1}\sum_{\mu=1}^{k-1} \cot (\pi h\mu/k)\Gamma'(\mu/k)/\Gamma(\mu/k)$, which is shown to be equivalent to an expression obtained by Rademacher for the Dedekind sums, namely: $s_1(h, k) = (4k)^{-1}\sum_{\mu=1}^{k-1} \cot (\pi h\mu/k) \cot (\pi\mu/k)$. The paper also includes an elementary derivation of this last equation using finite Fourier series. (Received October 22, 1951.)

76*t*. H. W. Becker: *Combinatory interpretations of the differences of the numbers of E. T. Bell.*

Apply the Steffensen operation $\nabla U_n = U_n - U_{n-1}$ to the elementary Bell numbers $@_n = e^{\Delta}0^n$. The first interpretations of $\nabla^m @_n$ were in terms of *forbidden* positions in rhyme schemes, distributions, and non-attacking rook patterns on a triangular chess board (Mathematical Magazine vol. 22 (1948) p. 25). Two new interpretations are: $\nabla^{n-p+1}@_n$ is the number of $n+1$ letter rhyme schemes whose last singleton, or else last "a," is in the pth position (except that the enumeration for one "a" is $@_n$, not $\nabla^n @_n$) with isomorphic distribution and rook meanings. These afford a combinatory proof that $\sum_0^n \nabla^m @_n - \nabla^{n+1}@_{n+1} = @'_{n+1}$, the number of $n+1$ letter rhyme schemes not without singletons. The iterated Bell numbers (Ann. of Math. vol. 39 (1938) p. 539; Amer. J. Math. vol. 61 (1939) p. 89) have analogous breakdowns, in terms of more complicated difference operators. (Received October 17, 1951.)

77. Anne C. Davis: *Cancellation theorems for products of order types.* I. Preliminary report.

Let $\alpha, \beta, \cdots, \kappa, \cdots$ denote arbitrary order types. The (ordinal) sum and the (ordinal) product of α, β are respectively denoted by $\alpha+\beta$ and $\alpha\ \beta$. A type κ is here called a *cancelling* type if $\kappa\ \alpha = \kappa\cdot\beta \rightarrow \alpha = \beta$ for arbitrary types α, β. Theorem I. *The following three conditions are equivalent*: (i) *κ is a cancelling type.* (ii) *$\kappa \neq \kappa\cdot\alpha$ for every type $\alpha \neq 1$.* (iii) *$\kappa \neq \kappa + \kappa\cdot\alpha + \kappa$ for every type α.* Theorem II. *If κ is a scattered type (i.e. the type of a scattered order set) and $\kappa \neq 0$, then $\kappa \neq \kappa + \mu + \kappa$ for every μ.* A consequence of Theorems I and II is Corollary III. *Every nonzero scattered type is a cancelling type.* Corollary III is an improvement of the result, published without proof by Lindenbaum (Comptes Rendus des Séances de la Société des Sciences et des Lettres de Varsovie vol. 19 (1926) p. 321) that every nonzero ordinal is a cancelling type. Another consequence of Theorem I is Corollary IV. *Every type κ of a nonempty ordered set without gaps and without first and last elements is a cancelling type.* Hence, in particular, the type λ of the set of real numbers is a cancelling type (a fact previously noted by Tarski). (Received October 15, 1951.)

78. D. G. Duncan: *A formula in Littlewood's algebra of S-functions.*

Let $t_i = \{\mu\} \otimes S_i$ where $\otimes$ denotes Littlewood's "new" multiplication of S-functions. In this note the generating function $\prod_{i=1}^{\infty} e^{(t_i/i)z^i} = \sum_{r=0}^{\infty} (\{\mu\} \otimes \{r\})z^r$ is used to establish the identity: $\{\mu\} \otimes \{2r\} = \sum_{j=1}^{r-1} (\{\mu\} \otimes \{2r-j\})(\{\mu\} \otimes \{j\})(-1)^{j+1} + 2^{-1}(\{\mu\} \otimes \{r\})^2(-1)^{r+1} + 2^{-1}\sum_{\alpha} (1/\alpha_1! \cdots \alpha_r!)(t_2/1)^{\alpha_1} \cdots (t_{2r}/r)^{\alpha_r}$. An application of this formula arises in invariant theory where a systematic study of $\{\mu\} \otimes \{r\}$ is desired. (Received October 23, 1951.)

79. Harley Flanders: *Generalization of a theorem of Ankeny and Rogers.*

The following result is proved: Let k be an algebraic number field and ν the largest integer such that $\cos(2\pi/2^r)$ is in k. Let a be a nonzero element of k, n a natural number, and assume that the congruence $x^n \equiv a \pmod{\mathfrak{p}}$ is solvable for almost all prime divisors $\mathfrak{p}$ of k. Then either a is a perfect nth power, $a = b^n$, or $a = b^n(2^{-1}(1+\cos(2\pi/2^r)))^{n/2}$ with b in k. The converse is true. Also a short proof of the Ankeny-Rogers result (k = rationals) is given, based on an extension of the Eisenstein irreducibility lemma. Finally a corresponding theorem for function fields is given. (Received October 29, 1951.)

80. Alfred Horn: *The normal completion of a subset of a complete lattice and lattices of continuous functions.*

Let C be a subset of a complete lattice B. A construction is given for a subset of B which is isomorphic with the normal completion $\tilde{C}$ of C. As an application, let B be the set of all real-valued (including $\pm\infty$) functions on a topologic space X. If C is the class of continuous members of B, then $\tilde{C}$ becomes the class of normal lower semicontinuous functions, provided that X is completely regular. Also treated are the cases where C is the class of finite-valued or the class of bounded continuous functions. The latter case was first discussed by R. P. Dilworth (Trans. Amer. Math. Soc. vol. 68 (1950) pp. 427–438). New characterizations of normal lower semicontinuous functions are obtained. These are applied to prove that for any two topologic spaces, the lattices of normal lower semicontinuous functions are isomorphic if and only if the lattices of regular open sets are isomorphic. The sufficiency was first proved in the completely regular case by Dilworth (loc. cit.). (Received October 17, 1951.)

81. Jan Kalicki: *On equational equivalence of abstract algebras.*

Terminology used will be that of Birkhoff (*On the structure of abstract algebras*, Proc. Cambridge Philos. Soc. vol. 31 (1935)). Two abstract algebras are equationally equivalent if their sets of laws coincide. A decision method is obtained for equational equivalence of finite abstract algebras. It is proved that if an equation $\phi = \psi$ is a law of a finite algebra A_1 of order m_1 without being a law of a finite algebra A_2 of order m_2, or vice versa, then there is an equation $\phi' = \psi'$ with the same property and with the number of different variables not exceeding $m = m_1 m_2$. Also if there is an equation with the properties of $\phi' = \psi'$, then there is an equation $\phi'' = \psi''$ with the same properties and such that the ranks of ϕ'' and ψ'' do not exceed the number $\sum_{i=1}^{m} m^{m^i}$. Since for given A_1 and A_2 the family of equations with the properties of $\phi'' = \psi''$ is determined and finite, the decision method follows. By applying the method to A_1 and to direct product $A_1 \times A_2$ it can be found whether the set of laws of A_1 is included in that of A_2. Similar results are obtained for logical matrices. (Received October 18, 1951.)

82*t*. Peter Scherk: *On sets of integers.* II.

(1) The results of the preceding abstract are extended from sets of non-negative integers to arbitrary sets $A, B, C, \cdots$ of integers. (2) The following corollary is noted: Let $A+B \subset C$, $r \subset A$, $s \subset B$. Let $k \geq 0$, $0 < \gamma < 1$, and $A(r, r+h) + B(s, s+h) \geqq \gamma h$ $[h = 0, 1, \cdots, k]$. Then $C(r+s, r+s+h) \geqq \gamma h$. Here $D(x, y) = \sum_{x < d \leqq y,\, d \subset D} 1$. (Received October 18, 1951.)

83. E. S. Selmer: *A conjecture of Mordell concerning rational points on cubic surfaces.* Preliminary report.

It is well known that the elementary congruence conditions (e.c.c.)—together with solubility in real numbers—are sufficient for the solubility in integers of a homogeneous quadratic equation in any number of variables. The author has recently (Acta Math. vol. 85 (1951)) shown the insufficiency of the e.c.c. for a homogenous *cubic* equation in *three* variables. Prof. L. J. Mordell (*Rational points on cubic surfaces*, Publ. Math. Debrecen vol. 1 (1949) p. 1) has conjectured that the e.c.c. are sufficient for solubility of a homogeneous cubic equation in *four* variables, i.e. for the existence of rational points on cubic surfaces. The author considers the purely cubic equation $a_1x_1^3+a_2x_2^3+a_3x_3^3+a_4x_4^3=0$, with the additional condition that (for instance) a_1a_2/a_3a_4 is a rational cube. In this case the equation can be written as $x^3+my^3=n(z^3+mu^3)$. By factorizing both sides in the purely cubic field $R(m^{1/3})$, the author proves Mordell's conjecture for this particular equation. To cover all cases, a deep result of H. Hasse is needed. The same method does not apply to the general purely cubic equation, but it seems very likely that Mordell's conjecture still holds. (Received November 29, 1951.)

84. E. G. Straus: *A problem of D. H. Lehmer.*

Lehmer conjectured that every number theoretical function $f(x)$ with the property that it maps every complete residue system modulo any prime, except for a finite number of exceptional primes, onto a complete residue system modulo this prime, is of the form $f(x)=ax+b$, where a is divisible only by exceptional primes and b is an arbitrary integer. This conjecture is proved and certain generalizations are established. (Received October 19, 1951.)

85. Alfred Tarski and F. B. Thompson: *Some general properties of cylindric algebras.* Preliminary report.

Given any ordinal α, let $\mathfrak{A}$ be a Boolean algebra $\mathfrak{B}=\langle A, +, \cdot, ^-, 0, 1\rangle$ with additional systems of unary operations C_ξ and distinguished elements $d_{\xi,\eta}$ satisfying (for $x, y\in A$ and $\xi, \eta, \zeta<\alpha$) the postulates: P$_1$. $C_\xi 0=0$; P$_2$. $x\cdot C_\xi x=x$; P$_3$. $C_\xi(x\cdot C_\xi y)=C_\xi x\cdot C_\xi y$; P$_4$. $C_\xi C_\eta x=C_\eta C_\xi x$; P$_5$. $d_{\xi,\xi}=1$; P$_6$. $\xi, \eta\neq\zeta\rightarrow d_{\xi,\eta}=C_\zeta(d_{\xi,\zeta}\cdot d_{\eta,\zeta})$; P$_7$. $\xi\neq\eta\rightarrow C_\xi(d_{\xi,\eta}\cdot x)\cdot C_\xi(d_{\xi,\eta}\cdot \bar{x})=0$. $\mathfrak{A}$ is called an *α-dimensional cylindric algebra with diagonal elements*—a CA_α; for examples see following abstract. An ideal in $\mathfrak{A}$ is a set X which is a Boolean-algebraic ideal in $\mathfrak{B}$ such that $C_\xi x\in X$ whenever $x\in X$, $\xi<\alpha$. Connections between ideals and homomorphisms in CA_α's are the same as in Boolean algebras. $\mathfrak{A}$ proves to be simple if and only if, for every $x\neq 0$, there is a finite sequence $\xi_0, \cdots, \xi_{n-1}<\alpha$ with $C_{\xi_0}\cdots C_{\xi_{n-1}}x=1$. For $x\in A$ let the *dimension index* $Dx=\mathrm{E}_\xi(\xi<\alpha,\ C_\xi x\neq x)$. Direct factors in $\mathfrak{A}$ prove to be sets of the form $\mathrm{E}_x\ (x\leq a)$ where Da is empty. $\mathfrak{A}$ is directly indecomposable if and only if Dx is nonempty for every $x\neq 0, 1$. A CA_α in which Dx is finite for every x is called *locally finitely-dimensional*—an FCA_α. Simple FCA_α's coincide with directly indecomposable FCA_α's. Hence, by a theorem of Birkhoff, every FCA_α is a subdirect product of simple FCA_α's. (Received October 16, 1951.)

86. Alfred Tarski: *A representation theorem for cylindric algebras.* Preliminary report.

For notations see preceding abstract. X being any set, X^α is the set of all sequences of length α with all terms in X. A *proper* CA_α is a system $\mathfrak{A}$ in which: 1 is a union of some pairwise disjoint sets Y_ι^α; 0 is the empty set; $+$ is set-addition, $\cdot$ set-multiplication, $^-$ complementation (relative to 1); $C_\xi X$ (for $X\subseteq 1$, $\xi<\alpha$) is the set of all sequences $y\in 1$ such that $y_\eta=x_\eta$ for some $x\in X$ and for every $\eta<\alpha$, $\eta\neq\xi$; $d_{\xi,\eta}$ $(\xi, \eta<\alpha)$ is the set of all $x\in 1$ with $x_\xi=x_\eta$; A is any field of sets $X\subseteq 1$ closed under the operations C_ξ and containing the sets $d_{\xi,\eta}$. Clearly, P$_1$–P$_7$ hold in every proper CA_α. By carrying over Godel's completeness theorem for predicate calculus to CA_α's the following result is obtained: *For $\alpha \geqq \omega$, every algebraic equation which holds (identically) in every proper FCA_α holds also in every FCA_α (and in fact in every CA_α).* The following theorem improving this result has been established: *Every FCA_α with $\alpha \geqq \omega$ is isomorphic to a proper FCA_α.* The problem is open whether this theorem extends to arbitrary CA_α with $\alpha \geqq \omega$. For $\alpha < \omega$ some results can be derived from what is available in the literature. (Received October 16, 1951.)

87*t*. R. L. Vaught: *On the equivalence of the Axiom of Choice and a maximal principle.*

As a particular case of Zorn's Lemma, one obtains the following statement: (1) Every family of sets has a maximal disjointed subfamily, i.e., a maximal subfamily having the property that any distinct two of its members are disjoint. Thus (1) is a consequence of the Axiom of Choice. The question was asked by Tarski whether conversely (1) implies the Axiom of Choice. The answer is affirmative. In fact, suppose $\mathcal{A}$ is any disjointed family of nonempty sets. Let $\mathcal{B}$ be the family of all sets of the form $\{A, \{x\}\}$ with $x \in A \in \mathcal{A}$. By (1), $\mathcal{B}$ has a maximal disjointed subfamily $\mathcal{M}$. Let M be the set of all x such that for some A, $\{A, \{x\}\} \in \mathcal{M}$. From the maximality of $\mathcal{M}$ it follows easily that M is a set having exactly one element in common with each member of $\mathcal{A}$. (Received October 19, 1951.)

ANALYSIS

88*t*. L. D. Berkovitz: *On double trigonometric integrals.*

Let ϕ be a complex-valued measure which is finite-valued on all bounded Borel sets in the plane and whose total variation over a circle with center at (u, v) and radius 1/2 is $o((u^2+v^2)^{-1/2})$. The first principal result of this paper is that there exists a trigonometric series $T=\sum_{m,n=-\infty}^{\infty} a_{mn}\, e^{i(mu+nv)}$ with coefficients $a_{mn}=o((m^2+n^2)^{-1/2})$ such that T is circularly equiconvergent with the integral $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(xu+yv)}d\phi(u, v)$. The second result is the analogous theorem stated for restricted equiconvergence, the conditions on the variations of ϕ and the order of the a_{mn} now being $o((|u|+1)^{-1/2} \cdot (|v|+1)^{-1/2})$ and $o((|m|+1)^{-1/2}(|n|+1)^{-1/2})$ respectively. To prove the two theorems the concept of formal product for integrals is defined and multiplication theorems analogous to those used in the Rajchman-Zygmund approach to localization for single trigonometric series are developed. Riemann formulae are then established and from these the principal results follow. (Received October 15, 1951.)

89. Paul Civin: *Multiplicative closure and the Walsh functions.*

A study is made of real orthogonal systems which are closed under multiplication and which contain a unit. Any such system which is infinite is shown to be a multiplicative group isomorphic with the group of the Walsh functions. Furthermore, there is a measurable transformation of the unit interval onto itself which transforms the Walsh functions into the given system of functions and such that the inverse image of a set of measure α is of measure α. (Received September 27, 1951.)

90*t*. Abolghassem Ghaffari: *Topological nature of hodograph transformation.*

When considering the flow around an aerofoil, one observes that there is usually a stagnation point at $z=0$. Let $\phi+i\psi=Az+Bz^2+\cdots$ represent the complex potential near the stagnation point. Since the velocity is zero at $z=0$, then $A=0$ and therefore

$d(\phi+i\psi)/dz=u-iv=2Bz+\cdots$, where $u-iv=w\exp(-i\theta)=w_m\zeta^{1/2}\exp(-i\theta)$ in which $\zeta=w^2/w_m^2$ and θ denote hodograph variables; u and v indicate the components of fluid velocity w; w_m is the maximum of w. It is found that $\phi+i\psi=C(u-iv)^2+\cdots$, where $C=1/4B$. Therefore $\phi+i\psi$ can be expanded in a power series in $\zeta^{1/2}$ without constant term and no term in $\zeta^{1/2}$; the first term involves ζ. At greater distance from the aerofoil the complex potential may be expanded in the form $\phi+i\psi=Az+B\log B+C/z+\cdots$;, where A is the complex velocity in the main stream and B depends on the circulation. It is shown that $\phi+i\psi\sim AB/(u-iv-A)+B\log B/(u-iv-A)+P(u-iv-A)$, where $P(u-iv-A)$ is a power series in $(u-iv-A)$, and hence the singularities in the hodograph plane at $u-iv=A$ are a combination of single poles and logarithmic singularities. Therefore no finite set of elementary solutions can represent the true state of the field of flow around an aerofoil. (Received October 11, 1951.)

91*t*. Joseph Kampé de Fériet: *Uniqueness theorem for the heat equation on an infinite rod.*

If $f(x)\in L^2$ and $u(x,t)$ satisfies in the half-plane $t>0$ the following conditions: (1) u_x and u_{xx} exist, u_{xx} being everywhere finite, (2) u and $u_{xx}\in L^2$ with respect to x for every t, (3) $u_{xx}(x,t)=\text{l.i.m.}\,[u(x,t+h)-u(x,t)]:h$ if $h\to 0$, (4) $f(x)=\text{l.i.m}\,u(x,t)$ if $t\to+0$, then: (5) u_t exists everywhere, (6) $u_t=u_{xx}$, (7) $u(x,t)$ is equal to the Poisson integral: $(4\pi t)^{-1/2}\int_{-\infty}^{+\infty}\exp\left[-(s-x)^2/4t\right]f(s)ds$. This proposition is an improvement of the uniqueness theorem given by S. Bochner, K. Chandrasekharan, *Fourier transforms*, p. 134. The proof is based on the following facts: (a) due to (1), (2): $u_x\in L^2$ and $(u, u_{xx})=-\|u_x\|^2$; (b) due to (3): the derivative of $\|u(x,t)-f(x)\|^2$ is equal to $2(u, u_t)\leqq 0$. (Received October 29, 1951.)

92*t*. M. S. Klamkin: *On series derived from* Ln$(1+x)$ *by successive integrations.*

By evaluating the p-fold integrals $\int_0^x\cdots\int_0^x(dx)^p/(1+x)$ and $\int_0^x\cdots\int_0^x(dx)^p/(1-x)$ we can determine the sums of series whose nth terms are given by $X^{p+n}n!/(p+n)!$, $(-1)^{n+1}X^{p+n}n!/(p+n)!$, $X^{p+2n}2n!/(p+2n)!$, and $X^{p+2n+1}(2n+1)!/(p+2n+1)!$. The integrals are evaluated by expressing each as a polynomial in x plus a logarithmic term. The coefficients of the polynomial are determined from a difference equation of the type $a_{r,p}=a_{r,p-1}/(p-r)+\phi(r,p)$ subject to the condition $a_{p,p}=0$ and where $\phi(r,p)$ is known. (Received October 18, 1951.)

93. Y. L. Luke: *An associated Bessel function.*

The purpose of this paper is to investigate the properties of the solution to $\Delta_\nu y=z^2d^2y/dz^2+zdy/dz-(\nu^2+z^2)y=e^{-z}z^{\mu+1}$. The technique follows the pattern for the development of the well known Lommel functions. The result is applied to evaluate explicitly some indefinite integrals involving Bessel functions of the type previously considered by the author (*Some notes on integrals involving Bessel functions*, Journal of Mathematics and Physics vol. 29 (1951) pp. 27–30). The analysis is extended to the evaluation of some infinite integrals of interest. Other applications are also given. (Received September 28, 1951.)

94*t*. František Wolf: *Rellich's theorem on analytic perturbation for operators in Banach space.*

Rellich's theorem (Math. Ann. vol. 113 (1936) pp. 600–619) is concerned with an operator $A(\lambda)$ in Hilbert space which is self-adjoint for real λ and depends analytically on the parameter λ. If $A(0)$ has an isolated spectral value μ_0 of finite multiplicity m, then the theorem asserts that there exist m analytic functions $\mu_k(\lambda)$, $k=1, \cdots, m$, $\mu_k(0)=\mu_0$, which for sufficiently small λ represent all spectral values of $A(\lambda)$ near μ_0. This theorem has been generalized by S. L. Jamison (Thesis, Berkeley, 1949) to operators in Hilbert space which are normal for real λ's. The author generalized this theorem (to appear in Math. Ann.) to operators in Banach space which have the following property. There exists a sequence $\{\lambda_n\}$, $\lim \lambda_n=0$, such that the Banach algebra generated by $A(\lambda_n)$ has no radical. (Received October 19, 1951.)

APPLIED MATHEMATICS

95*t*. Abolghassem Ghaffari: *On a solution of the hodograph equations in the supersonic region*. Preliminary report.

Let ψ and ϕ denote the stream function and the velocity potential of a compressible supersonic flow and let ζ, $1/6<\zeta<1/2$, and θ be the hodograph variables. The simplified form of the hodograph equations in the supersonic range is (1) $P\phi_\zeta=\psi_\theta$, $Q\psi_\zeta=\phi_\theta$, where $P=2\zeta(1-\zeta)^{\beta+1}[(2\beta+1)\zeta-1]^{-1}$, $Q=2\zeta(1-\zeta)^\beta$, and $\beta=1/(\gamma-1)$; γ is adiabatic index. The author gives a solution of (1) in the supersonic region. Considering the new variable t defined by $t=\int_{\zeta_0}(PQ)^{-1/2}d\zeta$ and setting $R=(Q/P)^{1/2}$, where $\zeta_0=1/6$ is the value of ζ at the sonic speed, the equations (1) become (2) $(R\psi_t)_t=R\psi_{\theta\theta}$, $(R^{-1}\phi_t)_t=R^{-1}\phi_{\theta\theta}$. It is found that for the approximate value $R\sim t^{1/3}$, the equations (2) transform into (3) $\psi_{tt}+1/3t\psi_t=\psi_{\theta\theta}$, $\phi_{tt}-1/3t\phi_t=\phi_{\theta\theta}$ which are satisfied by the solutions (4) $\psi=-t^{1/3}\sum A_mJ_{1/3}(mt)$, $\phi=t^{2/3}\sum B_mJ_{2/3}(mt)$, where the J's denote Bessel functions and A_m, B_m are arbitrary constants. (Received October 11, 1951.)

96. P. G. Hodge and William Prager: *Limit design of reinforcements of cut-outs in slabs*. Preliminary report.

A thin square slab with a central circular cut-out is subjected to a slowly increasing uniformly distributed tensile stress on the four edges. The flow limit is determined, i.e., that value of the tensile stress for which unrestricted plastic flow sets in. Reinforcing rings welded to the two sides of the slab are then designed so that the reinforced slab has the same flow limit as an unreinforced slab without cut-out. The analysis is based on the assumptions of plane stress and Tresca's yield criterion of constant maximum shearing stress. (The results presented in this paper were obtained in the course of research conducted under Contract N7onr-35810 between the Office of Naval Research and Brown University.) (Received October 15, 1951.)

97. František Wolf: *On Mann's singular integral equation.*

W. R. Mann and the author discussed (Quarterly of Applied Mathematics vol. 9 (1951) pp. 163–184) the integral equation $y(t)=\int_0^t\mathcal{G}(y(\tau))/(\pi(t-\tau))^{1/2}\,d\tau$ which arose from a problem in the conduction of heat with a nonlinear boundary condition. $\mathcal{G}$ is a nonincreasing continuous function in $(0, 1)$ with $\mathcal{G}(1)=0$. While the discussion arrived at satisfactory results in the case of $\mathcal{G}$ satisfying a Lipschitz condition, in the case of the more general conditions on $\mathcal{G}$ given above only existence of a solution was proved. The uniqueness of the solution seemed to lie rather deep. Later J. H. Roberts and W. R. Mann proved the last two results in a paper to be published in the Pacific Journal of Mathematics. They discuss the integral equation $y(t)=\int_0^tK(t-\tau)\mathcal{G}(y(\tau))d\tau$ with certain conditions on the kernel K. They attack the problem directly and while uniqueness proved easy, the increasing character of the solution required a long and delicate proof. The author was able to prove the same results by approximating the general problem by special problems where conditions imposed upon $\mathcal{G}$ and K make the results easily accessible. The proofs prove to be simple, using standard methods, and are so general that their ideas may very well apply to many similar problems. (Received October 19, 1951.)

GEOMETRY

98. D. B. Dekker: *The curves of Darboux and a classification of surfaces.*

On a surface in three-dimensional Euclidean space the differential equation for the Darboux curves is obtained in tensor form. The coefficients in the differential equation are found to be the triply covariant components of a third order tensor relative to transformations of the surface coordinates. This tensor is expressed in terms of the first and second fundamental second order surface tensors and the covariant derivative of the second fundamental tensor. It is then possible to study the Darboux curves on a surface without transforming to a particular type of surface coordinate system. Also a simple method of classifying surfaces by means of their first and second fundamental tensors is obtained by using the above third order tensor. (Received October 16, 1951.)

99. Milton Drandell: *Generalized convex sets in the plane.*

Let $\{C\}$ be a family of curves in the complex plane satisfying: (1) each member of $\{C\}$ is a closed Jordan curve passing through the point at infinity; (2) through any two finite points there passes a unique member of $\{C\}$. It is proved that if $\{p_n\}$ and $\{q_n\}$ converge to distinct finite points p_0 and q_0 respectively, and if $C_n \in \{C\}$ passes through p_n and q_n, $C_0 \in \{C\}$ passes through p_0 and q_0, then $\{C_n\}$ converges to C_0 in much the same sense as uniform convergence for real continuous single-valued functions. If $C \in \{C\}$ and p is a finite point not on C, then through p there passes a member of $\{C\}$ having no finite point in common with C. It is shown that this member is not unique. A set E is defined to be convex relative to $\{C\}$ in a manner analogous to the usual notion of convexity. If E is closed, bounded, possesses interior points, and is convex relative to $\{C\}$, then (a) E is the closure of a region; (b) the boundary of E is a closed Jordan curve. If F is compact, then its convex hull is compact. Several differences between this concept and ordinary convexity are discussed. (Received October 5, 1951.)

100*t*. Peter Scherk: *Convex bodies off center.*

Let $0<t<1$; $n \geqq 2$. Given a distance function $F(\mathfrak{p})$ in n-space. Suppose $F(\mathfrak{p}) \leqq 1$ defines a centrally symmetric convex body of volume J and suppose $\mathfrak{p}=0$ is the only integral vector satisfying $F(\mathfrak{p}) \leqq 2tJ^{-1/n}$. Then to every vector $\mathfrak{a}$ there is an integral vector $\mathfrak{g}$ such that $F(\mathfrak{a}-\mathfrak{g}) \leqq (1+(n-1)t^n)/t^{n-1}J^{1/n}$. This improves results by Mahler (Neder. Akad. Wetensch. vol. 51 (1938) pp. 634–637) and Störmer and Walter (Archiv der Mathematik vol. 2 (1950) pp. 346–348). (Received October 18, 1951.)

TOPOLOGY

101*t*. R. C. Blanchfield: *Self-linking invariants of homotopy chains.* Preliminary report.

Let G be a finitely generated additive abelian group, with the ring $\mathfrak{o}$ of a number field F as operator domain, which is paired to itself by a linking function V (see

Burger, Math. Zeit. vol. 52 (1949) pp. 217–255). Calculable invariants for the equivalence class of V, analogous to those defined for the rational case in a recent paper of Fox and Blanchfield (Ann. of Math. vol. 53 (1951) pp. 556–564), are defined. First the domain of operators is extended to a suitable principal ideal ring $\mathfrak{o}' \subset F$. The torsion coefficients of the group G with operators $\mathfrak{o}'$ are elements $\tau_1, \cdots, \tau_n$ of $\mathfrak{o}'$; say $\tau_{i+1} | \tau_i$ and let $\tau_{n+1} = 1$. For each r, $1 \leqq r \leqq n$, there exists an r-tuple $(a_1, \cdots, a_r)$ of elements of G for which the element $D(a_1, \cdots, a_r) = \tau_1 \cdots \tau_r \cdot \det \| V(a_i, a_j) \|$ of $\mathfrak{o}'$ is $\not\equiv 0 \pmod{\tau_r/\tau_{r+1}}$. If $(a_1, \cdots, a_r)$ and $(b_1, \cdots, b_r)$ are two such r-tuples, then $D(a_1, \cdots, a_r)$ and $D(b_1, \cdots, b_r)$ are equivalent in the sense that there exists an element $\xi \in \mathfrak{o}'$ such that $D(a_1, \cdots, a_r) \equiv \xi\bar{\xi} D(b_1, \cdots, b_r) \pmod{\tau_r/\tau_{r+1}}$. These equivalence classes are analogous to the quadratic residue classes and reduce to them in the rational case. Linking functions of the type considered arise in the study of $(4n+3)$-dimensional manifolds (see Burger, loc. cit.); the above mentioned equivalence classes are topological invariants of the manifold. (Received October 5, 1951.)

102. R. P. Dilworth: *The space of normal upper semicontinuous functions.*

Let $N(S)$ denote the set of normal upper semicontinuous functions on a completely regular space S. It has been shown (Trans. Amer. Math. Soc. vol. 68 (1950) pp. 427–438) that $N(S)$ is lattice isomorphic with the set of real continuous functions on the Boolean space $\mathfrak{B}$ associated with the Boolean algebra of regular open sets of S. Following Čech, the functions of $N(S)$ may be mapped into continuous functions on a compact Hausdorff space as follows: Let I_f denote the closed real interval $(\inf_S f, \sup_S f)$. Let $\mathfrak{S}$ denote the closure in the Cartesian product $\prod_f I_f$ of the points $\prod_f f(x)$ where $x \in S$. The correspondence $f \rightarrow F$ where $F(t) = t_f$ maps $N(S)$ into continuous functions on $\mathfrak{S}$. Now let $\mathfrak{S}$ be partially ordered by the relation $x \supseteq y$ if and only if $f(x) \geqq f(y)$ all $f \in N(S)$. The following results are obtained: (i) $\mathfrak{S}$, as a partially ordered set, has minimal elements, (ii) the minimal elements form a closed subset $\mathfrak{C}$ of $\mathfrak{S}$, (iii) $\mathfrak{C}$ is homeomorphic to $\mathfrak{B}$. (Received October 16, 1951.)

103*t*. S. T. Hu: *On homotopy groups of homogeneous spaces.*

Let G be a simply-connected topological group. If G is locally arcwise connected and locally simply-connected and Z is a discrete normal subgroup of G, then it is well known that the fundamental group of the topological quotient group $X = G/Z$ is isomorphic with the discrete group Z. Since, in this case, G is a covering space of X, it follows from a theorem of Hurewicz that $\pi_n(X) \approx \pi_n(G)$ for the higher homotopy groups. The purpose of this paper is to remove the inessential conditions imposed upon G and Z in these classical theorems. Indeed, the following generalization is the main theorem of the paper. If G is a simply-connected topological group and Z is either a zero-dimensional closed normal subgroup of G or a zero-dimensional locally compact closed subgroup of G, then the fundamental group of the homogeneous space $X = G/Z$ of the left cosets of Z in G is isomorphic with the abstract group Z and $\pi_n(X) \approx \pi_n(G)$ for the higher homotopy groups. (Received October 16, 1951.)

104. J. R. Jackson: *On homotopy groups of function spaces.*

Let X be an absolute neighborhood retract, X_0 be a retract of X, Y be an arbitrary topological space, and $y_0 \in Y_0 \subset Y$. Let Ω be the space of continuous functions on X into Y carrying X_0 into Y_0, Ω_0 be the space of continuous functions on X into Y

carrying X_0 to y_0, and Φ be the space of continuous functions on X_0 into Y_0. Each function space is given the compact-open topology. Each function space contains a function carrying all of the domain into y_0, and these functions we denote by y_0. Then the mth homotopy group $\Pi_m(\Omega, y_0)$ of Ω at base point y_0 is a split extension of a subgroup isomorphic to $\Pi_m(\Omega_0, y_0)$ by $\Pi_m(\Phi, y_0)$. In particular, if $m \geqq 2$, then $\Pi_m(\Omega, y_0)$ is isomorphic to the direct sum of $\Pi_m(\Omega_0, y_0)$ and $\Pi_m(\Phi, y_0)$. These results contain most of the known theorems on homotopy groups of function spaces, and also make it possible to determine the structures—in terms of the homotopy groups of Y and Y_0—of many homotopy groups inaccessible to previous results. (Received October 22, 1951.)

105. R. S. Pierce: *The Boolean algebra of regular open sets.*

Let S be a completely regular topological space. Denote by $C(S)$ the lattice of bounded, real-valued, continuous functions on S. A partial ordering of $C(S)$ is defined as follows: $f \supseteq g$ if, for all $h \in C(S)$, $f \wedge h \leqq 0$ implies $g \wedge h \leqq 0$. Writing $f \sim g$ if $f \supseteq g$ and $g \supseteq f$ defines a congruence relation on $C(S)$. Let L be the lattice obtained from $C(S)$ by identifying equivalent elements. The main result of the paper is that L is isomorphic to a sublattice of B, the complete Boolean algebra of regular open sets of S, and, moreover, that B is isomorphic to the normal completion of L. (Received October 19, 1951.)

W. T. PUCKETT,
Acting Associate Secretary

HARALD BOHR

April 22, 1887–January 22, 1951

Born in Copenhagen, Denmark, lived, learned and taught there, and died there; but liked to visit near and far. His father was Christian Bohr, Professor of physiology at the medical school; his mother was Jewish, and he had to flee the country in 1943, going then to Sweden for two years. During his last years he was Master of the one resident college in Copenhagen and he greatly enjoyed being surrounded by young people. Bohr was a very friendly person, kind and good-natured, chatty but not indiscreet, and he made and kept to his friends all his life. For all that, he was a shade melancholy, but this side of him was brightened by being married exceedingly happily. In his younger years he had been a football (or rather soccer) player of some renown, a rare enough occurrence among budding savants on the Continent. In after-years he would trounce a ball if he saw one, but he did not look much of an athlete.

Bohr's fame is his almost periodic functions. He did the theory in his mid-thirties, and very little in other topics afterwards. He wrote afterwards many papers in his theory, though, and rather substantial ones too. But the vim of the first papers did not return, and by now Bohr lives forth as much in the work of others as in his own. Privately, he gloried in the interest his theory has evoked and he would help to propagate it. He loved to lecture, and he lectured extremely well. In an individual lecture, no matter how learned the audience, he would first recite the definition and basic properties and make introductory comments; and he would do the same in print even when full references were included.

Whether he had an opinion as to future potentialities of the theory we do not know. Right now it is rather quiet about the theory, except for some distant rumbling in the field of automorphic functions, especially of several variables, which might precipitate in a new shower of almost periodicity. However that may be, it may be said that Harald Bohr has had the good fortune of having a very distinctive theory of magnitude very clearly and originally attached to his name, and this, after all, is as much as a mathematician may expect to secure for himself in a lifetime.

When Bohr was a student the study of mathematics at Copenhagen was not organized, and Bohr learned in a somewhat haphazard way from persons like the geometer Zeuthen, the astronomer Thiele, and the analyst Niels Nielsen (cylinder functions) who made a fierce play for mathematical rigor but not always very successfully; but he could not learn from J. L. W. V. Jensen, probably the greatest of them all, because he was an electrical engineer by profession and a self-taught mathematician, and had no connection with academic life. The course fare was generously supplemented by reading in books such as the second edition of C. Jordan's *Cours d'analyse* (of Jordan theorem fame) and the *Zahlentheorie* by Dirichlet and Dedekind.

Even before earning his masters degree in 1909 (the Ph.D. degree, which was quite a serious affair, followed in 1910) Bohr was led to the study of the theory of summability, mainly of Dirichlet series, and he dwelt on this topic quite a number of years. Summability, even in its pre-Tauberian era, used to be an important topic of study which even algebraists of the stature of Frobenius and Schur would dignify with their attention, and the reasons for this were manifold. First of all, to the uninitiated it exuded a certain mystique of esoterically conjuring up a limit where none had been before. In the second place (and this led to the most far-reaching consequences), Fejér had succeeded in "summing" the Fourier series of a continuous function towards its value, where DuBois Reymond had distressingly shown that the partial sums themselves need not so converge. And finally, and this must have been decisive for Bohr for the time being, it held out some hope of "moving-in" on the zeta function in its critical strip (or at least on L-series without a pole) where other means of intrusion had not been successful. This purpose proper was not to be attained, really, but a new view on summability did remain. Actually, Bohr's contribution was eventually absorbed in M. Riesz' more felicitous approach, and the resulting summability is named after Riesz, and not Bohr; but Bohr himself described his share in the theory in his article written jointly with H. Cramér, *Die neuere Entwicklung der analytischen Zahlentheorie* in Encyklopädie der Math. Wiss. Bd. II-3, Heft. 6, pp. 722–849.

We might mention that this early work of his brought Bohr soon in contact with E. Landau, and he spent 1909–1910 in Göttingen, which place he used to revisit frequently. He also became a very close friend of Hardy's, and also knew Littlewood very well, and stayed at Oxford and Cambridge in 1913–1914; and finally he went for several months in 1914 to Lebesgue in Paris to study his measure from him first-hand.

We think that the dominant interest of Harald Bohr was diophantine approximation and (equi-) distribution of numbers, but in a certain broad sense that would not lead him to a study of Weyl sums and the Waring problem, say. Somehow, to Bohr, the entire theory of almost periodic functions was an emanation from these sources, and this reflects itself best in his own proof for his "completeness theorem" which is so unlike any given afterwards and, in a nontechnical sense, is the most arithmetical among them all. Bohr was also infatuated with the classical theorem of Kronecker itself, and he repeatedly would give a new proof for it, which after all was not such an urgent task when compared with all the new problems suggesting themselves. Also, several years before arriving at the new theory, and repeatedly afterward, Bohr studied the zeta function and related functions as regards average frequency with which values are attained in the critical strip.

Bohr was the creator of the almost periodic functions, and he gave them such an appropriate name that no need arose for turning them into "fonctions bohriennes." Actually Bohr had a forerunner in the person of P. Bohl, who came within sight of the theory but missed it, although he has to his credit at least one nontrivial theorem which Bohr could take over virtually without change. One might even claim that Lagrange had anticipated a part of the theory by his study of mean motion. But after all this is conceded, the fact remains that almost periodic functions were Bohr's in the same sense as Lie groups were Lie's and in a stronger and much more significant sense than Banach spaces were Banach's, say. Bohr's high points of achievement consist of his having broken through towards a general definition for his functions such as he has found it, and of having then succeeded in proving his completeness theorem in his own fussy manner, such as he did; and as regards the genesis of his definition it is worth looking up a little known paper of his in which a tentative version is sought after (*Über eine quasi-periodische Eigenschaft Dirichletschen Reihen mit Anwendung auf die Dirichletschen L-funktionen*, Math. Ann. vol. 85, pp. 115–122). Bohr's personal triumph in attaining these achievements must be assessed against the fact that Bohr had been an arithmetico-analyst only, and always remained one only, and that nevertheless the true orientation of his achievements was towards topology, general group theory, homogeneous spaces, and such like directions, all of which had been rather alien to most arithmetico-analysts of his generation. We might perhaps point out for what it is worth that Hardy and Landau, both very prolific writers and close friends to Bohr, never took up the new topic themselves, or as much as published a note about it.

The brisk development of almost periodic functions on groups, as initiated by von Neumann, has by the very rapidity of its progress very quickly led to the sobering realization that after inherent compactifications have been introduced, the general theory is by actual scope, though not by initial intent, not larger than a theory of functions on compact groups only, or compact homogeneous spaces at best; and theories of functions on genuinely open groups are yet to be developed. In Euclidean space the current body of Fourier analysis including the theory of generalized Fourier integrals (alias "distributions tempérées") is a partial answer to the problem, perhaps. But on open semi-simple group spaces, say, the developments are not yet foreseeable, unless the recent work of Gelfand and Neumark and others is an indication of them.

If we contemplate Bohr's work the thought suggests itself that some of what is best in algebra, group theory, and topology has been coming forth from the fullness and fertility of analysis and has been stimulated if not always brought to surface by analysts themselves; and that perhaps analysis is the stuff of which our mathematics is made of, after all.

SALOMON BOCHNER

BOOK REVIEWS

The kernel function and conformal mapping. By Stefan Bergman. (Mathematical Surveys, vol. 5.) American Mathematical Society, New York, 1950. 8+160 pp. $4.00.

Orthogonal functions and the kernel function have become fundamental tools of the modern theory of functions; they allow one to consider the single-valued regular functions which are of integrable square over a given region as the elements of a Hilbert space.

The starting point for this development was L. Bieberbach's minimum problem connected with the Riemann mapping problem (see Rend. Circ. Mat. Palermo vol. 38 (1914)). The first researches on orthogonal functions in the complex domain originated under the influence of the Berlin school (particularly Erhard Schmidt and L. Bieberbach). Stefan Bergman, himself a descendent of this school, has intensively developed the theory of orthogonal functions and their kernels with great thoroughness during many years. He has now produced a text which provides an introduction to this now extensive field.

Chapter 1 begins with the following definition of the function class $\mathfrak{L}^2$ on which the whole theory rests: $f(z)$ is called of class $\mathfrak{L}^2$ in the given schlicht bounded region B if (1) $f(z)$ is regular and single-valued and

$$\text{(2)} \qquad \mathfrak{F}_B(f) = \iint_B |f(z)|^2 d\omega < \infty.$$

The orthonormal systems $\phi_\nu(z)$ are introduced in $\mathfrak{L}^2$, the Riesz-Fischer theorem is proved, and it is shown that there are closed orthonormal systems associated with B.

$$K(z, \bar{t}) = \sum_{\nu=1}^{\infty} \phi_\nu(z)\overline{\phi_\nu(t)}$$

is called the kernel. It is defined everywhere in B and is independent of the choice of the $\phi_\nu(z)$.

In Chapter 2 it is first shown that the uniquely determined function $f(z)$ in B, with $f(t) = 1$, $t \in B$, and $\iint_B |f(z)|^2 d\omega$ a minimum, is $f(z) = K(z, \bar{t})/K(t, \bar{t})$. Then some essentially more general minimum problems are solved.

Chapter 3 introduces an invariant metric with the aid of the kernel function. This is the Bergman metric, which is a true Riemannian metric and indeed a Kähler metric. It has turned out to be superior in many ways to the metric of H. A. Schwarz, Pick, and Carathéodory.

In Chapter 4 the author considers arbitrary classes of functions $f=\sum c_\nu f_\nu$, $\sum|c_\nu|^2<\infty$, where the f_ν form an orthonormal system. (The functions of these classes therefore represent a Hilbert space.) If then the kernel function

$$K(P, Q) = \sum_{\nu=1}^{\infty} f_\nu(P)\cdot\overline{f_\nu(Q)}$$

exists for arbitrary points P, Q of the fundamental region B, most of the results of the first three chapters can be extended to these more general classes of functions.

There is evidently a close relationship between statements about analytic functions and about sets of points in Hilbert spaces. This is developed for Vitali's theorem in a posthumous paper of Otto Toeplitz which appeared too late to be taken account of in this book (see Comment. Math. Helv. vol. 23 (1949) pp. 222–242).

Chapter 5 contains principally the important relationship among the Green's function, the Neumann function, and the kernel function for the class of harmonic functions,

$$k(z, \zeta) = (2\pi)^{-1}[N(z, \zeta) - G(z, \zeta)].$$

The next chapter uses these functions to establish the mappings of all p-ply (p finite) connected regions on the various types of canonical slit regions.

Chapter 7 takes up another orthogonalization (by integrals along the boundary). This yields a new kernel function $\hat{K}_B(z, \bar{\zeta})$. In simply connected regions

$$\left\{\frac{\hat{K}_B(z, \bar{\zeta})}{\hat{K}_B(\zeta, \bar{\zeta})}\right\}^2 = \frac{K_B(z, \bar{\zeta})}{K_B(\zeta, \bar{\zeta})}.$$

The close connection between this kernel function $\hat{K}$ and the class of bounded functions is studied. If $f(z)$ is regular and single-valued in B and $|f(z)|\leq 1$ there, then, for every $\zeta\in B$, $|f'(\zeta)|\leq 2\pi\hat{K}_B(\zeta, \bar{\zeta})$.

Chapter 8 is concerned with the dependence of the functions $K(z, \zeta)$, $G(z, \zeta)$, and $N(z, \zeta)$ on the region B. The variations of these functions are calculated when B is replaced by a somewhat larger region. Under certain assumptions,

$$\delta K(z, \bar{\zeta}) = \int_b K(z, \bar{t})K(t, \bar{\zeta})\delta n\cdot ds + o(\epsilon)$$

if the boundary b is displaced by an amount $\delta n = \epsilon\rho(s)$ in the direction of the inner normal. Later the more general variational method of M. Schiffer and the numerous results obtained by means of it are discussed.

While in the preceding chapters the existence of the Green's function, the Neumann function, etc., is assumed (which is permissible since the proofs can be found in many places in the literature), this borrowing is now dispensed with and the slit mapping is carried through without this assumption.

In Chapter 10 it is shown that the methods applied in the preceding chapters can also be applied to the solutions of partial differential equations of elliptic type. Here there are unexpected results from recent investigations of Bergman and of Bergman and Schiffer. The chapter ends with a treatment of the equation of elasticity, $\Delta\Delta\phi = 0$, in order to show how the process must be modified for equations of higher order.

The final chapter is concerned with functions of two complex variables and the analytic (pseudo-conformal) mappings generated by them. It is written for readers who are already familiar with the foundations of the theory of functions of several complex variables. First special regions (bicylinder, hypersphere, etc.) are treated; then the orthogonal functions are introduced for arbitrary schlicht bounded domains, and the mappings on representative regions by means of minimal functions and the invariant metric are set up. Finally the author discusses regions with distinguished boundary surfaces, the corresponding Bergman integral representations, and the "extended classes of functions." The choice of the topics in this chapter is perhaps somewhat too much oriented in the direction of the author's own extensive publications.

However, the book as a whole gives a distinguished introduction to the theory of orthogonal functions with its abundance of new results.

H. BEHNKE

Théorie des distributions. By Laurent Schwartz. (Publications de l'Institut de Mathématique de l'Université de Strasbourg, nos. 9 and 10; Actualités Scientifiques et Industrielles, nos. 1091 and 1122.) Vol. I, 1950, 148 pp. Vol. II, 1951, 169 pp.

In Euclidean E_k we consider a general function $\varphi(x) = \varphi(x_1, \cdots, x_k)$ which is defined and infinitely differentiable everywhere and is zero outside a bounded domain $D = D_\varphi$, and, as in a previous context, we call such a function a *testing function.* Next, if $F = F(x)$ is a fixed function in E_k which at first is continuously differentiable of order not less than n, then, by partial differentiation, we obtain, for $k=1$, for the integral

$$(1)\qquad \int_{-\infty}^{\infty} \varphi \frac{d^n F}{dx^n}\, dx$$

the value

$$(2)\qquad (-1)^n \int_{-\infty}^{\infty} F \frac{d^n \varphi}{dx^n}\, dx,$$

and for general k we obtain for

$$(3)\qquad \int_{E_k} \varphi \frac{\partial^{n_1+\cdots+n_k} F}{\partial^{n_1}x_1 \cdots \partial^{n_k}x_k}\, dv_k$$

the value

$$(4)\qquad (-1)^{n_1+\cdots+n_k} \int_{E_k} F \frac{\partial^{n_1+\cdots+n_k} \varphi}{\partial^{n_1}x_1 \cdots \partial^{n_k}x_k}\, dv_k.$$

Now, for the computation of the integrals (2), (4) the function F need not be differentiable and this leads to defining the symbols (1), (3) for testing functions φ (in terms of their values (2), (4)) even if the differentiation on F cannot be carried out literally. Such generalized integrals have been long in developing, and their systematic use was the very basis for the theory of generalized Fourier transforms as presented in the reviewer's book *Fouriersche Integrale*, 1932.

Now, the author proceeds as follows. On the distributive space of testing functions $\{\varphi\}$ introduce a limit topology by calling a sequence $\{\varphi_j\}$ convergent to φ if the given functions all vanish outside a joint bounded domain D_0, and if we have, uniformly in E_k,

$$(5)\qquad \lim_{j\to\infty} \frac{\partial^{n_1+\cdots+n_k}\varphi_j}{\partial^{n_1}x_1 \cdots \partial^{n_k}x_k} = \frac{\partial^{n_1+\cdots+n_k}\varphi}{\partial^{n_1}x_1 \cdots \partial^{n_k}x_k}$$

for all $n_1 \geqq 0, \cdots, n_k \geqq 0$; and, more generally, call the sequence n-convergent if (5) holds for $n_1 + \cdots + n_k \leqq n$. Obviously, (4) represents for every continuous F a distributive functional on $\{\varphi\}$ which is continuous in this topology, and for given n even n-continuous; and the author establishes the following converse which is the actual content of volume I. Given any continuous functional of this kind, then corresponding to any bounded D_0 there exist an F and a system $\{n_j\}$ such that for those testing functions which vanish outside D_0, the functional can be represented by the integral (4), and is thus also n-continuous in D_0, for some finite n.

In the first half of volume II this theorem is adapted to a Euclidean multi-torus, a testing function being any function

$$\varphi(x) = \sum a_{r_1\cdots r_k} e^{i(r_1x_1+\cdots+r_kx_k)} \tag{6}$$

which is infinitely differentiable, without its being required to vanish anywhere; it can also be characterized by the necessary and sufficient condition that for *every* $n > 0$ we have

$$a_{r_1\cdots r_k} = O(|r|^{-n}) \tag{7}$$

as $|r| \to \infty$, where $r = (r_1^2 + \cdots + r_k^2)^{1/2}$. Now, the topology for testing functions being introduced as before, the result is that any continuous functional is automatically n-continuous for some $n = n_0$, and that the functional has the value

$$\sum_{(r)} a_{r_1\cdots r_k} t_{r_1\cdots r_k} \tag{8}$$

where the "multiplier" $t_{r_1\cdots r_k}$ is the value of the functional for the particular testing function $e^{i(r_1x_1+\cdots+r_kx_k)}$. We may then also associate with the functional itself the unique expansion

$$\sum t_{r_1\cdots r_k} e^{i(r_1x_1+\cdots+r_kx_k)} \tag{9}$$

and a sequence of multipliers will constitute such an expansion if and only if for some $n_0 > 0$ we have

$$t_{r_1\cdots r_k} = O(|r|^{n_0}) \tag{10}$$

as $|r| \to \infty$. And finally it follows that the series (9) is convergent by *arbitrary* partial sums of itself in the (dual) topology of the space of functionals. The author rather prides himself on this last statement, but within the given context if amounts only to stating that if (10) holds, then for some other exponent $n = m$ we have

$$\sum{}' \frac{|t_{r_1\cdots r_k}|}{|r|^m} < \infty,$$

as we indeed do have for $m = n_0 + 2$, say. We might perhaps recall that in the L_2-theory, the multiple Fourier series likewise converges by arbitrary partial sums in the dual topology there given.

The second half of volume II is given over to generalized Fourier integrals, and there the analysis had always been very much subtler, and has so remained.

If $f(x)$ in $(-\infty, \infty)$ is $O(|x|^{-2})$ at infinity, say, then it has a transform

$$g(\alpha) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-ix\alpha}dx \tag{11}$$

from which it can be re-obtained as

$$f(x) \sim \int_{-\infty}^{\infty} e^{ix\alpha}g(\alpha)d\alpha. \tag{12}$$

If, now, for any $n>0$, $g_n(\alpha)$ denotes an nth indefinite integral of $g(\alpha)$, then we write for (12)

$$f(x) \sim \int_{-\infty}^{\infty} e^{ix\alpha}\frac{d^n g_n(\alpha)}{d\alpha^n}\,d\alpha, \tag{13}$$

and in a suitable normalization we have

$$\begin{aligned} 2\pi g_n(\alpha) = {} & \int_{-1}^{1} f(x)\,\frac{e^{-ix\alpha} - \sum_{k=0}^{n-1}\frac{(-ix\alpha)^k}{k!}}{(ix)^n}\,dx \\ & + \left(\int_{-\infty}^{-1} + \int_{1}^{\infty}\right) f(x)\,\frac{e^{-ix\alpha}}{(-ix)^n}\,dx. \end{aligned} \tag{14}$$

Now, this expression may be calculated even if $f(x)$ does not vanish at infinity; it suffices that we have $f(x) = O(|x|^l)$ for some finite l, and (14) may be then introduced, for $n \geq l+2$ at any rate. The resulting function $g_n(\alpha)$ will not be differentiable any more, but we nevertheless envisaged the relation (13) in a symbolic fashion, and these were the generalized Fourier integrals in our book cited. Actually we gave a first but different systematic version in an earlier paper,[1] and we also gave an application to almost periodic functions which ought to be generalized to several variables.[2] In these earlier papers we followed a lead of Norbert Wiener in considering functions which were locally L_2-integrable instead of L_1-integrable, and this made the theory so much more complicated than the one we arrived at eventually in *Fouriersche Integrale* that we even did not emphasize them afterwards; but for the present-day re-study of the entire topic they may contain view points yet explorable.

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[1] *Darstellung reellvariabler und analytischer Funktionen durch Verallgemeinerte Fourier- und Laplace-Integrale*, Math. Ann. vol. 97 (1927) pp. 635–662.

[2] *Über gewisse Differential- und allgemeinere Gleichungen, deren Lösungen fastperiodisch sind* I, II, III, Math. Ann. vol. 102 (1929) pp. 489–504; vol. 103 (1930) pp. 588–597; vol. 104 (1931) pp. 579–587.—For a later application of generalized Fourier integrals see also our paper jointly with R. P. Boas, Jr.: *Closure theorems for translations*, Ann. of Math. (2) vol. 39 (1938) pp. 287–300.

Now, turning again to our relation (13) we note that the author goes a step further than we did and he also differentiates the function $f(x)$ itself symbolically, thus

$$\frac{d^p f(x)}{dx^p} \sim \int_{-\infty}^{\infty} e^{ix\alpha}(i\alpha)^p \frac{d^n g_n(\alpha)}{d\alpha^n}\, d\alpha.$$

At first sight this still leaves the two sides unsymmetric in that, seemingly, on the right side the integrand is

$$(i\alpha)^p \frac{d^n g_n(\alpha)}{d\alpha^n},$$

with the unbalancing factor $(i\alpha)^p$ in front. It turns out however that for a suitable $G(\alpha)$ and m this can be symbolically written as

$$\frac{d^m G(\alpha)}{d\alpha^m}; \tag{15}$$

and in this way the author arrives at a symmetric Fourier transformational reciprocity between symbols

$$\frac{d^p F(x)}{dx^p}, \qquad \frac{d^m G(\alpha)}{d\alpha^m} \tag{16}$$

as it were, the functions $F(x)$, $G(\alpha)$ being arbitrary continuous functions in $(-\infty, \infty)$ which are $O(|x|^q)$ and $O(|\alpha|^n)$ at infinity, the indices p, m, q, n being unrestricted.

The operational foundation for this reciprocation is as follows. If $\varphi(x)$ is a testing function as before, then its transform

$$\psi(\alpha) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \varphi(x)e^{-ix\alpha}dx \tag{17}$$

will not vanish outside a finite interval. But if we assume only that $\varphi(x)$ is infinitely differentiable and that itself and each derivative is $O(|x|^{-l})$ at infinity for every finite l, then its transform $\psi(\alpha)$ has the same property. Thus, the class of these functions is self-inversible under Fourier transformation, and we shall call them ***semi-testing functions***. Now (11) and (17) imply by "convolution"

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\varphi(x)dx = \int_{-\infty}^{\infty} g(\alpha)\psi(-\alpha)d\alpha \tag{18}$$

and this suggests that for a pair of reciprocal symbols (16) we ought to have the relation

$$(19) \qquad \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{d^pF(x)}{dx^p}\varphi(x)dx = \int_{-\infty}^{\infty} \frac{d^pG(\alpha)}{d\alpha^p}\psi(-\alpha)d\alpha.$$

This is indeed so, provided we interpret the symbol

$$(20) \qquad \int_{-\infty}^{\infty} \frac{d^pF(x)}{dx^p}\varphi(x)dx,$$

and also the right side in (19), in the following manner. In the space of semi-testing functions we introduce a topology which is based on the following limit concept: a sequence $\{\varphi_j\}$ converges to 0 if for every $p \geqq 0$, $q \geqq 0$ the sequence

$$x^p \frac{d^q\varphi_j(x)}{dx^q}$$

is convergent to 0, as $j \to \infty$, uniformly in $(-\infty, \infty)$. This topology is different from the previous one even if applied to testing functions proper, and we now consider among the functionals originally given only such ones as are continuous with respect to the new topology of the testing functions as well. Such a functional can be extended to the semi-testing functions and remains continuous over them, and it is such a functional which can be represented in the form (20). These are the "distributions tempérées" of the author and they can be associated in pairs in such a manner that (19) shall hold for any pair of semi-testing functions themselves reciprocal, and two distributions thus associated are Fourier transforms of one another.

The author does all this for several variables as well, the class of semi-testing functions being again self-inversive. We note however that the resulting self-inversiveness of the class of distributions, interesting as it is, is only the "dual" to the self-inversiveness of the semi-testing functions themselves, which latter self-inversiveness is a rather obvious phenomenon and, for instance, cannot compare in subtlety to the self-inversiveness of the Plancherel transforms, say, where a natural norm is preserved as well.

The author also gives an extension of the reviewer's theorem on positive definite functions

$$(21) \qquad P(x_1, \cdots, x_k) = \int_{E_k} e^{i(x_1\alpha_1 + \cdots + x_k\alpha_k)} dT(\alpha_1, \cdots, \alpha_k)$$

for positive masses $dT \geqq 0$, from such ones for which the total mass is bounded

$$\int_{E_k} dT(\alpha) < \infty,$$

to more general ones for which the total mass is only "slowly" bounded, that is

$$\int_{E_k} \frac{dT(\alpha)}{(1+\alpha_1^2+\cdots+\alpha_k^2)^l} < \infty$$

for some l. The transform $P(x)$ is then not a function but a distribution of the author's kind, and he adapts our necessary and sufficient condition of positive-definiteness to his distribution so as to secure again the representation (21). But we might state that, contrary to what the author seems to think, we ourselves gave the theorem not only for one variable in *Fouriersche integrale*, but also for several variables immediately afterwards.[3]

A dominant analytical tool in the work is a certain "smoothing" process (in French "régularisation") which is used both to localize pieces of a spectrum or of a functional, and to approximate to a distribution by a function. As an analytical tool it is older than sometimes realized and it has been constantly used by us both for generalized Fourier integrals and almost periodic functions; and the closely related "partition of unity," so-called, which is gaining in importance in the cohomology theory of differential forms was introduced for the first time for just such a purpose in our note: *Remark on the theorem of Green*, Duke Math. J. vol. 3 (1937) pp. 333–338. And as regards the novelty of introducing "distributions" which are more general than Stieltjes integrals, say, we think that the credit for it ought to be assigned to Riemann who in his paper on trigonometric series interprets a series

$$\sum_{1}^{\infty} (\lambda_n \cos nx + \mu_n \sin nx), \tag{22}$$

with only $\lambda_n \to 0$, $\mu_n \to 0$, as a symbol

$$\frac{d^2F}{dx^2}$$

where $F(x)$ is defined as the uniformly convergent series

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[3] *Monotone Funktionen, Stieltjessche Integrale und harmonische Analyse*, Math. Ann. vol. 108 (1933) pp. 378–410.

$$-\sum_{1}^{\infty} \frac{\lambda_n \cos nx + \mu_n \sin nx}{n^2}$$

and then "convolves" the series (22) with that of a testing function in the appropriate manner.

We have recounted all this with a view to suggesting that it would not be easy to decide what the general innovations in the present work are, analytical and even conceptual, and that it is in order to appraise the value of the book by its specific results, such as we have extracted above; and of such let the author produce many more, by all means.

S. BOCHNER

Tables relating to Mathieu functions. Characteristic values, coefficients, and joining factors. Prepared by the Computation Laboratory of the National Applied Mathematics Laboratories, National Bureau of Standards. New York, Columbia University Press, 1951. 48+278 pp. $8.00.

Since its foundation (January 1938) the New York Unit of the Computation Laboratory of the National Applied Mathematics Laboratories, a division of the National Bureau of Standards—until July 1947 it was called the Mathematical Tables Project—has been very active in producing extensive and accurate numerical tables of important mathematical functions. Besides a series of tables of the elementary transcendents, they have published almost a dozen tables relating to the higher transcendents such as sine, cosine, and exponential integrals, probability functions, Bessel functions, Legendre functions. No matter how high-speed electronic calculating machinery may be further developed, applied mathematicians will always owe a great debt to these and other table-makers.

This particularly holds for the fascinating tables under review. As Professor Erdélyi emphasizes in the foreword, the comparatively slender numerical material available for Mathieu functions shows the urgency of the task undertaken by the National Bureau of Standards. The more so, since several important problems of applied mathematics and theoretical physics involving Mathieu functions have so far received only little attention because of lack of adequate numerical data. These problems include all types of vibrational, wave and diffusion problems connected with ellipses or elliptic cylinders, as well as stability investigations of various mechanical systems, the theory of frequency modulation, and loud-speaker theory.

Bessel functions and the like are rather elementary compared to Mathieu functions. The latter functions have no really simple representations such as power series with rational coefficients. On the other hand, a characteristic feature of Mathieu functions is that they are (in various ways) expansible into infinite series with a relatively simple three-term linear recurrence relation between successive coefficients. It is this feature (which the Mathieu functions share with the spheroidal wave functions) which renders possible the numerical calculation of Mathieu functions by the method of continued fractions. This technique was introduced by E. L. Ince and later on refined by others.

An excellent introduction, covering all the important numerical aspects of Mathieu functions and giving a wealth of material not easily to be found elsewhere, is presented by Gertrude Blanch of the Computation Laboratory. The standard form adopted for Mathieu's differential equation is (*) $y''+(b-s\cos^2 x)y=0$. Only real values of the parameters b and s are considered. It is known that (*) possesses periodic solutions of period π or 2π for the so-called characteristic values $b=\mathrm{be}_r$ or bo_r which are functions of s. These solutions are even or odd in x. There are altogether four different types of periodic Mathieu function, viz ($r=0, 1, 2, \cdots$)

$$\mathrm{Se}_{2r}(s, x) = \sum_{k=0}^{\infty} \mathrm{De}_{2k}^{(2r)} \cos 2kx \qquad [\text{period } \pi,\ b = \mathrm{be}_{2r}(s)],$$

$$\mathrm{Se}_{2r+1}(s, x) = \sum_{k=0}^{\infty} \mathrm{De}_{2k+1}^{(2r+1)} \cos (2k+1)x \qquad [\text{period } 2\pi,\ b = \mathrm{be}_{2r+1}(s)],$$

$$\mathrm{So}_{2r}(s, x) = \sum_{k=1}^{\infty} \mathrm{Do}_{2k}^{(2r)} \sin 2kx \qquad [\text{period } \pi,\ b = \mathrm{bo}_{2r}(s),\ r \neq 0],$$

$$\mathrm{So}_{2r+1}(s, x) = \sum_{k=0}^{\infty} \mathrm{Do}_{2k+1}^{(2r+1)} \sin (2k+1)x \qquad [\text{period } 2\pi,\ b = \mathrm{bo}_{2r+1}(s)].$$

The coefficients De and Do satisfy simple three-term recurrence relations. The characteristic values form an unbounded countably infinite set occurring in the order ($s>0$) $\mathrm{be}_0<\mathrm{bo}_1<\mathrm{be}_1<\mathrm{bo}_2<\mathrm{be}_2<\mathrm{bo}_3<\mathrm{be}_3< \cdots$. The periodic Mathieu functions are normalized by the conditions

$$(**) \qquad \mathrm{Se}(s, 0) = 1, \qquad \left[\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{So}(s, x)\right]_{x=0} = 1.$$

The coefficients De and Do are sufficient to characterize the *complete* solution of (*) for $b=\mathrm{be}$ or bo. For details concerning the nonperiodic solutions corresponding to the characteristic values the reader should consult the introduction. The same holds for power series and asymptotic expressions for the characteristic values, relations between solutions corresponding to the parameters s and $-s$, differences in notation and normalization as compared to earlier work, etc.

There is one obvious disadvantage in adopting (*) as the standard Mathieu equation, so far as the characteristic values are concerned. It would have been more convenient to have a table of $a = b - s/2$, corresponding to McLachlan's canonical form $y'' + (a - 2q \cos 2x)y = 0$, $s = 4q$. The normalization (**) seems the most satisfactory, in particular with a view to extension of (**) to functions of unrestricted complex s.

The characteristic values $\mathrm{be}_r(s)$ are tabulated to eight decimals for the following values of r and s:

$$r = 0,\ s = 0(0.2)20(0.5)37(1)100;\ r = 1,\ s = 0(0.2)15(0.5)50(1)100;$$
$$r = 2,\ s = 0(0.2)40(0.5)77(1)100;\ r = 3,\ s = 0(0.2)13(0.5)56(1)100;$$
$$r = 4(1)6,\ s = 0(0.5)100;\ r = 7(1)8,\ s = 0(1)100;\ r = 9(1)15,$$
$$s = 0(2)100.$$

The characteristic values $\mathrm{bo}_r(s)$ are tabulated to eight decimals for

$$r = 1(1)4,\ s = 0(0.5)100;\ r = 5(1)7,\ s = 0(1)100;\ r = 8(1)15,$$
$$s = 0(2)100.$$

In both cases modified second differences are given. Two graphs of the characteristic values are included (stability chart). For negative s the characteristic values follow from $\mathrm{be}_{2r}(-s) = \mathrm{be}_{2r}(s) - s$; $\mathrm{bo}_{2r}(-s) = \mathrm{bo}_{2r}(s) - s$; $\mathrm{be}_{2r+1}(-s) = \mathrm{bo}_{2r+1}(s) - s$.

The greater part of the tables under review is devoted to the coefficients De_k and Do_k, at an interval in s sufficiently small to yield a set of key values of the functions Se_r and So_r amenable to subtabulation. (The Mathieu functions themselves are not tabulated.) The coefficients De_k of Se_r are tabulated for

$$r = 0,\ s = 0(0.2)8(0.5)20(1)100;\ r = 1,\ s = 0(0.5)20(1)40(2)100;$$
$$r = 2,\ s = 0(0.2)8(0.5)15(1)40(2)100;\ r = 3,\ s = 0(0.2)20(1)40(2)100;$$
$$r = 4,\ s = 0(1)60(2)100;\ r = 5,\ s = 0(1)100;\ r = 6(1)7,\ s = 0(2)100;$$
$$r = 8,\ s = 0(4)68(2)100;\ r = 9,\ s = 0(4)100;\ r = 10,\ s = 0(5)100;$$
$$r = 11(1)15,\ s = 0(10)100.$$

The coefficients Do_k of So_r are tabulated for

$r = 1, s = 0(0.5)10(1)40(2)100$; $r = 2, s = 0(1)40(2)100$;

$r = 3, s = 0(1)20(2)68(4)100$; $r = 4, s = 0(1)14(2)40(4)100$;

$r = 5, s = 0(2)60(4)100$; $r = 6, s = 0(2)100$; $r = 7(1)9, s = 0(4)100$;

$r = 10, s = 0(5)100$; $r = 11(1)15, s = 0(10)100$.

Most values are given to 9 or 10 decimals, though for the smaller values of k a constant number (9 or 10) of significant figures is provided so as to allow the complete solution of (*) to be calculated to a high degree of accuracy. The remaining tables contain joining factors and auxiliary functions, a description of which is beyond the scope of this review.

In conclusion, the reviewer thinks that the computation of the several Mathieu functions themselves would be very welcome. At least for the periodic Mathieu functions proper, this would not be too difficult for table-makers using high-speed electronic computers!

C. J. BOUWKAMP

Die zweidimensionale Laplace-Transformation. Eine Einführung in ihre Anwendung zur Lösung von Randwertproblemen nebst Tabellen von Korrespondenzen. By D. Voelker and G. Doetsch. (Lehrbücher und Monographien aus dem Gebiete der Exakten Wissenschaften, Mathematische Reihe, vol. 12.) Basel, Birkhäuser, 1950. 259 pp. 43 Swiss fr.

The motivation of this book is described by the authors in the preface in the following words. "The classical one-dimensional Laplace transformation now belongs to the common heritage of mathematicians and technologists, and since the publication of the monograph *Theorie und Anwendung der Laplace-Transformation*, books have been written on it in almost all cultured languages. By contrast, the two-dimensional (or double) Laplace transformation has been used only occasionally, in some memoirs. There is no systematic presentation of its theory and applications, or an elucidation of its distinctive features. Such a presentation is offered in this book which consists largely of unpublished material." (Reviewer's translation.)

The emphatic reference to the senior author's well known treatise should not mislead the reader into expecting the same sort of book here. On the one hand, the present book is based on Lebesgue's integration theory while Riemann integrals were used in Doetsch's book; and on the other hand, in contradistinction to the basic character of the earlier work, the orientation of the book under review is towards the applications, especially partial differential equations. This more practical orientation is shown by leaving aside basic questions of purely mathematical interest (for instance the highly complex behaviour of non-absolutely convergent double Laplace integrals). It is also shown by a mellower attitude towards the point of view of the engineer. On page 20 it is pointed out that even if one does not know Lebesgue's theory of integration, one can use with impunity theorems based on that theory as long as one follows the "rules of the game"; and readers of Doetsch's earlier work will be surprised to see that on page 51 the description of the formal process of solving boundary value problems by means of the double Laplace transformation is merely followed by the remark that a *careful* worker (reviewer's emphasis) will then find out under what conditions and in which sense the formal solution actually solves the boundary value problem. The most conspicuous, and the most welcome, sign of this practical orientation is the collection of tables of operations and transform pairs which occupies almost one half of the book.

The relation

$$(1) \qquad f(s) = \int_0^\infty e^{-st} F(t)\,dt$$

is indicated either as $f(s) = \mathfrak{L}\{F(t)\}$ or, more often, as $f(s) \bullet - \circ F(t)$ or $F(t) \circ - \bullet f(s)$. If necessary, the variables involved in the transformation are noted. Thus

$$f(u, y) \overset{u}{\bullet} - \overset{x}{\circ} F(x, y)$$

means

$$f(u, y) = \int_0^\infty e^{-ux} F(x, y)\,dx.$$

The double Laplace transform

$$(2) \qquad f(u, v) = \int_0^\infty \int_0^\infty e^{-ux-vy} F(x, y)\,dx\,dy$$

is written as

$$f(u, v) \underset{v}{\overset{u}{\bullet}} = \underset{y}{\overset{x}{\circ}} F(x, y)$$

or simply as $f(u, v) \bullet = \circ F(x, y)$. There are already several symbols for the Laplace transformation and the introduction of a new notation is not in itself desirable, yet in the present case it can be defended on the grounds of its being the only notation which lends itself conveniently to an indication of the number and names of the variables involved in the transformation.

The whole work is organized in two parts. Part I (137 pp.), on the theory and application to boundary value problems of the double Laplace transformation, is by both authors; Part II (107 pp.), containing the tables, is compiled by Voelker.

In Chapter 1 of Part I first the basic properties of the Laplace transform (1) are recalled (§1). Then the double Laplace transform is introduced (in §2) as the absolutely convergent Lebesgue double integral (2). For this, there are associated abscissae of (absolute) convergence, with properties similar to the associated radii of convergence of double power series. By Fubini's theorem, the double Laplace transform is reduced to a succession of two Laplace transforms (§3), and it is shown (in §4) that in case either of the two parameters u, v is fixed, the double Laplace transform is an analytic function of the other, regular in a half-plane which is determined by the associated abscissa of convergence. This leads to the expression of the double Laplace transform of $x^m y^n F(x, y)$ in terms of f. Then follow the so-called rules (§§5–7).

Almost the whole of the remainder of Part I is devoted to partial differential equations. The domain is always the quadrant $x \geqq 0$, $y \geqq 0$ which arises in many problems, and especially naturally with parabolic partial differential equations. From the work of §3 it follows that in principle all the work could be based on the (one-dimensional) Laplace transformation, and some of the problems presented here have previously been so handled; but the authors point out certain advantages of the double Laplace transformation. The general method is as follows. First the double Laplace transformation is applied formally to obtain a tentative solution expressed in terms of the coefficients of the partial differential equation and certain boundary (and initial) values, and then an independent investigation is carried out to show under which conditions the formal result represents an actual solution of the problem.

Chapters 2–5 are concerned with partial differential equations in two independent variables. In Chapter 2, after some general remarks (§8), the partial differential equation

$$F_x + pF_y + qF = \Phi(x, y), \qquad x \geqq 0,\ y \geqq 0,$$

is discussed (§9) where p and q are constants, and $p \neq 0$. With the boundary values

$$F(x, 0) = A(x) \circ\!-\!\bullet\, a(u), \qquad F(0, y) = B(y) \circ\!-\!\bullet\, b(v)$$

the double Laplace transform of the solution is

$$f(u, v) = \frac{\phi(u, v) + pa(u) + b(v)}{u + pv + q}. \tag{3}$$

From here on two cases must be distinguished. If $p>0$, the denominator in (3) is always positive for sufficiently large (positive) u and v, and the solution can be obtained, without any quantitative restrictions on the data, by means of the rules. On the other hand, if $p<0$ the condition

$$\phi(-pv - q, v) + pa(-pv - q) + b(v) = 0 \tag{4}$$

must be satisfied identically in v to make (3) regular for all sufficiently large u, v, and this condition imposes a quantitative restriction on the data. The situation is further elucidated by reference to the theory of characteristics. As an application of the double Laplace transformation the work is not very impressive, for the solution can be obtained by elementary methods as easily as, if not more easily than, by the double Laplace transformation. Yet the example is well chosen in that it aptly illustrates the general method, and also shows that in the application of the double Laplace transformation to boundary value problems one needs general operational rules, such as the interpretation of (4), even more than a large number of particular transform pairs.

In Chapter 3, §10 contains a similar discussion of the parabolic partial differential equation

$$F_{xx} - F_y + \Phi(x, y) = 0.$$

At first three boundary values,

$$F(0, y) \circ\!-\!\bullet\, b(v), \qquad F_x(0, y) \circ\!-\!\bullet\, b_1(v), \qquad F(x, 0) \circ\!-\!\bullet\, a(u)$$

enter, and

$$f(u, v) = \frac{-a(u) + ub(v) + b_1(v) - \phi(u, v)}{u^2 - v}.$$

The regularity of f demands

$$-a(v^{1/2}) + v^{1/2}b(v) + b_1(v) - \phi(v^{1/2}, v) = 0 \tag{5}$$

and thus sets up a compatibility condition. Various boundary value problems are discussed. Other partial differential equations in this chapter are: the wave equation (§11), $F_{xy}+pF+\Phi(x, y)=0$ (§12), the telegraphist's equation (§13), and Poisson's equation (§14). In all these cases the authors discuss those boundary value problems which can be handled most conveniently by the operational calculus, rather than those which arise most naturally in physical problems. As is well known, the discrepancy is greatest in the case of elliptic equations.

The general linear partial differential equation of order two (in two independent variables) with constant coefficients is discussed in Chapter 4. §15 correlates the double Laplace transform method with Green's formula, §16 deduces compatibility conditions (whose number depends on the coefficient matrix of the equation). In the remaining paragraphs (§§17–20) the discussion of hyperbolic and parabolic equations is taken up in more detail.

In Chapter 5 (§§21–24) the discussion of the system of partial differential equations with constant coefficients

$$\begin{aligned} F_x + p_1F_y + q_{11}F + q_{12}G &= \Phi(x, y), \\ G_x + p_2G_y + q_{21}F + q_{22}G &= \Psi(x, y) \end{aligned}$$

is taken up. The number of compatibility conditions is 0, 1, 2 according as $0 \leqq p_1 \leqq p_2$, $p_1 < 0 \leqq p_2$, $p_1 \leqq p_2 < 0$.

Chapter 6 is on partial differential equations in three independent variables. After a general introduction (§25), the partial differential equation

$$(6) \qquad F_{xx} - F_y + F_{zz} = 0, \qquad x \geqq 0,\ y \geqq 0,\ 0 \leqq z \leqq l,$$

is discussed (§26). The double Laplace transformation is applied with respect to x and y, and (6) is transformed into an ordinary differential equation which can be solved explicitly. The inverse transformation is considerably more difficult in this case. Three methods of it are illustrated in §§27–29.

Part I concludes with Chapter 7 (§§30, 31) on functional relations involving Bessel functions and Laguerre polynomials. A brief list of references is added.

Although Part I is unlikely to become as basic as the senior author's well known treatise on the Laplace integral, in its more restricted sphere it should prove useful.

In the preface the authors explain that one-dimensional operational calculus was for a long time handicapped by the absence of an adequate "dictionary." In order to forestall a similar situation with regard to the double Laplace transformation, they provide extensive tables in Part II, by Voelker.

Table A contains the "grammar" or operational rules. 1. Fundamental operations (64 entries). Beside the formulas for double Laplace transforms of partial derivatives (general formulas, and explicit relations up to order 4), there are other useful formulas of which a sample is

$$(x+y)^{-1}F(x,y) \circ\!=\!\bullet \int_u^\infty f(\lambda, v+\lambda-u)\,d\lambda.$$

2. Double Laplace transforms obtained from Laplace transforms (86 entries). Sample:

$$v^{-1}\phi(u+v^{1/2}) \bullet\!=\!\bullet\ \Phi(x)\operatorname{erfc}\frac{x}{2y^{1/2}} \quad\text{where}\quad \phi(u)\bullet\!-\!\bullet\ \Phi(x).$$

3. Difference quotients of Laplace transforms (63 entries). Samples:

$$\frac{\phi(u)-\phi(v)}{u-v}\bullet\!=\!\circ -\Phi(x+y) \text{ where } \phi(u)\bullet\!-\!\circ\ \Phi(x),$$

$$\frac{\phi(u)-\phi((v^2+\alpha^2)^{1/2})}{(v^2+\alpha^2)^{1/2}(u-(v^2+\alpha^2)^{1/2})}\bullet\!=\!\circ -\int_0^y J_0(\alpha(y^2-\xi^2)^{1/2})\,\Phi(x+\xi)\,d\xi.$$

4. More involved operations (134 entries). Sample:

$$\frac{1}{u}f\left(u+\frac{\alpha}{v},\ v\right)\bullet\!=\!\circ\int_0^x J_0(2(\alpha\xi(x-\xi))^{1/2})F(\xi,y)\,d\xi.$$

5. Difference quotients of double Laplace transforms (41 entries). Sample:

$$\frac{f(u,v)-f(v,v)}{u-v}\bullet\!=\!\circ -\int_0^y F(x+\eta,\ y-\eta)\,d\eta.$$

6. Laplace transforms derived from double Laplace transforms (10 entries). Sample:

$$f(u,0)\ \overset{u}{\bullet}\!-\!\overset{\bullet}{\circ}\int_0^\infty F(x,\eta)\,d\eta.$$

Table B contains the "dictionary" or transform pairs. These are arranged according to the $f(u, v)$. 1. Rational functions (177 entries). 2. Irrational algebraic functions and powers with arbitrary index (97 entries). 3. Logarithms (27 entries). 4. Exponential functions (28 entries). 5. Hyperbolic functions (13 entries). 6. The exponential integral function and similar functions (11 entries). 7. Confluent hypergeometric functions (13 entries). 8. Other functions (10 entries).

Table C contains a list of notations for special functions (110 entries).

These very detailed tables are likely to become an indispensable tool for the practical application of the double Laplace transformation.

The publishers must be congratulated on the excellent performance of a typographical job (the tables) which presents considerable technical and financial difficulties.

A. ERDÉLYI

Mathematische Grundlagenforschung. By Arnold Schmidt. (Enzyklopädie der Mathematischen Wissenschaften, Band I_1, Heft 1, Teil II.) Leipzig, 1950. 48 pp.

This article was originally written in 1939, but was revised in 1948–49, so as to include reference to more recent contributions. It consists of an exposition of "those parts of foundation studies which are either directly concerned with the construction of mathematics, or in which the application of mathematical methods has proved fruitful." The author confines himself almost entirely to the foundations of arithmetic, and does not attempt to deal with such topics as set theory, group theory, or geometry, nor with the problems proper to mathematical logic itself.

The article consists of four parts: (A) *Axiomatik und allgemeine Beweistheorie*; (B) *Kodifikation und Beweistheorie der Zahlenlehr*; (C) *Die logische Begrundung der Mathematik*; and (D) *Intuitionistische Mathematik*.

In Part (A), the author explains what is meant by the formalization of a mathematical system, and introduces some metamathematical terms. He then shows (following Gödel) that every system which contains arithmetic also contains its own syntax in arithmetical form, and proceeds to sketch a proof of Gödel's theorem (as strengthened by Rosser): that a system which contains its own syntax in arithmetical form cannot be both consistent and complete. Among other results of a negative character mentioned here are a second theorem of Gödel, that a consistent system containing its own syntax in arithmetical form cannot be shown to be consistent by any proof which can be formalized within the system, and the theorem of Tarski that, in a system which contains its own syntax in arithmetical form, one cannot define truth for the system itself.

Part (B) begins with a brief sketch of the theory of recursive functions. The result of Péter is cited, that one can keep on getting new recursive functions by increasing the number of variables in primitive recursions. Some of the various equivalent methods of defining general recursive functions are indicated: the original method of Herbrand and Gödel, Turing's notion of computability, and Church's notion of λ-definability. The author mentions Bernays' proof that every function whose values can be calculated in some consistent deductive calculus is general recursive, and describes the normal form of Kleene for general recursive functions. The relationship between general recursive functions and decision methods is pointed out, and reference is made to Church's proof that there is no decision method for the restricted predicate calculus; the more recent extensions of this result by Mostowski, Post, Julia Robinson, R. M. Robinson, and Tarski are not mentioned, however, as they have appeared since Schmidt's article.

The remainder of this part is taken up with an exposition of consistency proofs of various fragments of arithmetic, Gentzen's consistency proof of all of arithmetic using transfinite induction, and a sketch of some unpublished transfinite proofs (due to Lorenzen and Schütte) of the consistency of part of analysis.

Part (C), which is very short, is devoted to a résumé of the theory of types, and of the attempt of Russell and Whitehead (following Frege) to construct arithmetic from logic alone. The author criticizes this attempt on the ground that Russell and Whitehead were unable to prove certain theorems of arithmetic without using axioms which hardly seem to be part of "pure logic" (the axiom of infinity and the axiom of choice).

Part (D) is devoted to an exposition of the intuitionistic mathematics of Brouwer and his followers. The philosophical background of this movement is made clearer here than in the usual presentations.

J. C. C. McKinsey

Dirichlet's principle, conformal mapping, and minimal surfaces. By R. Courant. With an appendix by M. Schiffer. (Pure and Applied Mathematics, vol. 3.) New York, Interscience, 1950. 14+330 pp. $4.50.

At the beginning of its history, Dirichlet's principle was one of the occasions for the ascetic enterprise of revising and clarifying the foundations of the calculus of variations. In its latest stage it has proved a powerful tool for the solution of one of the most interesting, difficult, and colorful problems, a solution which could be carried to a generality which, a quarter of a century ago, even the most fanciful optimism would hardly have dared to dream of. No other mathematician is more competent to write a presentation of this subject than is the author of the present book: His first steps in research were still linked to the first part of the history of Dirichlet's principle, and he has been one of the most successful promoters of the development during its latest stage.

Naturally, the scope of the book is limited to Dirichlet's principle in two dimensions. The treatment mainly follows the author's own papers and those of his school, including even the most recent results. As the author stresses in the preface, his foremost aim has not been merely to "present the crystallized product of his thoughts," but he is anxious also to shed light on the development of those thoughts from their very sources. So he does not spare heuristic considerations, motivations of problems, reference to special questions; also many open problems are pointed out. It is needless to say how much this characteristic of the book is to the profit of every reader who is more interested in information than in austere beauty.

In the reviewer's opinion, the geometric aspects of Plateau's and of Douglas's problems come off somewhat badly; it is to be regretted, e.g., that no systematic report on the author's interesting soapfilm experiments has been included.

An appendix, written by M. Schiffer, gives a good picture of the main ideas underlying the recent development of the theory of conformal mapping. Conciseness is gained not by dropping the proofs but by selecting typical cases.

In Chap. I, Dirichlet's principle in its relations to boundary value problems of potential theory is motivated and stated; two proofs are developed in extenso. The simplest applications to conformal mapping are made.

Chap. II deals with an extension which is applicable to conformal mapping of domains of multiple (even of infinite) connectivity onto parallel slit domains. The variational problem and its solution are not restricted to genus zero; in the case of higher genus the result is a representation on a simple parallel slit domain, part of whose slits are ideally coordinated, thus becoming interior lines of a Riemannian domain.

Chap. III is concerned with Plateau's problem in its original form: To span a minimal surface (eventually with the additional condition that it be the surface of least area) into a closed Jordan curve in Euclidean m-space, $m \geq 2$. The solution is based on a variational problem: To minimize the Dirichlet integral $\iint_{u^2+v^2\leq 1}(\mathfrak{x}_u^2+\mathfrak{x}_v^2)\,du\,dv$ within a class of vector functions $\mathfrak{x}(u, v)$, each of which represents an admissible surface. The existence of a solution is proved, and afterwards it is identified as a minimal surface of least area. That it is a minimal surface may be proved, rather quickly, by the use of existence theorems on conformal mapping, or, avoiding such theorems, by consideration of the "first variation of Dirichlet's integral." For the proof of the least area property, theorems on conformal mapping are indispensable. The first variation also yields an independent existence proof, by the method of "safe descent" (an analogue to "steepest descent" for functions of a finite number of variables), whose result, however, cannot be identified as a surface of least area or even as a surface with a relative minimum of area.

Chap. IV is devoted to a generalization of Plateau's problem which was first, and with great success, considered by J. Douglas and is, therefore, here called Douglas's problem: Given any number of Jordan curves in Euclidian m-space; to find a minimal surface bounded by these curves with a prescribed Euler characteristic and with a prescribed character of orientability. Again the problem is replaced by a variational problem analogous to the former one; its main new feature is the variability also of the parameter domain within an appropriate class (e.g., all circular rings with outer radius 1 in the case of an orientable surface of genus 0 with two boundary curves). It is rather obvious that one cannot always expect a solution of the problem, since among the limits obtainable from the class of admissible surfaces there are degenerate cases which are not admissible: surfaces with a smaller characteristic or consisting of several parts. So one of the main concerns of this chapter is the discussion of conditions for the existence of a solution.

In the foregoing the solutions of Plateau's and of Douglas's problems are always given by a vector function, harmonic in a certain plane parameter domain, giving a conformal map of the latter on the surface in m-space. In the case $m=2$, one finds a conformal representation of the parameter domain onto the domain bounded by the given curves. So proofs are obtained for the existence of a conformal mapping of any multiply-connected domain of genus 0 onto an appropriate member of a certain class of normal domains, admitted as parameter domains. This method is developed in Chap. V. Of course, this time the proof of the "minimal surface character," i.e., of the property that the mapping is conformal, has to be based on the consideration of the first variation of Dirichlet's integral, avoiding all use of existence theorems on conformal mapping. Among the types of normal domains there are very general ones, as yet not accessible to other methods.

The last chapter is devoted to two different problems whose treatment is naturally not as well rounded by far as that of the former ones; many attractive questions are still open. In the first problem, part of the boundary (eventually, with suitable other restrictions, the whole boundary) is left free on some manifold of less than m dimensions. Only special types of this problem are considered. The second question is that of unstable minimal surfaces with prescribed boundary, which has been considered by Morse and Tompkins, and, with another approach, by Shiffman. The short exposition follows the latter author's lines.

Schiffer's appendix deals with such problems as: The relations between Green's function of a multiply-connected domain, Neumann's function, the harmonic measures of the boundary curves, certain special mapping functions, kernel functions, etc.; certain extremal problems; the variation of Green's function considered as a function of the domain, with applications.

Two bibliographies, one for the main part and one for the appendix, are inserted.

It may well be hoped that this lucid presentation of results obtained up to now will prove a sound basis and a stimulus for further research.

H. GRUNSKY

Mathematische Grundlagen der höheren Geodäsie und Kartographie. Vol. I. *Das Erdsphäroid und seine konformen Abbildungen.* By R. König and K. H. Weise. Berlin, Springer, 1951. 18+522 pp. 49.60 DM.

The mathematical theory of cartography and related problems in higher geodesy has provided frequent inspiration for basic studies in geometry, function theory, and conformal transformation. Most mathematicians have been inclined however to dismiss the subject as a fairly elementary exercise in analysis and have left its problems to the geographers and map makers. A small group has nevertheless maintained a continuing interest in this field which was set on a sound mathematical basis by the early work of Lambert, Lagrange, and Gauss.

The present work represents a significant contribution to the few outstanding works in this field. It stands along with the three major references: M. A. Tissot, *Mémoire sur la représentation des surfaces géographiques et des projections des cartes*, Paris, 1881; L. Driencourt and J. Laborde, *Traité des projections des cartes géographiques*, Paris, 1932; and W. Jordan and O. Eggert, *Handbuch der Vermessungskunde*, 8th ed., Stuttgart, 1939–1941. These three together with the present treatment provide a comprehensive background and guide for research and application.

The authors make their contribution primarily in establishing a more generalized mathematical basis for their subject. Rather than starting with the most familiar projections, they have started with certain general analytical formulations from which the several projections arise as specializations. The tone of the work is indicated by the fact that not a single world map appears while a great many diagrams illustrating the pattern of various conformal transformations are included.

This volume is the first of two with the second forthcoming. The first part as indicated by the subtitle is restricted to a closely coordinated, comprehensive study of the conformal mapping of the spheroid on the sphere and plane. Considerable emphasis is made of the use of complex variables to their fullest extent and on the characteristics of the projection in the large. The second volume is to be devoted largely to problems of higher geodesy such as the geodetic line and its conformal representation, and to the geodetic triangle.

The first and third chapters are introductory. The earth as an ideal ellipsoid of revolution is described and its geometrical characteristics are catalogued. A number of analytical and series relations are listed for future reference. The third chapter reviews compactly the basic principles of conformal transformation between two planes by use of various complex variable transformations.

In the second chapter there are described the so-called three fundamental surface variables $\boldsymbol{A}=\boldsymbol{M}$, $\boldsymbol{B}$, $\boldsymbol{\Gamma}$, representing the complex extension of the isometric latitude, the geographic latitude, the meridian arc length respectively. As complex variables they can be interpreted to give respectively the Mercator projection, the ellipsoidal transverse Mercator projection, and the Gauss-Kruger projection. The analytical and geometrical relations between these three variables are considered in chapters four and five. Various analytical correlations are developed explicitly by methods from the complex function theory. The several pairs of coordinate transformations are examined in relation to behavior of coordinate net distortion, rotation, and curvature in the manner of conventional conformal transformation and differential geometry theory.

Chapter six provides a description of the basic conformal projections on a plane, sphere, and spheroid derived from the three fundamental variables. Questions of scale variation and rotation, and series evaluation of projection coordinates are considered. The conformal projection on a sphere provides a convenient intermediate step for subsequent projection transformations. The projection from one spheroid to another is of concern in evaluating the effect of modification of the reference ellipsoid.

The application of exponential transformations to the three fundamental variables is the subject of chapter seven. The complex variable $H = -\epsilon^{-M}$ and its linear transformations determines the stereographic projections. Similarly $\Lambda = -\epsilon^{-aM}$ provides the conic projections.

In chapter eight is outlined the method of conformal displacement on the spheroid from one reference point to arbitrarily selected alternative reference points. Chapter nine surveys briefly some of the more geographical aspects of the several types of projections and mentions briefly other than conformal types. The work is concluded with a chapter providing an extensive set of auxiliary formulae and analytical relations.

This contribution represents an unusually exhaustive treatment of a fairly practical application of differential geometry and conformal transformation. The organization is good though overburdened with multiplicity of formulas. This is somewhat mitigated by a formula summary at the end of each chapter. The important characteristics of conformal maps are treated so as to provide an adequate basis for any further work. This volume and apparently the one to follow refers to the conformal projection almost exclusively. While this is of primary concern in higher geodesy it would seem appropriate in a basic treatise on cartography to give more than a cursory treatment to equal-area and the several geometrically defined projections such as the polyconic.

Newman A. Hall

Brief Mention

Problèmes de propagations guidées des ondes électromagnétiques. 2d ed. By L. de Broglie. Paris, Gauthier-Villars, 1951. 8+118 pp. 1100 fr.

This text is concerned with the classical phases of the theory of guided electromagnetic waves and as such does not discuss the developments made in the United States and Great Britain during the past ten years. Chapter I summarizes the basic facts regarding Maxwell's equations. They are written in Cartesian as well as orthogonal curvilinear coordinates. Complex representation of the field quantities and some of the useful potentials are discussed. Attention is turned to wave guides in Chapter II. For purposes of illustration four different cross sections are discussed: rectangular, circular, coaxial, and elliptic. The chapter closes with a brief but informative section on methods of excitation as well as transient effects. Chapter III is concerned with characteristic frequencies of electromagnetic cavities of several shapes: the parallelepiped, right circular cylinder, torus of rectangular and circular cross sections, and sphere.

In Chapter IV, account is taken of losses in guides. First the effect of a nonperfect conductor is discussed with reference to the skin-effect. This leads to the study of attenuation losses which are discussed for some special cases in an approximate and an exact fashion. Chapter V deals with a discussion of propagation in horns. Several geometries are considered, the right circular cone, biconical horn, the sectoral horn and the parabolic mirror. For these geometries, the solution of Maxwell's equations are given in detail. The closing chapter deals with Huyghens' principle and the diffraction of electromagnetic waves at the opening of a guide or a horn. Kottler's representation for the electromagnetic field is given and the chapter closes with some specific applications. A good fraction of the references pertain to post-war contributions of the French school.

ALBERT E. HEINS

Elements of the topology of plane sets of points. 2d ed. By M. H. A. Newman. Cambridge University Press, 1951. 8+209 pp.

This is a new edition of Newman's very useful and readable book on the topology of the plane published in 1939 (see the review in Bull. Amer. Math. Soc. vol. 45). The text has been revised extensively and there are a number of changes in the choice and arrangement of topics. For example, the sections on the boundary elements of domains and on the connectivities of certain closed sets have been replaced by a section on the orientation and intersection of plane curves. The first part of the book, dealing with the general topology of metric spaces, has been augmented by a section on complete spaces and by discussions of the properties of various vector spaces of infinite dimension. In the chapter on separation theorems the author has added a short proof of the implicit function theorem as an application of Brouwer's theorem on the invariance of regionality (established for arbitrary dimension despite the fact that the book deals primarily with the plane). The proof of the so-called stronger form of Cauchy's integral theorem has been agreeably shortened. On the other hand the proof of the Jordan separation theorem—essentially that of Alexander—seems a trifle longer. Here no doubt is a reflection of the author's purpose of presenting the proof of this famous theorem with such care as to make it accessible to mathematicians generally, not merely to those who will read the book as an admirable introduction to a systematic training in topology.

P. A. SMITH

Über Kurven und Flächen in Allgemeinen Räumen. By P. Finsler. A reprint of the 1918 dissertation with a detailed bibliography by H. Schubert. Basel, Birkhäuser, 1951. 12+160 pp. 14.80 Swiss fr.

This unaltered photographic reproduction of Finsler's thesis will be welcome to many mathematicians. For, as Ostrowski states in his preface, this is one of the few first works which have had a great and lasting influence, and, being a thesis, existed in only very few copies many of which are owned by Finsler's friends. But the book will be more than a classic in a geometer's library. The reviewer concludes from his own sad experience that there must be many instances where apparently new results were found to be already in Finsler's thesis after the latter finally became available through interlibrary loan. Moreover, there are many ideas which may facilitate later work. An example, where this could have, but did not, happen because the thesis was so inaccessible, is furnished by the expression (58), p. 59, for curvature, which can be applied without change to general metric spaces, and was later proved by Haantjes to be essentially equivalent to Menger's curvature. In addition to the thesis the present edition contains a comprehensive list of books and papers concerning Finsler spaces (until 1949) compiled by H. Schubert. It has a wide scope and comprises references to tensor calculus, ordinary and Riemannian geometry, the geometry of paths, the calculus of variations, spaces of infinite dimension, which are ordinarily not considered as contributions to Finsler spaces. This bibliography will prove very helpful, provided the user is aware of the fact that it is not complete, even as to papers which contain results on Finsler spaces in the narrow sense.

H. BUSEMANN

Integral transforms in mathematical physics. By C. J. Tranter. New York, Wiley, 1951. 10+118 pp. $1.50.

A variety of linear integral transformations that are useful in solving boundary value problems in differential equations is presented briefly here. In addition to the usual Laplace, Fourier, Mellin, and Hankel transformations, finite Fourier, Hankel, and Legendre transformations are considered. The inverses of those transforms are given, accompanied in most cases by formal derivations. Applications of those transformations to the solution of boundary value problems in partial differential equations of physics or engineering are illustrated. Short chapters are included on numerical evaluation of integrals and on a combination of relaxation and transformation methods. Each transformation reduces a characteristic differential form in the original function to an algebraic form in the transform. The algebraic form also involves characteristic boundary values of the original function. Those characteristic forms and boundary values determine the types of boundary value problems that can be solved directly by the particular transformation. They are indicated here, sometimes not explicitly, but they are not emphasized. In fact, at the beginning of the second chapter the author states that his examples will serve as a guide in selecting appropriate transforms for the solution of boundary value problems. Some of the examples presented display no advantage in using the transformation that is being illustrated, or in using any transformation. The book gives some good indications of the importance of various integral transformations in the solution of boundary value problems; it could have presented a still more convincing case for such transformations.

R. V. CHURCHILL

Atti del Terzo Congresso dell' Unione Matematica Italiana tenuto in Pisa nei giorni 23–26 Settembre 1948. Rome, Edizioni Cremonese della Casa Editrice Perrella, 1951. 4+265 pp. 1600 lire.

This volume contains lectures by Severi, Terracini, Martinelli, Cesari, Agostini, and Tolotti, and about 100 abstracts of papers presented to the Congress.

Opere. By Felice Casorati. Vol. 1. Rome, Edizioni Cremonese della Casa Editrice Perrella, 1951. 16+420 pp. 4000 lire.

This volume, prepared by the Unione Matematica Italiana, contains a bibliography, a life by E. Bertini, two lectures, three papers on geodesy, and fourteen papers on the theory of functions of a complex variable.

Moderna teoria delle funzioni di variabile reale. By G. Vitali and G. Sansone, 3d ed. Part 1, by Giuseppe Vitali. *Aggregati, analisi delle funzioni, integrazione, derivazione.* Bologna, Zanichelli, 1951. 8+222 pp. 3000 lire.

The first edition (1935) was reviewed in Bull. Amer. Math. Soc. vol. 42 (1936) p. 15. Little change has been made in the later editions.

NOTES

The editors of the Bulletin and the Proceedings wish to make grateful acknowledgment of the services rendered by the following persons who have refereed papers: R. P. Agnew, L. V. Ahlfors, A. A. Albert, Richard Arens, Reinhold Baer, E. G. Begle, J. H. Bell, R. E. Bellman, Stefan Bergman, Garrett Birkhoff, Salomon Bochner, T. A. Botts, A. T. Brauer, Richard Brauer, R. H. Breusch, R. H. Bruck, R. C. Buck, R. H. Cameron, Leonard Carlitz, R. E. Carr, Lamberto Cesari, Herman Chernoff, C. C. Chevalley, W. L. Chow, Sarvadaman Chowla, K. L. Chung, R. V. Churchill, Paul Civin, A. H. Clifford, A. B. Coble, I. S. Cohen, L. W. Cohen, A. H. Copeland, Philip Davis, M. M. Day, J. A. Dieudonné, R. P. Dilworth, J. L. Doob, F. G. Dressel, R. J. Duffin, Nelson Dunford, Ben Dushnik, Aryeh Dvoretzky, W. F. Eberlein, L. P. Eisenhart, G. C. Evans, Herbert Federer, William Feller, Werner Fenchel, D. T. Finkbeiner, M. K. Fort, R. H. Fox, Bernard Friedman, K. O. Friedrichs, W. H. Fuchs, W. B. Fulks, R. E. Fullerton, P. R. Garabedian, A. M. Gleason, Oscar Goldman, H. H. Goldstine, Michael Golomb, R. A. Good, Marshall Hall, P. R. Halmos, Harish-Chandra, Philip Hartman, A. E. Heins, M. R. Hestenes, J. D. Hill, Einar Hille, I. I. Hirschman, S. P. Hoffman, T. R. Hollcroft, Eberhard Hopf, A. S. Householder, C. C. Hsiung, Ralph Hull, G. A. Hunt, Witold Hurewicz, G. E. Hutchinson, D. H. Hyers, R. C. James, J. A. Jenkins, Roberta F. Johnson, Bjarni Jónsson, Mark Kac, Shizuo Kakutani, Irving Kaplansky, P. W. Ketchum, Fred Kiokemeister, V. L. Klee, E. R. Kolchin, R. E. Langer, D. H. Lehmer, W. W. Leutert, Norman Levinson, D. C. Lewis, Hans Lewy, B. W. Lindgren, L. H. Loomis, E. R. Love, Charles Loewner, R. E. Lowney, N. H. McCoy, E. J. McShane, C. C. MacDuffee, G. W. Mackey, G. R. MacLane, Saunders MacLane, Wilhelm Magnus, Szolem Mandelbrojt, H. B. Mann, Morris Marden, M. H. Martin, F. I. Mautner, W. H. Mills, E. E. Moise, Deane Montgomery, A. P. Morse, Tadasi Nakayama, John Nash, Zeev Nehari, J. D. Novak, Yoshitomo Okada, Oystein Ore, O. G. Owens, George Piranian, Harry Pollard, C. R. Putnam, J. F. Randolph, Anatol Rapoport, R. M. Redheffer, P. V. Reichelderfer, Eric Reissner, C. E. Rickert, Arthur Rosenthal, J. B. Rosser, Walter Rudin, R. D. Schafer, I. J. Schoenberg, W. T. Scott, I. E. Segal, Atle Selberg, H. N. Shapiro, M. F. Smiley, A. H. Smith, Frank Smithies, Andrew Sobczyk, M. H. Stone, Walter Strodt, D. J. Struik, J. L. Synge, Otto Szász, Gabor Szegö, Alfred Tarski, A. H. Taub, Olga Taussky, J. M. Thomas, R. M. Thrall, F. G. Tricomi, C. A. Truesdell, A. W. Tucker, D. F. Votaw, R. J. Walker, J. L. Walsh, Morgan Ward, S. E. Warshawski, C. P. Wells, J. G. Wendel, Herman Weyl, Hassler Whitney, G. T. Whyburn, D. V. Widder, Norbert Wiener, Albert Wilansky, L. R. Wilcox, R. L. Wilder, V. M. Wolontis, Jacob Wolfowitz, Bertram Yood, L. C. Young, J. W. T. Youngs, Antoni Zygmund.

The editors of the Transactions wish to acknowledge the services of the following persons, not members of the Editorial Committee, who have been consulted regarding papers submitted for publication: A. A. Albert, Warren Ambrose, Lipman Bers, R. P. Boas, H. W. Brinkmann, K. L. Chung, I. S. Cohen, J. A. Dieudonné, Orrin Frink, Hilda Geiringer, Franklin Haimo, Einar Hille, Vaclav Hlavatý, Nathan Jacobson, Bjarni Jónsson, L. M. Kelly, M. S. Knebelman, M. M. Loève, Kurt Mahler, Morris Marden, E. E. Moise, Tadasi Nakayama, Zeev Nehari, R. S. Phillips, C. E. Rickart, J. B. Rosser, Arthur Sard, A. C. Schaeffer, R. D. Schafer, I. J. Schoenberg, I. E. Segal, Abraham Seidenberg, Atle Selberg, Max Shiffman, M. F. Smiley, E. H. Spanier, Gabor Szegö, W. J. Thron, A. D. Wallace, J. L. Walsh, S. E. Warschawski, P. A. White, Aurel Wintner, Bertram Yood, H. J. Zassenhaus, Daniel Zelinsky, Antoni Zygmund.

The attention of the members of the Society is called to the Group for Academic Freedom. This organization is composed of faculty members of the University of California who refused to sign the special oath required by the Board of Regents of the University and who are taking legal action against the Board. The cost of such legal action is heavy. Those who wish to help defray these expenses may send checks, payable to the Group for Academic Freedom, to this Group at the Shattuck Hotel, Berkeley, California.

A new mathematical organization, the Tensor Society, has been formed at Hokkaido University. Membership is open to all interested persons. The Society's Journal (*Tensor, New Series*) will be published in two or three issues annually. Further information can be obtained from Professor H. V. Craig, Department of Applied Mathematics, University of Texas, Austin, Texas.

Professor Hans Freudenthal of Utrecht University has been elected to ordinary membership in the Netherlands Academy of Sciences.

Professors Norbert Wiener, Solomon Lefschetz, and Garrett Birkhoff have been awarded the degree of Doctor Honoris Causa by the National University of Mexico.

Professor H. B. Curry of Pennsylvania State College had the title Visiting Professor Honoraire conferred upon him by the University of Louvain, Belgium.

Professor W. R. Transue of Kenyon College is on leave of absence and is in Italy on a Fulbright grant.

Dr. K. A. Hirsch of King's College, University of Durham, has been appointed a reader at Queen Mary College, University of London.

Mr. D. M. Merriell of the University of Chicago has been appointed to an assistant professorship at Robert College, Istanbul, Turkey.

Professor B. R. Seth of Hindu College has been appointed to a professorship at the Indian Institute of Technology, Kharagpur, West Bengal.

Dr. D. M. Adelman of the University of Connecticut has accepted a position as mathematician with the United States Naval Ordnance Test Station, Pasadena, California.

Dr. E. L. Arnoff of the California Institute of Technology has accepted a position as Aeronautical Research Scientist with the National Advisory Committee for Aeronautics.

Dr. Miriam C. Ayer of Wellesley College has accepted a position with the Sandia Corporation, Albuquerque, New Mexico.

Assistant Professor Frederick Bagemihl of the University of Rochester has resigned.

Associate Professor D. H. Ballou of Middlebury College is on leave of absence and has been appointed to a visiting associate professorship at Yale University in the program of Internships in General Education sponsored by the Carnegie Foundation.

Mr. J. R. Belford of Johns Hopkins University has accepted a position as design engineer with the International Business Machines Corporation, Endicott, New York.

Mr. P. H. Berning of Johns Hopkins University has accepted a position as mathematician with the Bausch and Lomb Optical Company, Rochester, New York.

Professor B. A. Bernstein of the University of California has retired with the title emeritus.

Professor Harry Birchenough of New York State College for Teachers has retired with the title emeritus.

Dr. Jerome Blackman of Cornell University has accepted a position as associate research engineer with the York Company, York, Pennsylvania.

Mr. Joseph Blum of the University of Virginia has accepted a position as analyst with the Armed Forces Security Agency, Washington, D. C.

Dr. F. E. Bothwell of Northwestern University has accepted a position as consultant with the United States Naval Ordnance Test Station, China Lake, California.

Assistant Professor N. A. Brigham of the University of Maryland has accepted a position as mathematician with the Applied Physics Laboratory, Johns Hopkins University, Silver Spring, Maryland.

Mr. R. L. Broussard of Louisiana State University and Agricultural and Mechanical College has accepted a position as mathematician with the Douglas Aircraft Company, Inc., Santa Monica, California.

Dr. H. E. Campbell of the University of Pennsylvania has been appointed to an assistant professorship at Emory University.

Associate Professor Virginia Carlton of Northwestern State College has been appointed to a professorship at Wesleyan College.

Assistant Professor J. O. Chellevold of Lehigh University has been appointed to a professorship at Wartburg College.

Professor J. A. Clarkson of Tufts College has accepted the position of Executive Secretary of the Division of Mathematics, which was recently established by the National Research Council.

Professor Emeritus A. B. Coble of the University of Illinois has been appointed to a visiting professorship at the University of North Carolina.

Mr. Paolo Comba of the California Institute of Technology has been appointed to an assistant professorship at the University of Hawaii.

Professor N. A. Court of the University of Oklahoma has retired with the title emeritus.

President H. N. Davis of Stevens Institute of Technology has retired with the title president emeritus.

Dr. R. F. Dressler of New York University has accepted a position as mathematics consultant with the National Bureau of Standards, Washington, D. C.

Dr. F. E. Ehlers of Oregon State College has accepted a position as aerodynamicist with the Boeing Aircraft Company, Seattle, Washington.

Mr. C. C. Elgot of Columbia University has accepted a position as mathematician with the Naval Ordnance Laboratory, White Oak, Maryland.

Assistant Professor E. S. Elyash of the University of Pittsburgh is on leave of absence and has been appointed to an assistant professorship at Louisiana State University.

Mr. Trevor Evans of the University of Wisconsin has been appointed to an assistant professorship at Emory University.

Associate Professor Howard Eves of Oregon State College has been appointed to a professorship at Champlain College.

Dr. R. S. Finn of Syracuse University has been appointed a member of the Institute for Advanced Study.

Mr. Harold Forstat of Indiana University has accepted a position as physicist at Camp Detrick, Frederick, Maryland.

Dr. M. S. Friberg of the University of Minnesota has been appointed to an assistant professorship at the New Mexico Institute of Mining and Technology.

Mr. Seymour Ginsburg of the University of Michigan has been appointed to an assistant professorship at the University of Miami.

Mrs. Ralph Grimble has accepted a position as statistician with North American Aviation, Inc., Columbus, Ohio.

Dr. W. T. Guy, Jr. of California Institute of Technology has been appointed to an assistant professorship of applied mathematics and astronomy at the University of Texas.

Assistant Professor S. W. Hahn of Wittenberg College has been appointed to a professorship at Winthrop College.

Dr. I. N. Herstein of Ohio State University has been appointed a research associate with the Cowles Commission, University of Chicago.

Mr. F. A. Hiersch has accepted a position as assistant chief engineer at the American Air Filter Company, the Herman Nelson Division, Moline, Illinois.

Associate Professor G. P. Hochschild of the University of Illinois is on leave of absence and has been appointed to a visiting associate professorship at Yale University.

Dr. A. J. Hoffman of the Institute for Advanced Study has accepted a position as staff member in the division of applied mathemathematics, National Bureau of Standards, Washington, D. C.

Dr. J. X. Jamrich of Northwestern University has been appointed Dean of Students at Coe College.

Dr. Meyer Jerison of the University of Illinois has been appointed to an assistant professorship at Purdue University.

Associate Professor L. W. Johnson of Purdue University has been appointed to a professorship at Oklahoma Agricultural and Mechanical College.

Mr. T. L. Jordan has been appointed to an associate professorship at Wofford College.

Mr. L. D. Kovach of Purdue University has accepted a position as research engineer at Douglas Aircraft Company, El Segundo, California.

Mr. R. J. Lambert of Iowa State College of Agriculture and Mechanic Arts has accepted a position as mathematician with the Defense Department, Washington, D. C.

Dr. B. W. Lindgren of Massachusetts Institute of Technology has accepted a position as mathematician with the Minneapolis Honeywell Regulator Company, Minneapolis, Minnesota.

Professor J. J. Livers of Montana State College is on leave of absence and has been appointed an educational consultant at Boeing Aircraft Company, Seattle, Washington.

Mr. K. M. McMillin of the University of Minnesota has been appointed to an assistant professorship at the United States Air Force Institute of Technology.

Dr. A. V. Martin of San Jose State College has been appointed to a visiting assistant professorship at the University of California.

Mr. J. E. Martin of the University of Pittsburgh has accepted a position as senior project analytical engineer, with the Chance Vought Aircraft, Dallas, Texas.

Professor J. N. Michie has received the title professor emeritus from Texas Technological College.

Mr. Benjamin Evans Mitchell of the University of Wisconsin has been appointed to an assistant professorship at Alabama Polytechnic Institute.

Mr. Don Mittleman of Columbia University has accepted a position as mathematician with the National Bureau of Standards, Washington, D. C.

Mr. O. B. Moan of Purdue University has accepted a position as quality control analyst with Julius Hyman and Company, Denver, Colorado.

Professor P. B. Norman of Wagner College has accepted a position as project engineer with the Curtiss-Wright Corporation, Carlstadt, New Jersey.

Dr. Ilse L. Novak has been appointed a member of the Institute for Advanced Study.

Dr. Fritz Oberhettinger of the California Institute of Technology has been appointed to a professorship at American University.

Miss Virginia A. Ohlson of Purdue University has accepted a position as mathematician with the Douglas Aircraft Corporation, Santa Monica, California.

Mr. Philip Rabinowitz of the University of Pennsylvania has accepted a position as mathematician with the National Bureau of Standards, Washington, D. C.

Mr. B. A. Rattray of Princeton University has been appointed a sessional lecturer at McGill University.

Professor D. E. Richmond of Williams College is on leave of absence and has been appointed a lecturer at Massachusetts Institute of Technology.

Miss Jean E. Sammet of the University of Illinois has accepted a position as computer with the Metropolitan Life Insurance Company, New York, New York.

Mr. Judson Sanderson of the University of Michigan Research Center has been appointed to an assistant professorship at the United States Air Force Institute of Technology, Wright Field, Ohio.

Assistant Professor R. H. Scherer of Macalester College has been appointed to an assistant professorship at Coe College.

Mr. Stewart Schlesinger of Illinois Institute of Technology has accepted a position as research assistant at the Los Alamos Scientific Laboratory, Los Alamos, New Mexico.

Professor I. J. Schoenberg of the University of Pennsylvania is on leave of absence at the Institute for Numerical Analysis, Los Angeles, California.

Mr. R. C. Seber of Rockford College has been appointed to an assistant professorship at Wisconsin State College.

Professor Max Shiffman of Stanford University has been appointed a member of the Institute for Advanced Study.

Visiting Assistant Professor Edward Silverman of Kenyon College has accepted a position as staff member with the Sandia Corporation, Albuquerque, New Mexico.

Dean T. M. Simpson of the University of Florida has retired.

Mr. M. B. Sledd of Emory University has been appointed to an associate professorship at Georgia Institute of Technology.

Mr. P. C. Sweetland of Michigan State College has been appointed to an assistant professorship of research at Pennsylvania State College.

Mr. F. B. Thompson of the University of California has accepted a position as mathematician with Rand Corporation.

Dr. J. H. Wahab of the University of North Carolina has been appointed to an assistant professorship at Georgia Institute of Technology.

Dr. H. C. Wang of Louisiana State University has been appointed a member of the Institute for Advanced Study.

Mr. H. S. Wolf of the University of Illinois has accepted a position as assistant statistician with the Employees Life Insurance Company, Chicago, Illinois.

Associate Professor R. K. Zeigler of the University of California has accepted a position as statistician with the U. S. Atomic Energy Commission, Oak Ridge, Tennessee.

The following promotions are announced:

Joshua Barlaz, Rutgers University, to an associate professorship.

A. L. Blakers, Lehigh University, to an associate professorship.

G. M. Bloom, Miami University, to an assistant professorship.

R. C. Campbell, United States Naval Postgraduate School, Annapolis, Maryland, to an associate professorship.

A. B. Carson, United States Air Force Institute of Technology, to a professorship.

Nathaniel Coburn, University of Michigan, to an associate professorship.

Esther Comegys, University of Maine, to an associate professorship.

Byron Cosby, State University of Iowa, to an associate professorship.

J. B. Diaz, University of Maryland, to an associate research professorship.

James Dugundji, University of Southern California, to an associate professorship.

A. L. Foster, University of California, to a professorship.

K. A. Fowler, University of Michigan, to an assistant professorship.

D. B. Goodner, Florida State University, to an associate professorship.

H. G. Haefeli, Boston College, to a professorship.

A. W. Jacobson, Wayne University, to an associate professorship.

R. B. Johnson, Clemson Agricultural College, to an assistant professorship.

G. K. Kalisch, University of Minnesota, to an associate professorship.

R. G. Langebartel, University of Illinois, to an assistant professorship.

B. J. Lockhart, United States Naval Postgraduate School, Annapolis, Maryland, to an associate professorship.

D. C. Morrow, Wayne University, to a professorship.

E. A. Nordhaus, Michigan State College, to an associate professorship.

Rufus Oldenburger, Woodward Governor Company, Rockford, Illinois, to Chief Mathematician.

Daniel Orloff, Southern Illinois University, to an assistant professorship.

Helen G. Russell, Wellesley College, to a professorship.

C. L. Seebeck, Jr., University of Alabama, to a professorship.

C. P. Wells, Michigan State College, to a professorship.

The following appointments to instructorships are announced: Cornell University: Mr. D. E. van Tijn; University of Kansas: Mr. A. J. Zeichner; Lamar State College of Technology: Mr. E. B. West; University of Michigan: Mr. Ràoul Bott; Michigan State College: Mr. K. H. Carlson, Mr. J. G. Hocking; Ohio State University: Mr. R. F. Reeves; University of Saskatchewan: Dr. James Sanders; Seattle Pacific College: Miss Clarice MacDonald; Simmons College: Miss Edith Moss; Southern Illinois University: Mr. M. R. Kenner; University of Utah: Dr. C. E. Burgess; Wayne University: Mr. M. T. Wechsler.

Professor Emeritus Abraham Cohen of Johns Hopkins University died on April 25, 1951 at the age of eighty years. He had been a member of the Society for thirty-eight years.

Dr. C. L. Poor of Columbia University died on September 27, 1951 at the age of eighty-five years. He had been a member of the Society for thirty-eight years.

Professor J. B. Rosenbach of the Carnegie Institute of Technology died on November 6, 1951 at the age of fifty-four years. He had been a member of the Society for thirty-three years.

NEW PUBLICATIONS

Anós y Díaz de Arcaya, A. See Azorín Poh, F.

Azorín Poh, F., Ríos García, S., Cansado Maceda, E., and Anós y Días de Arcaya, A. *Conferencias de preparacion matemática y estadística.* Madrid, Instituto Nacional de Estadística, 1950. 11+186 pp.

Batchelor, G. K. *Turbulent diffusion* (mimeographed notes). (Institute for Fluid Dynamics and Applied Mathematics, Lecture Series no. 4.) College Park, University of Maryland, 1951. 11 pp. $.40.

Blaschke, W. *Geometría diferencial moderna.* (Conferencias de Matemática, vol. I.) Madrid, Instituto de Matemáticas "Jorge Juan," 1950. 1+43 pp.

Burgers, J. M. *Nonuniform propagation of shock waves* (mimeographed notes). (Institute for Fluid Dynamics and Applied Mathematics, Lecture Series no. 10.) College Park, University of Maryland, 1951. 67 pp. $1.40.

Buros, O. K. See *Statistical methodology reviews.*

Cansado Maceda, E. See Azorín Poh, F.

Collatz, L. *Numerische Behandlung von Differentialgleichungen.* (Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen mit besonderer Berücksichtigung der Anwendungsgebiete, vol. 60.) Berlin, Springer, 1951. 13+458 pp., 1 plate. 48 DM.

DeLury, D. B. *Values and integrals of the orthogonal polynomials up to* $n=26$. University of Toronto Press, 1950. 5+33 pp. $1.25.

Èl'sgol'c, L. È. *Obyknovennye differencial'nye uravneniya.* Moscow, Gosudarstvennoe Izdatel'stvo Tehniko-Teoreticeskoi Literatury, 1950. 220 pp.

Freudenthal, H. *Oktaven, Ausnahmegruppen und Oktavengeometrie.* Utrecht, Mathematisch Instituut der Rijksuniversiteit te Utrecht, 1951. 1+49 pp.

Gel'fand, I. M. *Lekcii po lineinoi algebre.* 2d ed. Moscow, Gosudarstvennoe Izdatel'stvo Tehniko-Teoreticeskoi Literatury, 1951. 252 pp.

Goldstein, S. *Statistical theory of turbulence* (mimeographed notes). (Institute for Fluid Dynamics and Applied Mathematics, Lecture Series no. 6.) College Park, University of Maryland, 1951. 45 pp. $.80.

de Groot, S. R. *Thermodynamics of irreversible processes.* (Selected Topics in Modern Physics, vol. 3.) New York, Interscience, 1951. 16+242 pp. $5.00.

Herland, L. *Dictionary of Mathematical Sciences.* Vol. 1. German-English. New York, Ungar, 1951. 2+235 pp. $3.25.

Hodge, P. G. See Prager, W.

Kampé de Fériet, J. *Atmospheric turbulence* (mimeographed notes). (Institute for Fluid Dynamics and Applied Mathematics, Lecture Series no. 7.) College Park, University of Maryland, 1951. 41 pp. $.65.

Kampé de Fériet, J. *Introduction to statistical theory of turbulence* (mimeographed notes). (Institute for Fluid Dynamics and Applied Mathematics, Lecture Series no. 8.) College Park, University of Maryland, 1951. 166 pp. $2.25.

Littlewood, D. E. *A university algebra.* Melbourne, Heinemann, 1950. 8+292 pp. 25 s.

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Nagell, T. *Introduction to number theory.* New York, Wiley, 1951. 309 pp. $5.00.

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Privalov, I. I. *Graničnye svoĭstva analitičeskih funkciĭ*. 2d ed. Moscow, Gosudarstvennoe Izdatel'stvo Tehniko-Teoretičeskoĭ Literatury, 1950. 336 pp., 1 plate.

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Sade, A. *Quasigroupes*. Marseille, 1950. 16 pp.

Sierpiński, W. *Algèbre des ensembles*. (Monografie Matematyczne, vol. 23.) Warsaw, Polskie Towarzystwo Matematyczne, 1951. 2+205 pp.

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Smart, E. H. *Advanced dynamics*. Vol. I. *Dynamics of a particle*. Vol. II. *Dynamics of a solid body*. London, Macmillan, 1951. Vol. I, 11+419 pp. Vol. II, 11+420 pp. $12.00.

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Synge, J. L. *Hamilton's methods in geometrical optics* (mimeographed notes). (Institute for Fluid Dynamics and Applied Mathematics, Lecture Series no. 9.) College Park, University of Maryland, 1951. 66 pp. $1.70.

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Journals Published by the
American Mathematical Society

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This journal is the official organ of the Society. It reports official acts of the Society and the details of its meetings, with abstracts of all research papers presented. It contains some of the officially invited addresses presented before the Society. It contains also announcements and notes on matters of current interest, and reviews of advanced mathematical books.

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PUBLICATION CHARGES

The American Mathematical Society has been led with considerable reluctance to establish a publication charge to help meet the high and rapidly increasing cost of publishing mathematical manuscripts.

The publication charge is applicable to contributed papers published in the Proceedings or in the Transactions which are credited to non-member institutions in the North American area. The charge for publication in these journals is, at present, $5.00 per page and is used to help defray the cost of printing and of providing the author with the minimum number of reprints.

The institution credited with the research is requested to pay the publication charge. The bill, in each case, is sent initially to the author of the paper with the request that he present it to the institution for payment.

The establishment of a publication charge was approved by the Council and the Board of Trustees of the Society in the fall of 1949 in connection with a revision of the dues and privileges of institutional members. The charge became effective in 1950.

Institutional members of the Society pay dues roughly proportional to the research publication of the members of their respective staffs. These dues are based on the same page rate, namely, $5.00, as the publication charge and are in lieu of the publication charge. Institutional memberships were established in 1934, and most of the larger mathematical research centers in the North American area are institutional members. Through membership, they enjoy additional privileges and avoid the inconvenience of paying a separate charge for each article published.

The cost of publishing research papers in mathematics is high and usually cannot be met from the sale of the publication. One of the major activities of the Society since its founding has been to provide appropriate media of publication for all contributed manuscripts meeting its editorial standards. During recent years, papers of high quality have been submitted and accepted at a faster rate than the Society could afford to publish them and a considerable backlog of papers meeting the editorial requirements of the Society now awaits publication. It is hoped that the publication charge will help meet this serious financial problem and make possible the acceptance and prompt publication of all original research papers submitted which meet the Society's editorial standards.

E. G. BEGLE,
Secretary

ERGODIC SETS

JOHN C. OXTOBY

Introduction. Ergodic sets were introduced by Kryloff and Bogoliouboff in 1937 in connection with their study of compact dynamical systems [16]. The purpose of this paper is to review some of the work that has since been done on the theory that centers around this notion, and to present a number of supplementary remarks, applications, and simplifications. For simplicity we shall confine attention to systems with a discrete time. Continuous flows present no difficulty, but the development of a corresponding theory for general transformation groups is still in an incomplete stage. An example due to Kolmogoroff (see [5]) shows that such an extension cannot be made without sacrificing either the invariance or the disjointness of ergodic sets.

In §§1 and 2 we give a brief, but self-sufficient, development of the basic theorems of Kryloff and Bogoliouboff. In §3 we collect some auxiliary results for later use. In §4 a simple characterization of transitive points is obtained. In §5 the distinctive properties of some special types of systems and subsystems are discussed, and in §6 these results are used to discover conditions under which the ergodic theorem holds uniformly. In §7 a generalization to noncompact systems is considered, and in §§8 and 9 some known representation theorems are obtained as an application of ergodic sets. In §10 there is given an example of a minimal set that is not strictly ergodic, similar to one constructed by Markoff.

1. **Some corollaries of the ergodic theorem.** We shall use the following notations: If $f(p)$ is a real-valued function on a set Ω, and if T is a $1:1$ transformation of Ω onto itself, then

$$M(f, p, k) = f_k(p) = \frac{1}{k}\sum_{i=1}^{i=k} f(T^i p) \quad (k = 1, 2, \cdots)$$

and

$$M(f, p) = f^*(p) = \lim_{k\to\infty} M(f, p, k)$$

in case this limit exists. The characteristic function of a set E is

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An address delivered before the Washington meeting of the Society, October 27, 1951, by invitation of the Committee to Select Hour Speakers for Eastern Sectional Meetings; received by the editors November 12, 1951.

denoted by χ_E. The *upper density* of a set E of integers is defined by $\delta^*(E) = \limsup_{k\to\infty} (1/k) \sum_{n=1}^{n=k} \chi_E(n)$.

Let T be a 1:1 measure preserving transformation of a measure space (Ω, μ) onto itself, and assume that $\mu(\Omega) = 1$. The ergodic theorem states:

(1.1) *If $f \in L_1$, then the sequence $\{f_k(p)\}$ converges a.e. to a limit function $f^* \in L_1$ such that $f^*(Tp) = f^*(p)$ and $\int f^* d\mu = \int f d\mu$.*

(1.2) *If $f \in L_1$ and f is non-negative, then for almost all p either* $f^*(p) > 0$ or $f(p) = 0$.

Proof. Let $\chi(p)$ be the characteristic function of the set on which $f^*(p)$ is either zero or undefined, and apply (1.1) to the function χf.

A second corollary is the mean ergodic theorem:

(1.3) *If $f \in L_q$, $1 \leq q < \infty$, then $f^* \in L_q$ and $\|f_k - f^*\|_q \to 0$ as $k \to \infty$.*

Proof. For any bounded function f the conclusion follows immediately from (1.1). For any other $f \in L_q$ it follows from the inequality $\|f_k\|_q \leq \|f\|_q$ $(k = 1, 2, \cdots)$ and the fact that bounded functions are dense in L_q.

A less familiar corollary is the following:

(1.4) *If $f \in L_2$, then the limit functions $f^*(p)$ and $\phi^k(p) = \lim_{n\to\infty} (1/n) \sum_{i=1}^{i=n} (f_k(T^i p) - f^*(p))^2$ $(k = 1, 2, \cdots)$ are defined a.e., and $\int \phi^k d\mu \to 0$ as $k \to \infty$.*

Proof. By (1.3), $f_k - f^* \in L_2$, hence $(f_k - f^*)^2 \in L_1$. Since $\phi^k(p) = M((f_k - f^*)^2, p)$, it follows from (1.1) that $\phi^k(p)$ is defined a.e. and that $\int \phi^k d\mu = \int (f_k - f^*)^2 d\mu$. The last conclusion then follows from (1.3).

A system (Ω, T, μ) (and the measure μ) is called *ergodic* if Ω cannot be split into two disjoint invariant measurable sets of positive measure. For such systems (1.1) implies that f^* is constant a.e., and (1.4) can be strengthened as follows:

(1.5) *If the system (Ω, T, μ) is ergodic, and if $f \in L_2$, then as $k \to \infty$ the sequence $\{\phi^k(p)\}$ defined in (1.4) converges to 0 uniformly a.e., and for any $\alpha > 0$ the sequence $\psi^k(p, \alpha) = \delta^*\{n : |f_k(T^n p) - \int f d\mu| > \alpha\}$ likewise converges to 0 uniformly a.e.; both sequences converge uniformly on the set Ω_1 of points where $f^*(p) = \int f d\mu$ and $\phi^k(p) = \int (f_k - f^*)^2 d\mu$ $(k = 1, 2, \cdots)$.*

Proof. For an ergodic system, $\mu(\Omega_1) = 1$, and the first conclusion follows from (1.3). The second conclusion then follows from the inequality

$$\alpha^2 \delta^*\{n : |f_k(T^n p) - f^*(p)| > \alpha\} \leq \phi^k(p),$$

which is implied by the definition of ϕ^k.

This result shows that for ergodic systems the convergence of f_k

to its constant limiting value f^* is *nearly uniform* on almost every orbit, in the sense that if $\epsilon > 0$ and k is sufficiently large, then for almost all p the mean square difference between $f_k(T^n p)$ and $f^*(p)$ as n ranges over the positive integers is less than ϵ, and for all n except a set of small upper density the absolute difference is less than ϵ.

2. **The theory of Kryloff and Bogoliouboff.** Assume now that Ω is a compact metric space and that T is a homeomorphism of Ω onto itself. Denote by $C(\Omega)$ the space of real-valued continuous functions f on Ω, with the norm $\|f\| = \max |f(p)|$. Any finite Borel measure μ in Ω determines a corresponding positive linear functional $L(f)$ on $C(\Omega)$, namely,

$$L(f) = \int f d\mu \qquad (f \in C(\Omega)).$$

The Riesz theorem [28; 12] asserts that, conversely, any positive linear functional on $C(\Omega)$ corresponds in this way to one and only one finite Borel measure μ in Ω. Invariant measures correspond to invariant functionals, and normalized measures to functionals for which $L(1) = 1$. It will be seen that the theory of Kryloff and Bogoliouboff is essentially a series of corollaries of this theorem and the ergodic theorem.

(2.1) *Any compact system* (Ω, T) *admits at least one normalized invariant Borel measure. If* K *is a compact subset of* Ω, *then either* $\mu(K) > 0$ *for some such measure* μ *or else* $M(\chi_K, p) = 0$ *for every* p *in* Ω.

Proof. Choose any point $p \in \Omega$ and determine (by a diagonal process) an increasing sequence of positive integers k_i such that the limit

$$L(f) = \lim_{i \to \infty} M(f, p, k_i)$$

exists for a countable dense set of functions f in $C(\Omega)$. Then the limit $L(f)$ exists for every $f \in C(\Omega)$ and defines an invariant positive linear functional such that $L(1) = 1$. Hence $L(f) = \int f d\mu$ for some normalized invariant Borel measure μ. If K is compact and $M(\chi_K, p)$ is not identically zero, the point p and the sequence $\{k_i\}$ can be so chosen that $\alpha = \limsup_{i \to \infty} M(\chi_K, p, k_i) > 0$. This implies that $L(f) \geqq \alpha$ whenever $f \geqq \chi_K$, and therefore $\mu(K) \geqq \alpha$.

A point p in Ω is called *quasi-regular* ($p \in Q$) if the mean value $M(f, p)$ is defined for every $f \in C(\Omega)$. A Borel subset E of Ω is said to have *invariant measure one* if $\mu(E) = 1$ for every normalized invariant Borel measure μ. It follows from (2.1) that any such set is nonempty.

(2.2) *Q is a Borel set of invariant measure one.*

PROOF. In order that p belong to Q it is sufficient that the limit $M(f, p)$ exist for every f belonging to a countable dense subset of $C(\Omega)$. For any $f\in C(\Omega)$ the set on which $M(f, p)$ is defined is an $F_{\sigma\delta}$ subset of Ω [17, p. 274], and by (1.1) this set has invariant measure one. Because Q is a countable intersection of such sets, it has the same two properties.

To each quasi-regular point p corresponds uniquely a normalized invariant Borel measure μ_p in Ω, such that

$$M(f, p) = \int f d\mu_p$$

for every $f\in C(\Omega)$. A quasi-regular point p is called a *point of density* ($p\in Q_D$) if $\mu_p(U)>0$ for every open set U containing p. A quasi-regular point p is called *transitive* ($p\in Q_T$) if μ_p is an ergodic measure. A point p is called *regular* if it belongs to the set $R=Q_D\cap Q_T$.

(2.3) *Q_D is a Borel set of invariant measure one.*

PROOF. Let $\{U_i\}$ be a countable base of open sets in Ω, and for each i let f^i be a continuous function on Ω such that $f^i(p)=0$ on $\Omega-U_i$ and $f^i(p)>0$ on U_i. Then the set E_i on which either $M(f^i, p)>0$ or $f^i(p)=0$ is an F_σ relative to the set on which $M(f^i, p)$ is defined, therefore an $F_{\sigma\delta}$ in Ω. By (1.2), E_i has invariant measure one. Since $Q_D=Q\cap\bigcap_{i=1}^{\infty} E_i$, the conclusion follows. This result can be looked upon as a refinement of Poincaré's recurrence theorem (cf. [14]).

(2.4) *Q_T is a Borel set of invariant measure one.*

PROOF. A quasi-regular point p is transitive if and only if for each f belonging to some countable dense subset D of $C(\Omega)$, $M(f, q)=\int f d\mu_p$ for all q except a set of μ_p-measure zero. Consequently, a point p is transitive if and only if $p\in Q$ and

$$\int_Q (f^*(q) - f^*(p))^2 d\mu_p(q) = 0$$

for every $f\in D$. For any $f\in C(\Omega)$, $f_k(q)$ converges boundedly on Q to $f^*(q)$. Hence for any $p\in Q$,

$$\begin{aligned}\int_Q (f^*(q) - f^*(p))^2 d\mu_p(q) &= \lim_k \int_Q (f_k(q) - f^*(p))^2 d\mu_p(q)\\ &= \lim_k \lim_n \frac{1}{n}\sum_{i=1}^{i=n} (f_k(T^i p) - f^*(p))^2.\end{aligned}$$

The subset of Q on which this iterated limit vanishes is an $F_{\sigma\delta}$ set, and (1.4) shows that it has invariant measure one.

(2.5) *R is a Borel set of invariant measure one.*

(2.6) *For any bounded Borel measurable function f on Ω, $\int f d\mu_p$ is a Borel measurable function of p on Q, and*

$$\int f d\mu = \int_Q \left(\int f d\mu_p \right) d\mu(p)$$

for every finite invariant Borel measure μ.

Proof. For any positive number M, the class of Borel measurable functions f on Ω, bounded by M, for which the assertions of (2.6) are true is closed with respect to pointwise convergence, and includes all continuous functions f with $\|f\| \leq M$.

The next two theorems follow from (2.5) and (2.6).

(2.7) *For any Borel set $E \subset \Omega$, $\mu_p(E)$ is Borel measurable on Q, and $\mu(E) = \int_R \mu_p(E) d\mu(p)$ for every finite invariant Borel measure μ.*

(2.8) *A Borel set $E \subset \Omega$ has invariant measure zero if and only if $\mu(E) = 0$ for every ergodic measure μ.*

For any ergodic measure μ, it follows from (1.1) that $\mu_p = \mu$ for all p except a set of μ-measure zero. The set of all such quasi-regular points is called the *quasi-ergodic set* [4] corresponding to μ, and the part of this set contained in R is called the *ergodic set* corresponding to μ. Each of these sets is invariant and $F_{\sigma\delta}$ (see (3.1)). Distinct ergodic measures correspond to disjoint quasi-ergodic sets. The ergodic sets (and the quasi-ergodic sets) stand in 1:1 correspondence with the totality of ergodic measures, and each ergodic measure vanishes outside the corresponding set. The ergodic and quasi-ergodic sets corresponding to a measure do not differ very essentially, since $Q - R$ has invariant measure zero, but both notions are convenient. It should be noted that the points of any ergodic set all have the same orbit closure, whereas the points of a quasi-ergodic set need not. The ergodic sets constitute a partition of R, the quasi-ergodic sets a partition of Q_T.

3. Some additional properties of the set Q.

(3.1) *There exists a Borel measurable function $f(p)$ from Q to the unit interval such that $f(p) = f(q)$ if and only if $\mu_p = \mu_q$.*

Proof. Let $\{f^i\}$ be a generating sequence in $C(\Omega)$, that is, a sequence that spans a linear manifold dense in $C(\Omega)$, and assume further that $\|f^i\| \leq 1$ for every i. In order that $\mu_p = \mu_q$ it is necessary and sufficient that $M(f^i, p) = M(f^i, q)$ for every i. For any $p \in Q$ the sequence $\{M(f^i, p)\}$ defines a point $\phi(p)$ of the Cartesian product of $\aleph_0$ intervals each equal to $[-1, 1]$, and ϕ is of first Baire class on

Q. (It follows that each quasi-ergodic set is G_δ relative to Q, and therefore $F_{\sigma\delta}$ in Ω.) This product space can, in turn, be mapped onto $[0, 1]$ by a 1:1 Borel measurable mapping [17, p. 358]. By composition one obtains a function $f(p)$ with the required properties.

For any $p \in Q$, the equation $M(f, p) = \int f d\mu_p$ holds for every $f \in C(\Omega)$, by definition. We now show that it holds also for some other functions.

(3.2) *If* $p \in Q$, *the equation* $M(f, p) = \int f d\mu_p$ *holds for any bounded Borel measurable function* f *whose upper and lower limit functions are equal almost everywhere* (μ_p). *In particular,* $\mu_p(E) = M(\chi_E, p)$ *for any Borel set* E *whose boundary has* μ_p*-measure zero.*

Proof. For any such f, and any $\epsilon > 0$, there exist continuous functions f_1 and f_2 such that $f_1(p) \leqq f(p) \leqq f_2(p)$ and $\int f_2 d\mu_p < \int f_1 d\mu_p + \epsilon$, and the conclusion follows immediately.

The class of sets E for which $\mu_p(E)$ can be identified with $M(\chi_E, p)$ is somewhat more extensive than is indicated by (3.2), although simple examples show that it does not necessarily include all compact sets. In this connection the following result is useful. We denote by $O^+(p)$ the positive semi-orbit of p, that is, the set $O^+(p) = \bigcup_0^\infty \{T^i p\}$.

(3.3) *For any compact set* $K_0 \subset \Omega$ *and any quasi-regular point* p *there exists a compact set* K *such that* $K_0 \subset K \subset K_0 \cup O^+(p)$ *and such that* $M(\chi_K, p) = \mu_p(K) = \mu_p(K_0)$.

Proof. If p is periodic, we may take $K = K_0$. Otherwise, let U_k denote the $1/k$-neighborhood of K_0, and for each positive integer k let f^k be a continuous function such that $f^k(q) = 1$ on K_0, $f^k(q) = 0$ on $\Omega - U_k$, and $0 \leqq f^k(q) \leqq 1$ on Ω. Let $\{N_k\}$ be an increasing sequence of positive integers such that $M(f^k, p, n) > \mu_p(K_0) - 1/k$ for all $n > N_k$. Put

$$K = K_0 \cup \bigcup_{k=1}^{\infty} \left(\bigcup_{i=1}^{N_{k+1}} \{T^i p\} \right) \cap U_k.$$

Since K has only a finite number of points outside U_k, it follows that K is compact and that $\limsup_n M(\chi_K, p, n) \leqq \mu_p(K_0)$. If $0 < i \leqq N_{k+1}$, then either $f^k(T^i p) = 0$ or $T^i p \in K$, hence $M(f^k, p, n) \leqq M(\chi_K, p, n)$ for $N_k < n \leqq N_{k+1}$, $\mu_p(K_0) - 1/k < M(\chi_K, p, n)$ for all $n > N_k$, and therefore $\mu_p(K_0) = M(\chi_K, p)$. The equation $\mu_p(K_0) = \mu_p(K)$ results from the assumption that p is nonperiodic.

(3.4) *If* f *is bounded and Borel measurable on* Ω, *if* p *is quasi-regular, and if for every* $\epsilon > 0$ *there exists a Borel set* E *containing* $O^+(p)$ *such that* $\mu_p(E) > 1 - \epsilon$ *and such that the contraction of* f *to* E *is continuous, then* $\int f d\mu_p = M(f, p)$.

PROOF. Let A be an upper bound of $|f(q)|$ on Ω. Choose a compact set $K_0 \subset E$ with $\mu_p(K_0) > 1-\epsilon$ (cf. [12, §43(3)]) and construct a compact set K with the properties stated in (3.3). Since $K \subset E$, the contraction of f to K is continuous and there exists a function $g \in C(\Omega)$ such that $g = f$ on K and $\|g\| \leqq A$. Since $|f-g| \leqq 2A\chi_{\Omega-K}$, we have

$$\begin{aligned} \left| \int f d\mu_p - f_k(p) \right| &\leqq \left| \int f d\mu_p - \int g d\mu_p \right| \\ &\quad + \left| \int g d\mu_p - g_k(p) \right| \\ &\quad + | g_k(p) - f_k(p) | \\ &\leqq 2A\mu_p(\Omega - K) + \left| \int g d\mu_p - g_k(p) \right| \\ &\quad + 2A \cdot M(\chi_{\Omega-K}, p, k). \end{aligned}$$

The last term is equal to $2A(1-M(\chi_K, p, k))$ and therefore tends to $2A(1-\mu_p(K))$ as $k \to \infty$. Hence the right member tends to a limit less than $4A\epsilon$, and the conclusion follows. For later reference (§7) we record the following corollary:

(3.5) *If f is bounded and Borel measurable on Ω, if p is quasi-regular, and if there exists an invariant Borel set E containing p such that $\mu_p(E) = 1$ and such that the contraction of f to E is continuous, then $\int f d\mu_p = M(f, p)$.*

4. Two characteristic properties of transitive points. For any $p \in Q$, for any $f \in C(\Omega)$, for any positive integer k, and for any $\alpha > 0$, we may define

$$\phi^k(p) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{i=n} (f_k(T^i p) - f^*(p))^2$$

and

$$\psi^k(p, \alpha) = \delta^*\{n: | f_k(T^n p) - f^*(p) | > \alpha\}.$$

(4.1) *A quasi-regular point p is transitive if and only if $\phi^k(p) \to 0$ as $k \to \infty$ for every $f \in C(\Omega)$. Moreover, $\phi^k(p)$ tends to zero uniformly on any quasi-ergodic set.*

PROOF. The first assertion was proved in the course of proving (2.4), the second follows from (1.5).

From the defintions of $\phi^k(p)$ and $\psi^k(p, \alpha)$ it is easy to verify the inequalities

$$\alpha^2\psi^k(p, \alpha) \leqq \phi^k(p) \leqq \alpha^2 + 2\|f\|\psi^k(p, \alpha).$$

Hence (4.1) implies the following theorem:

(4.2) *A quasi-regular point p is transitive if and only if $\psi^k(p, \alpha)\to 0$ as $k\to\infty$, for every $\alpha>0$ and for every $f\in C(\Omega)$. The convergence is uniform on any quasi-ergodic set.*

The obvious analogy of the conditions stated in (4.1) and (4.2) to mean convergence and convergence in measure suggests the following definitions:

DEFINITION. *A sequence $\{F_k(n)\}$ of functions of an integer n converges in the mean if the limit $F(n)=\lim_k F_k(n)$ exists for every n, and the mean square difference*

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{n=N} (F_k(n) - F(n))^2$$

exists for every k and tends to zero as $k\to\infty$.

DEFINITION. *A sequence $\{F_k(n)\}$ of functions of an integer n converges in density if the limit $F(n)=\lim_k F_k(n)$ exists for every n, and $\delta^*\{n: |F_k(n)-F(n)|>\alpha\}\to 0$ as $k\to\infty$, for every $\alpha>0$.*

(4.3) *In order that a point $p\in\Omega$ be transitive it is necessary that for every $f\in C(\Omega)$ the sequence $f_k(T^n p)$, regarded as a sequence of functions of n, should converge in density and in the mean, and it is sufficient that convergence in either sense should hold for a generating sequence of functions in $C(\Omega)$.*

We remark that in general convergence in the mean and convergence in density are independent conditions, but that they are equivalent for any bounded convergent sequence such that the mean square difference exists for every k.

5. **Uniquely ergodic and strictly ergodic systems.** We shall call a system (Ω, T) *uniquely ergodic* if it has a unique normalized invariant Borel measure, or equivalently, if it has only one ergodic set. Following Nemyckiĭ and Stepanov [23], a system (Ω, T) is called *strictly ergodic* if Ω consists of a single ergodic set. (Note that this usage differs slightly from that of Kryloff and Bogoliouboff [16].) Obviously any strictly ergodic system is uniquely ergodic, and any uniquely ergodic system is ergodic with respect to its unique measure.

(5.1) *If (Ω, T) is uniquely ergodic with measure μ, then for any $f\in C(\Omega)$ the sequence $\{f_k(p)\}$ converges uniformly on Ω to $\int f d\mu$, as $k\to\infty$.*

PROOF. Suppose the conclusion false. Then for some $g\in C(\Omega)$ and some number $\alpha\neq\int g d\mu$ there exists a sequence $\{p_i\}\subset\Omega$ and a sequence $\{k_i\}$ of positive integers such that $k_i\to\infty$ and $g_{k_i}(p_i)\to\alpha$. We may assume in addition that the $\lim_i f_{k_i}(p_i)$ exists for every f

belonging to some countable dense set in $C(\Omega)$. It follows that the limit $L(f)=\lim_i f_{k_i}(p_i)$ exists for every $f\in C(\Omega)$ and defines a positive linear functional on $C(\Omega)$. By the Riesz theorem, $L(f)=\int f d\nu$ for some normalized Borel measure ν, and ν is invariant since $L(fT)=L(f)$, hence $\nu=\mu$. In particular, taking $f=g$, we find that $L(g)=\int g d\mu$, and therefore $\alpha=\int g d\mu$, contrary to hypothesis.

It follows that in a uniquely ergodic system $Q_T=Q=\Omega$, $Q_D=R$, and Ω consists of a single quasi-ergodic set. In this case R is closed, since $\Omega-Q_D$ is the union of all open sets of measure zero, and because R consists of a single ergodic set it follows that R is a minimal set [16]. (A nonempty compact set is called ***minimal*** if it is equal to the orbit closure of each of its points.) There can be no other minimal set, since (2.1) and (2.5) imply that R intersects every minimal set, hence:

(5.2) *A uniquely ergodic system has only one minimal set and only one ergodic set and these sets coincide, both being equal to* R.

A number of partial converses of (5.1) can be stated. It is convenient to formulate a chain of equivalent conditions.

(5.3) *In a compact system* (Ω, T) *the following conditions are pairwise equivalent*:

(i) (Ω, T) *is uniquely ergodic*,

(ii) *for each* $f\in C(\Omega)$, $f_k(p)$ *converges uniformly on* Ω *to a constant*,

(iii) *for each* $f\in C(\Omega)$ *there is a sequence* $\{k_i\}$ *of positive integers such that* $f_{k_i}(p)$ *converges pointwise on* Ω *to a constant*,

(iv) Ω *contains only one minimal set, and for each* $f\in C(\Omega)$ *the sequence* $\{f_k(p)\}$ *converges uniformly on* Ω,

(v) R *is a minimal set.*

PROOF. That (i) implies (ii), (iv), and (v) follows from (5.1) and (5.2). The implications (ii)→(iii)→(i) are obvious. Any compact invariant set contains a minimal set, consequently if Ω contains only one minimal set then every invariant continuous function is constant, hence (iv) implies (ii). That (v) implies (i) is shown by the following theorem.

(5.4) *A minimal set is either an ergodic set or else it contains at least one non quasi-regular point.*

PROOF. Suppose that every point of a minimal set E is quasi-regular. For each $f\in C(\Omega)$ the limit function f^* is defined at every point of E and constant on each orbit. Since the orbit of each point of E is dense in E, it follows that f^* is either constant on E or else the contraction $f^*|E$ is everywhere discontinuous. The latter alternative cannot occur, since $f^*|E$ is the limit of a convergent sequence of continuous functions on a compact set. Consequently f^* is constant on E and each point of E has the same corresponding measure μ_p, equal to μ, say. E contains at least one regular point p_0, and the orbit of p_0 is dense in E, therefore μ is ergodic and positive for every open set that intersects E. Hence every point of E belongs to the ergodic set E_μ corresponding to μ, and since $\Omega - E$ is open and $\mu(\Omega - E) = 0$ it follows that $E = E_\mu$.

(5.5) *In a compact system* (Ω, T) *the following conditions on a point* $p \in \Omega$ *are pairwise equivalent*:

(i) *for each* $f \in C(\Omega)$ *the sequence* $f_k(T^n p)$ *converges uniformly in* n *as* $k \to \infty$,

(ii) *for each* $f \in C(\Omega)$ *the functions* $f_k(q)$ $(k = 1, 2, \cdots)$ *are equiuniformly continuous on* $\bar{O}(p)$ (= *the orbit closure of* p),

(iii) *the subsystem* $(\bar{O}(p), T)$ *is uniquely ergodic.*

PROOF. (i) implies that for all sufficiently large k, $f_k(q)$ is nearly constant on $\bar{O}(p)$, hence (i) implies (ii). (ii) implies that some subsequence $\{f_{k_i}(q)\}$ converges uniformly on $\bar{O}(p)$ to a constant [9, p. 304], hence (ii) implies (iii). That (iii) implies (i) follows from (5.1).

We shall call a point p *strictly transitive* if it satisfies condition (i) of (5.5). (4.2) shows that any such point is transitive, but the converse is not generally true (see §10 for a counterexample). Following Hedlund [15] a point p is called *almost periodic* if for every neighborhood U of p the set $\{n : T^n p \in U\}$ is relatively dense. According to a theorem of Gottschalk [8], $\bar{O}(p)$ is a minimal set if and only if p is almost periodic. From this result, together with (5.2) and (5.5), it is easy to deduce the following theorem:

(5.6) *In a compact system* (Ω, T) *the following conditions on a point* $p \in \Omega$ *are pairwise equivalent*:

(i) $\bar{O}(p)$ *is an ergodic set,*

(ii) *the subsystem* $(\bar{O}(p), T)$ *is strictly ergodic,*

(iii) p *is almost periodic and strictly transitive,*

(iv) p *is regular and strictly transitive,*

(v) p *belongs to a closed ergodic set.*

A system (Ω, T) is called *uniformly-L-stable* ("stable in the sense of Liapounov" [6]) if the powers of T are equiuniformly continuous on Ω, that is, if for every $\epsilon > 0$ there is a $\delta > 0$ such that $d(p, q) < \delta$ implies $d(T^n p, T^n q) < \epsilon$ for every integer n.

(5.7) *If a compact system is uniformly-L-stable, then every point is regular and strictly transitive.*

PROOF. From uniform-L-stability it follows that every point p satisfies (5.5) (ii), and that if $q \in \bar{O}(p)$, then $p \in \bar{O}(q)$. Consequently every point is strictly transitive and its orbit closure is a minimal set.

The conclusion then follows from (5.6), and has the following corollary:

(5.8) *If a compact system* (Ω, T) *is uniformly-L-stable and has at least one dense orbit, then the system is strictly ergodic.*

It is well known [13] that any system that fulfills the hypothesis of (5.8) is homeomorphic to a system (G, T), where G is a (monothetic) compact topological group and T is a translation of G by an element whose powers are dense in G. Hence a proof of (5.8) can also be based on the uniqueness of Haar measure.

6. Uniform ergodic limits. An important difference between the Birkhoff ergodic theorem (1.1) and the mean ergodic theorem (1.3) is that the latter implies that for any $f \in L_2$ the averages

$$f_{n,m}(p) = \frac{1}{n - m} \sum_{i=m+1}^{i=n} f(T^i p) \qquad (n > m)$$

converge (in L_2) as $n - m \to \infty$, in other words, the sequence $f_k(T^n p)$ converges (in L_2) *uniformly in* n as $k \to \infty$. On the other hand, (1.1) implies only that this sequence converges (a.e.) as $k \to \infty$ *for any fixed value of* n, and simple examples show that in general there is no uniformity in n, even for strictly ergodic systems. (For example, let T be an irrational rotation of the circle, let U_n be the union of the first 2^{n+1} images of some interval of length 4^{-n-1}, and let $f(p)$ be the characteristic function of the set $U = \bigcup_1^\infty U_n$. For every point p and positive integer k there is a positive integer n such that $f_k(T^n p) = 1$, but for almost all p the $\lim_k f_k(p) = m(U) \leqq 1/2$. Consequently the convergence of $f_k(T^n p)$ is nonuniform in n for almost all p.) Nevertheless, it is reasonable to ask for what compact systems it can be asserted that for any continuous function f the averages $f_{n,m}(p)$ converge a.e. as $n - m \to \infty$. An answer to this question can be formulated as follows:

(6.1) *In any compact system* (Ω, T) *with normalized invariant Borel measure* μ *the following conditions are pairwise equivalent*:

(i) *for each* $f \in C(\Omega)$, $\{f_{n,m}(p)\}$ *converges a.e. as* $n - m \to \infty$,

(ii) *almost all points are strictly transitive,*

(iii) *almost all points belong to closed ergodic sets.*

Proof. (i) implies that there is a set E with $\mu(E) = 1$ such that for each $p \in E$ and for each f belonging to a countable dense subset of $C(\Omega)$ the sequence $f_k(T^n p)$ converges uniformly in n as $k \to \infty$. It follows that the same conclusion holds for every $f \in C(\Omega)$, hence every point of E is strictly transitive, and (i) implies (ii). (ii) implies (iii) by (2.5) and (5.6). The reverse implications are obvious.

(6.2) *In any compact system* (Ω, T) *the following conditions are pairwise equivalent*:

(i) *for any* $f \in C(\Omega)$ *and for any finite invariant Borel measure* μ, $\{f_{n,m}(p)\}$ *converges a.e. as* $n-m\to\infty$,

(ii) *every ergodic set is closed*,

(iii) *the family of minimal sets and the family of ergodic sets coincide.*

The following definition was introduced by Fomin [6].

Definition. *A compact system* (Ω, T) *is mean-L-stable* ("*stable in the mean in the sense of Liapounov*") *if for every* $\epsilon>0$ *there is a* $\delta>0$ *such that* $d(p, q)<\delta$ *implies* $d(T^n p, T^n q)<\epsilon$ *for all* n *except a set of upper density less than* ϵ.

(6.3) *If a compact system is mean-L-stable, then for each* $f\in C(\Omega)$ *the sequence* $\{f_k(p)\}$ *is equiuniformly continuous and uniformly convergent on* Ω, *every ergodic set is closed, every quasi-ergodic set is closed, and every point is strictly transitive.*

Proof. The equiuniform continuity of the functions $f_k(p)$ $(k=1, 2, \cdots)$ is an easy consequence of mean-L-stability. From (5.5)(ii) it follows that every point is strictly transitive, and the uniform convergence of $\{f_k(p)\}$ then follows from equicontinuity and pointwise convergence. Consequently $f^*\in C(\Omega)$ whenever $f\in C(\Omega)$, and therefore every quasi-ergodic set is closed. That every ergodic set is closed follows from (5.6)(iv).

The next two theorems are corollaries of (6.3).

(6.4) *If a compact mean-L-stable system has at least one dense orbit, then it is uniquely ergodic.*

(6.5) *If a minimal set is mean-L-stable, then it is strictly ergodic* (cf. [6]).

The last three results should be compared with (5.7) and (5.8). Trivial examples (e.g. a system consisting of two invariant points p and q and an orbit $\{p_n\}$ such that $p_n\to p$ and $p_{-n}\to q$) show that mean-L-stability is not a necessary condition for every ergodic set to be closed, and (same example with $p=q$) that the hypothesis of (6.4) does not imply the conclusion of (6.5). Interesting examples of minimal sets that are mean-L-stable but not uniformly-L-stable are provided by the Sturmian minimal sets studied by Hedlund [15].

7. Noncompact systems. If T is a homeomorphism of a complete separable metric space Ω, the system (Ω, T) need not admit a finite invariant Borel measure. However, Fomin [4] has shown that in case such a measure exists, the main theorems of Kryloff and Bogoliouboff remain true provided certain definitions are appropriately generalized. (For a generalization in another direction, see

[29].) In this section we shall show how the study of such systems can be reduced to that of compact systems. At the same time a slightly greater degree of generality is attained, since it turns out to be unnecessary to postulate completeness.

By a *Borel system* we shall mean a system (Ω, T) where Ω is a Borel subset of some complete separable metric space, and T is a homeomorphism of Ω onto itself. Two such systems (Ω, T) and (Ω', T') are called *homeomorphic* if there exists a homeomorphism of Ω onto Ω' under which T corresponds to T'. (Ω', T) is called a *subsystem* of (Ω, T) in case Ω' is a Borel subset of Ω invariant under T.

(7.1) *Any Borel system* (Ω, T) *is homeomorphic to a subsystem of the compact system* (Ω^*, S) *where* Ω^* *is the Cartesian product of* $\aleph_0$ *Hilbert cubes, and* S *is the shift transformation on* Ω^*, *i.e.,* $S\{\omega_n\} = \{\omega_{n+1}\}$.

Proof. By Urysohn's theorem, we may assume that Ω itself is a subset of the Hilbert cube. Then the correspondence $p \rightarrow \{T^n p\}$ defines a 1:1 bicontinuous mapping of Ω onto a Borel subset Ω' of Ω^* under which T corresponds to S.

It is clear that the invariant Borel measures of the system (Ω, T) correspond to those of the system (Ω^*, S) for which $\mu(\Omega^* - \Omega') = 0$. Consequently, a necessary and sufficient condition that (Ω, T) admit no finite invariant Borel measure is that the set Ω' have invariant measure zero. This will be the case if and only if every compact subset of Ω' has invariant measure zero, and for this it is necessary and sufficient that $M(\chi_{K'}, p') = 0$ for every compact set $K' \subset \Omega'$ and for every $p' \in \Omega'$, by (2.1) and (1.2). Hence we obtain the following generalization of (2.1) (cf. [26]).

(7.2) *A necessary and sufficient condition that a Borel system* (Ω, T) *admit no finite invariant Borel measure is that* $M(\chi_K, p) = 0$ *for every compact set* $K \subset \Omega$ *and for every point* $p \in \Omega$.

Let us now denote by $C(\Omega)$ the space of all *bounded* real-valued continuous functions on Ω. The definition of a quasi-regular point requires modification, since it is no longer true that $M(f, p)$ can always be represented as an integral. We follow Fomin [4] in adopting the following definition:

Definition. *A point* $p \in \Omega$ *is quasi-regular with respect to the Borel system* (Ω, T) *if*

(1) *the mean value* $M(f, p)$ *exists for each* $f \in C(\Omega)$, *and*

(2) *for every* $\epsilon > 0$ *there is a compact set* $K \subset \Omega$ *such that* $M(\chi_K, p) > 1 - \epsilon$.

It is clear that this definition is equivalent to that of Kryloff and Bogoliouboff (cf. §2) in case the system is compact. That it is an appropriate generalization is shown by the following theorem.

(7.3) *If* (Ω', T) *is a subsystem of a compact system* (Ω^*, T), *then a point* $p \in \Omega'$ *is quasi-regular with respect to* (Ω', T) *if and only if* p *is a quasi-regular point of* (Ω^*, T) *and* $\mu_p(\Omega') = 1$.

PROOF. The necessity of both conditions is clear; that they are sufficient is shown by (3.3) and (3.5).

It follows that with any quasi-regular point of the Borel system (Ω, T) there is associated uniquely a normalized invariant Borel measure μ_p such that $\int f d\mu_p = M(f, p)$ for every $f \in C(\Omega)$, and that the quasi-regular points are the only points for which $M(f, p)$ is defined on $C(\Omega)$ and representable in this way. We remark that (2) cannot be weakened to the requirement that $M(\chi_K, p) > 0$ for some compact set K, since it is possible for a quasi-regular point p in Ω^* to be such that $0 < \mu_p(\Omega') < 1$.

The set Q' of quasi-regular points of (Ω^*, T) for which $\mu_p(\Omega') = 1$ is a Borel subset of Ω^*, by (2.7), and therefore the set Q of quasi-regular points of (Ω, T) is a Borel subset of Ω. If E is any ergodic set of (Ω^*, T), with ergodic measure μ, then either $\mu(E \cap \Omega') = 1$ or $\mu(E \cap \Omega') = 0$; hence $\mu(\Omega' - Q') = 0$. It follows from (2.8) that $\Omega' - Q'$ has invariant measure zero, and therefore Q has invariant measure one relative to (Ω, T). Similar reasoning shows that if points of density, transitive points, and regular points are defined as in §2, then not only (2.2) but also (2.3), (2.4), and (2.5) hold for any Borel system. The remaining theorems of §2 then follow at once from the corresponding theorems for compact systems. It should be emphasized, however, that all of these theorems are vacuous unless the Borel system admits at least one finite invariant measure.

8. Decomposition into a direct sum of ergodic systems. One of the most fruitful conceptions in ergodic theory has been the idea that any system can be regarded as a combination, in some sense, of ergodic systems. The first result of this sort was obtained by von Neumann [24]. For compact systems, the theory of Kryloff and Bogoliouboff gave a still more detailed analysis. Subsequently, Halmos, Ambrose, and Kakutani [10; 2] obtained von Neumann's theorem in a purely measure-theoretic form, with no explicit topological assumptions. It was for this purpose that Halmos introduced the notion of a direct sum of measure spaces. More recently, the work of Dieudonné [3; 11], Nikodým [25], and Maharam [18] on Boolean σ-algebras has led to a more precise version of the decomposition theorem for measure spaces, in which some of the direct summands are combined into a direct product. In this section and the next it will be shown how some of these decomposition theorems for measure spaces can also be obtained from the ergodic set decomposition.

Let (Ω, T) be a compact system, and let R be the set of regular points. Let f be a Borel measurable mapping of R into the unit interval, such that $f(p)=f(q)$ if and only if $\mu_p=\mu_q$ (cf. (3.1)), and let X denote the range of f. Let g be a 1:1 Borel measurable mapping of R onto the unit interval [17, p. 358]. Then the equations $x=f(p)$, $y=g(p)$ define a 1:1 Borel measurable mapping ϕ of R onto a Borel subset Y of the unit square, such that for each $x \in X$ the ergodic set $f^{-1}(x)$ is mapped onto the x-section Y_x of Y. Let ν_x denote the Borel measure in Y_x corresponding to the ergodic measure in $f^{-1}(x)$. Let $\mathcal{X}$ be the Borel field consisting of all sets $A \subset X$ such that $f^{-1}(A)$ is a Borel subset of R. Corresponding to any invariant Borel measure μ in Ω define $\nu(A)=\mu f^{-1}(A)$, for every $A \in \mathcal{X}$. Then for any Borel set $E \subset Y$ we have, by (2.7),

$$\begin{aligned}\mu(\phi^{-1}E) &= \int_R \mu_p(\phi^{-1}E)d\mu(p) \\ &= \int_R \nu_{f(p)}(E\cdot Y_{f(p)})d\mu(p) = \int_X \nu_x(E\cdot Y_x)d\nu(x).\end{aligned}$$

Thus ϕ represents the measure space (R, μ) as a direct sum of the measure spaces (Y_x, ν_x) over $(X, \mathcal{X}, \nu)$. T corresponds to a Borel measurable transformation $T'=\phi T\phi^{-1}$ of Y that leaves invariant each x-section Y_x, and on Y_x is ergodic with respect to ν_x. Thus ϕ represents (R, T, μ) as a direct sum of ergodic systems. It should be noted that *the same decomposition serves for every finite invariant Borel measure* μ, the only variable term being the measure function ν. Moreover, every finite measure ν on $\mathcal{X}$ corresponds in this way to one and only one invariant measure μ. For if E is any Borel subset of R, then $\mu_p(E)$ is measurable relative to $f^{-1}(\mathcal{X})$, hence $\nu_x(\phi(E)\cdot Y_x)$ is measurable $\mathcal{X}$, and for any finite measure ν on $\mathcal{X}$ the equation

$$\mu(E) = \int_X \nu_x(\phi(E)\cdot Y_x)d\nu(x)$$

defines a corresponding invariant Borel measure in R. A 1:1 correspondence is thus established between the invariant Borel measures μ of the system (Ω, T) and the finite measure functions ν on the measurable space $(X, \mathcal{X})$.

This correspondence between μ and ν determines at the same time a correspondence between their completions μ^* and ν^*, and ϕ represents (R, μ^*) as a direct sum of the measure spaces (Y_x, ν_x^*) over

(X, ν^*), in the sense that if E is measurable μ^* then $\phi(E)$ is measurable ν_x^* for all x except a set of ν^*-measure zero, and $\nu_x^*(\phi(E)\cdot Y_x)$ is a ν^*-measurable function of x whose integral is equal to $\mu^*(E)$.

Now let T be any 1:1 measure-preserving transformation on a *normal* measure space (Ω, m^*). (A normal measure space is one that is isomorphic to the unit interval, cf. [13]. Two measure spaces are said to be isomorphic if there exists a 1:1 measure preserving transformation from almost all of one to almost all of the other. Two systems (Ω, T, μ) and (Ω', T', μ') are isomorphic if T corresponds to T' under some isomorphism.) We are seeking an isomorphic representation of (Ω, T, m^*), hence we may assume from the outset that Ω is the unit interval and that m^* is Lebesgue measure. Moreover, up to isomorphism, T can be replaced by an equivalent 1:1 Borel measurable transformation. For each $p \in \Omega$, the sequence of numbers $\{T^n p\}$ may be regarded as a point $\phi_1(p)$ of the $\aleph_0$-dimensional cube Ω^∞, and then ϕ_1 is a 1:1 Borel measurable mapping of Ω onto a Borel subset of Ω^∞ (cf. [17, pp. 290, 397]). Under ϕ_1, the transformation T corresponds to the shift transformation S, which is a homeomorphism on Ω^∞. If μ^* denotes the completed Borel measure in Ω^∞ that corresponds to m^* under ϕ_1, then the system (Ω, T, m^*) is isomorphic to $(\Omega^\infty, S, \mu^*)$. We have already shown how the latter system can be represented as a direct sum of ergodic systems. In this way the following theorem of Ambrose, Halmos, and Kakutani [2] is obtained as a consequence of the ergodic set decomposition.

(8.1) *Any normal system* (Ω, T, m^*) *is isomorphic to a direct sum of ergodic systems.*

We remark that if T is a 1:1 Borel measurable transformation on a Borel subset Ω of a complete separable metric space, the system (Ω, T) can be mapped by a 1:1 Borel measurable mapping onto a subsystem (Ω', T') of a compact system by the device used in §7. The representation of a compact system described above yields a corresponding representation of (Ω, T) as a direct sum of ergodic systems, *simultaneously for every finite invariant Borel measure.* In other words, the possibility of a universal decomposition into ergodic parts is not limited to compact systems, but extends to any system that is "Borel isomorphic" to a subsystem of a compact system.

It should also be remarked that the possibility of representing a normal system by an isomorphic continuous transformation is not limited to systems with a discrete time; Ambrose and Kakutani [1] have shown that any measurable flow in a normal measure space is isomorphic to a continuous flow.

9. **Direct product representation.** The following theorem is the principal special case of a theorem of Maharam [18, Theorem 6]. The proof given here was suggested by a somewhat similar proof obtained by Kakutani.

(9.1) *If* (Ω, T, m^*) *is a normal system such that* (1) *no subsystem is ergodic relative to* m^*, *and* (2) *almost all points are nonperiodic, then* (Ω, T, m^*) *is isomorphic to a system* (Ω', T', m_2^*), *where* Ω' *is the unit square with plane measure* m_2^*, *and* T' *is a transformation that leaves every x-section of* Ω' *invariant and on it is ergodic with respect to linear measure.*

It should be remarked that both conditions (1) and (2) are obviously necessary, but that neither of them is seriously restrictive. Without them, it is not hard to see that the system can be decomposed into a direct sum of systems all but one of which are comparatively trivial, namely, either periodic or ergodic on an interval. The remaining summand, if present, is covered by the above theorem.

In §8 it was shown that any normal system is isomorphic to a compact system $(\Omega^\infty, S, \mu^*)$. Under the added hypotheses (1) and (2) each ergodic set of the latter system has μ^*-measure zero, and the set of periodic points has μ^*-measure zero. Hence (9.1) is a consequence of the following slightly more precise theorem.

(9.2) *If* (Ω, T, μ) *is a compact system with normalized invariant Borel measure* μ, *and if each ergodic set and the set* P *of periodic points has* μ-*measure zero, then there exists a* 1:1 *Borel measurable mapping of almost all of* R *onto a Borel subset* W *of the unit square such that* (i) μ *corresponds to plane Borel measure* m_2 *in* W, (ii) T *corresponds to a transformation of* W *that leaves every x-section of* W *invariant and on it is ergodic with respect to linear Borel measure* m, *and* (iii) m *is the only normalized invariant Borel measure on any x-section of* W.

Proof. We start with the direct sum representation already obtained in §8, but restrict ϕ to the set $R_0 = R - P$. Under present hypotheses the measures ν and ν_x vanish on every finite subset of X and Y_x respectively. For any $(x, y) \in Y$ define

$$F(x) = \nu(X \cdot [0, x))$$

and

$$G(x, y) = \nu_x(Y_x \cdot [0, 1] \times [0, y)).$$

Then $F(x)$ is a monotone increasing continuous function of x on X, and for each $x \in X$, $G(x, y)$ is a monotone increasing continuous function of y on Y_x. Both functions are Borel measurable on Y, the first because it is continuous, and the second because it corresponds under ϕ to the Borel measurable function $\mu_p(g^{-1}[0, g(p)))$ on R_s (cf. [17, p. 285]). Consequently the equations $x' = F(x)$, $y' = G(x, y)$ define a Borel measurable mapping ψ of Y onto a subset Z of the unit square. ψ is not necessarily 1:1, but we shall show that it can be made so by suitably contracting its domain. We first show that it preserves measure.

Consider any rectangle $J = [0, a) \times [0, b)$, where $0 < a < 1$ and $0 < b < 1$. Let $a' = \sup \{x : F(x) < a\}$, and for each $x \in X$ let $b'(x) = \sup \{y : (x, y) \in Y_x, G(x, y) < b\}$. Then $\psi^{-1}J$ consists of all points $(x, y) \in Y$ such that $0 \leqq x < a'$ and $0 \leqq y < b'(x)$. For any $x \in X$, using the fact that each point has ν_x-measure zero, we have $\nu_x(Y_x \cdot [0, 1] \times [0, b'(x)) = b$. Consequently $\mu(\psi^{-1}J) = \int_{X \cdot [0, a')} (\nu_x \cdot [0, 1] \times [0, b'(x))) d\nu(x) = b \cdot \nu(X \cdot [0, a')) = b \cdot a = m_2(J)$. Let $\mathcal{E}$ be the class of Borel subsets E of the square such that $\mu(\psi^{-1}E) = m_2(E)$. We have just shown that $\mathcal{E}$ contains all rectangles J, and since $\mathcal{E}$ contains proper differences, countable decreasing intersections, and countable disjoint unions of its members, it follows that $\mathcal{E}$ includes all Borel subsets of the square [12, §5(2), §6(3)].

Similarly, if x' is any value for which $x = F^{-1}(x')$ is single-valued, and if $\mathcal{E}_x$ denotes the class of Borel subsets E of $\{x'\} \times [0, 1]$ such that $\nu_x(\psi^{-1}E) = m(E)$, then $\mathcal{E}_x$ includes all intervals $\{x'\} \times [0, b)$, $0 < b < 1$, and has closure properties similar to those of $\mathcal{E}$. It follows that $\nu_x(\psi^{-1}E) = m(E)$ for every Borel subset E of $Z_{x'}$.

Let Z_0 be the set of all points of Z where ψ^{-1} is multiple-valued. The monotoneity of F and G shows that for all but countably many values of x', $F^{-1}(x')$ is a single x-section of Y, and that on any such x'-section of Z, Z_0 is at most countable. Since Z_0 is an analytic set [17, p. 402] it follows that $m_2^*(Z_0) = 0$. Enclose Z_0 in a Borel set B_0 with $m_2(B_0) = 0$, and choose B_0 so that every x'-section either contains $Z_{x'}$ or has linear measure zero. (It follows that B_0 includes all x'-sections $Z_{x'}$ for which $f^{-1}(x')$ is multiple-valued.) Put $Y_1 = Y - \psi^{-1}(B_0)$ and $Y_2 = \bigcap_{-\infty}^{\infty} (T')^n Y_1$. Then Y_2 is a Borel subset of Y with $\mu(Y_2) = 1$, and ψ is a 1:1 Borel measurable mapping of Y_2 onto a Borel subset $Z_2 = \psi(Y_2)$ of the unit square. Define $T'' = \psi T' \psi^{-1}$ on Z_2. Then the systems (Y_2, T', μ) and (Z_2, T'', m_2) correspond under ψ, and likewise $((Y_2)_x, T', \nu_x)$ and $((Z_2)_{x'}, T'', m)$ for every x for which $(Y_2)_x$ is nonempty. Moreover, m is the only normalized invariant Borel measure in $(Z_2)_{x'}$, since ν_x is the only such measure in $(Y_2)_x$. To complete the proof of (9.2) it suffices to follow ψ by a trivial 1:1 Borel measurable transformation of the square that involves only the values of x' on a nullset and maps Z_2 onto a set W with the property that W_x is nonempty for every x in $0 \leqq x \leqq 1$.

10. The shift transformation on sequence space. Let X be the set of all mappings $x(n)$ of the integers into the set $\{0, 1\}$, and let T be the transformation that takes $x(n)$ into $x(n+1)$. If we define $d(x, y) = \max \{1/(|n|+1): x_n \neq y_n\}$ for any two points $x \neq y$, then X is a compact metric space and T is a homeomorphism. In this system it is easy to characterize points of the various kinds discussed in §§2 and 5. For instance, a point x is quasi-regular if and only if each block of terms that appears in the sequence $x(n)$ recurs with a definite limiting frequency; x is strictly transitive if and only if for each block B and $\epsilon > 0$ there is a positive integer k such that in any sequence of k successive terms of $x(n)$, B occurs with a frequency that differs by less than ϵ from the frequency with which B occurs in any other sequence of k successive terms of $x(n)$. On the basis of (4.3) it is easy to formulate similar characterizations of transitive points and points of density.

The almost periodic points of this system are the same as ***recurrent sequences*** in the sense of Birkhoff (cf. [15]). A number of examples and classes of such sequences have been defined and studied by Morse [19], Morse and Hedlund [20; 21; 22], Garcia and Hedlund [7], and Robbins [27]. In most of these examples it is not hard to verify that the sequence defines a point that is not only almost periodic but also strictly transitive. Hence the orbit closure of such a point is a minimal set on which T is strictly ergodic. However, it is also possible to exhibit a minimal set that is not strictly ergodic. The first such example was constructed by Markhoff (cf. [23, p. 533]). We shall give a similar example in the system (X, T). For this purpose it suffices to define a point x that is almost periodic but not quasi-regular (cf. (5.4)).

Let $\{k_i\}$ $(i \geqq 0)$ be a sequence of positive integers such that k_i divides k_{i+1} and such that $\sum_{i=1}^{\infty} k_{i-1}/k_i \leqq 1/12$ (e.g., the sequence $k_i = 2^{i(i+9)/2}$). Let n and m denote variable integers, and for each positive integer i define

$$E_i = \bigcup_{-\infty < m < \infty} \{n : |n - mk_i| \leqq k_{i-1}\}.$$

Since E_i includes all n with $|n| \leqq k_{i-1}$, and since $k_i \to \infty$, it is clear that for each integer n there is a least positive integer $p = p(n)$ such that n belongs to E_p. Define $x(n) = 0$ or 1 according as $p(n)$ is even or odd. We shall show that this point x has the required properties.

In the first place, each of the sets $E_1, E_2, \cdots, E_p$ is invariant under translation by k_p. Hence if $p(n) = p$ and $m \equiv n \pmod{k_p}$ then $p(m) = p$. Consequently, if $m \equiv n \pmod{k_i}$ and $|n| \leqq k_{i-1}$, then

$p = p(n) \leq i$, k_p divides k_i, $m \equiv n \pmod{k_p}$, $p(m) = p(n)$, and therefore $x(m) = x(n)$. It follows that $d(x, T^n x) \leq 1/(1+k_{i-1})$ for all $n \equiv 0 \pmod{k_i}$. This shows that x is almost periodic (in fact, *regularly almost periodic*, as defined in [7, p. 957]).

Secondly, if $1 \leq j \leq i$ the number of elements of E_j in the interval $0 < n \leq k_i$ is exactly $(k_i/k_j)(2k_{j-1}+1)$. Hence an upper bound to the number of elements of $E_1 \cup E_2 \cup \cdots \cup E_i$ in the interval $0 < n \leq k_i$ is

$$\sum_{j=1}^{i} \frac{3k_i k_{j-1}}{k_j} \leq 3k_i \sum_{j=1}^{\infty} \frac{k_{j-1}}{k_j} < \frac{1}{4} k_i.$$

It follows that $p(n) = i+1$ for at least 3/4 of the numbers n in the interval $0 < n \leq k_i$. Consequently,

$$\left| \frac{1}{k_i} \sum_{n=1}^{k_i} x(n) - \frac{1}{k_{i+1}} \sum_{n=1}^{k_{i+1}} x(n) \right| \geq \frac{1}{2} \qquad (i = 1, 2, \cdots).$$

But $x(n) = \chi(T^n x)$, where χ denotes the characteristic function of the set of points $y \in X$ for which $y(0) = 1$, and therefore $M(\chi, x)$ is undefined. Since χ is a continuous function on X this shows that x is not quasi-regular.

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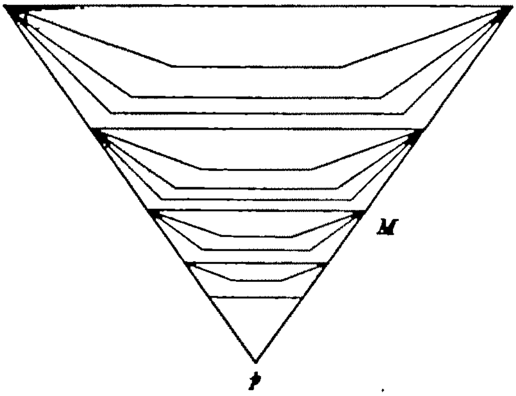
BRYN MAWR COLLEGE

CONCERNING APOSYNDETIC AND NON-APOSYNDETIC CONTINUA

F. BURTON JONES

Introduction. One might judge from the title that I am going to discuss *continua*. For is not a continuum either aposyndetic or non-aposyndetic? What I intend to do is to consider continua from a certain point of view, and from this point of view continua may be classified in a rough sort of way. This system of classification (and the basic concept upon which it rests) is only in its infancy. Here then is a report upon the beginning rather than the completion of an interesting and, I trust, useful field of study.



EXAMPLE 1

To avoid any confusion, I shall confine this discussion to continua lying in a compact metric space which in most cases is the complex number sphere (or a topological 2-sphere, S^2). Hence all continua are connected, closed, and compact sets.

Consider the difference between the familiar concepts of a continuum being connected im kleinen *at a point* and a continuum being *locally connected* at a point.[1] A continuum M is locally connected at a point p of M provided that if R is a region containing p, there exists a connected open subset U of M such that $R \supset U \supset p$. The continuum M is connected im kleinen at p provided that if R is a region

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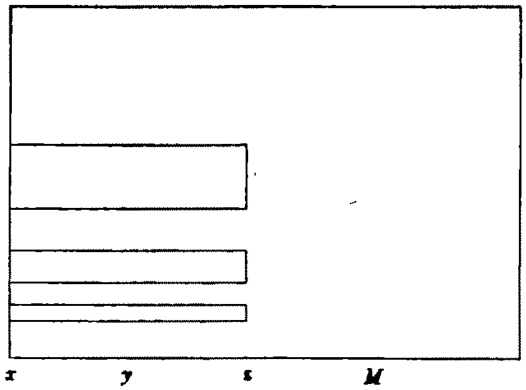
[1] For the definition of certain terms and phrases see [11]. Numbers in brackets refer to the bibliography at the end of this paper.

containing p, there exist a subcontinuum H of M and an open subset U of M such that $R \supset H \supset U \supset p$. Now as a *point-wise* property connectedness im kleinen is the weaker of the two. This may be seen in Example 1. In this example the plane continuum M consists of a simple infinity of contiguous and similar trapezoids (which form a sequence converging to p) together with a simple infinity of arcs lying in the interior of each trapezoid as indicated. A little study will show that M is connected im kleinen at the point p but M is not locally connected at p.[2]



EXAMPLE 2

In order to generalize the weaker of these two notions, I shall rephrase the definition of connectedness im kleinen. Let M be a continuum and let G denote the collection of all closed subsets of M. Then M is connected im kleinen at a point p of M provided that if K belongs to G and does not contain p, there exist a subcontinuum H of M and an open subset U of M such that $M-K \supset H \supset U \supset p$. Now let G' be the subcollection of degenerate elements of G, and substitute G' in the previous statement for G. Thus M is ***aposyndetic*** at p provided that if k is a point of M distinct from p, there exist a subcontinuum H of M and an open subset U of M such that $M-k \supset H \supset U \supset p$. The term aposyndetic means bound-together-away-from.[3] So for each point k in $M-p$, M is bound together at p away from k because the continuum H lies in $M-k$ and contains a (relative) neighborhood of p.

It may be helpful to consider the following simple example in the plane (Example 2). In this example the continuum M consists of a simple closed curve together with a simple infinity of rectangles as indicated, these rectangles forming a sequence which converges to the interval xyz. This continuum M *is* aposyndetic at x, and M is aposyndetic at y with respect to z but M is *not* aposyndetic at y with respect to x because if U is an open subset of M containing y, any subcontinuum of M which contains U would have to contain x.

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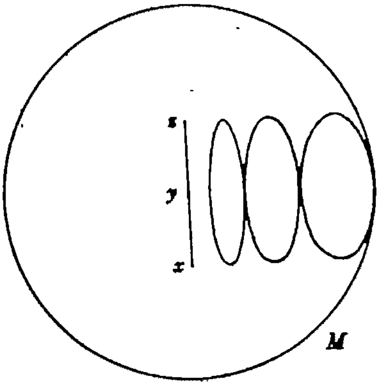
[2] An example of this nature has been given by Ayres in [1].

[3] Gk. apo = *away from*, syn = *together*, deo = *to bind*. See [4].

Boundary point theorems. The usefulness of the property of a continuum being aposyndetic *at a point* is illustrated by the following two theorems.



EXAMPLE 3

THEOREM 1. *In S^2 let D be a complementary domain of a continuum M. Then S^2-D is connected im kleinen at every point where M is aposyndetic.*[4]

A simple special case of Theorem 1 may serve to make the theorem clearer: If the continuum M does not separate S^2, then M is connected im kleinen at exactly those points where M is aposyndetic. It does not follow that M is locally connected at these points. In Example 1, throw away every point to the left of a vertical line through p. The remaining points form a continuum (not separating S^2) which is aposyndetic at p but which is not locally connected at p.

However consider the following theorem:

THEOREM 2 (BING). *If a continuum M is not separated by any of its subcontinua, then M is locally connected at every point where M is aposyndetic* [3].

Here is a test for local connectedness—at least in some cases.

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[4] Theorem 1 is a corollary of Theorem 10 in [4]. *A complementary domain* of a continuum is a *component* of its complement. The term *domain* will be used in general as equivalent to *open set* and is not necessarily connected.

Return once more to a consideration of Theorem 1. Suppose that (in S^2) D is a connected domain and M is the boundary of D. Is M connected im kleinen at every point where M is aposyndetic? *No.* For consider Example 3 in which M is the sum of a simple infinity of tangent ellipses as indicated together with their sequential limiting set xyz. Even though M is aposyndetic at y, M is *not* connected im kleinen at y. But M is not aposyndetic at y with respect to $x+z$. This observation may be formalized as follows:

THEOREM 3. *Suppose that y is a point of the boundary M of a connected domain in S^2. Suppose further that if x and z are points of $M-y$ there exist a subcontinuum H of M and an open subset U of M such that $M-(x+z)\supset H\supset U\supset y$. Then M is connected im kleinen at y.*

Continuous curves, aposyndetic continua. If a continuum were aposyndetic at each of its points, one would guess that it would in many respects resemble a continuous curve (this is, a continuum which is connected im kleinen at *each* of its points and, hence, locally connected). This is particularly true of topological properties in the large. For instance consider the Torhorst Theorem.

TORHORST THEOREM. *The boundary of a complementary domain of a continuous curve in S^2 is itself a continuous curve* [13].

The "continuous curve" in the hypothesis may be replaced by "aposyndetic continuum," so that we have the following:

BOUNDARY THEOREM (WILDER-WHYBURN). *The boundary of a complementary domain of an aposyndetic continuum in S^2 is a continuous curve* [4; 14; 17].

This, of course, follows immediately from Boundary Point Theorem 1. Priority of discovery belongs to Wilder for the cyclic case [16]. G. T. Whyburn proved the theorem independently for the general non-cyclic case [14]. Wilder's argument with slight modifications was extended by Wilder to the general case—and to higher dimensions [17]. My work generalized the theorem to more abstract two-dimensional spaces and did away with some of the compactness requirements [4]. Wilder's and Whyburn's arguments are quite analogous and depend upon a sort of mass behavior of the points. So it may be instructive to compare the various notions involved here as well as the terminology.

Whyburn defined a continuum M to be *semi-locally-connected* (*s-l-c*) at a point p provided that if U is an open subset of M containing p, there is an open subset V of M lying in U and containing

p such that $M-U$ is covered by a finite number of components of $M-V$. Wilder used essentially the same idea. Wilder used the term *almost i-avoidable* (i being associated with dimension). As a point-wise property these notions are not directly related to connectedness im kleinen. For in Example 2, M is s-l-c at z where M is *not* connected im kleinen and M is connected im kleinen at x where M is not s-l-c. The situation will be a little clearer if the definition of s-l-c is rephrased as follows: the continuum M is s-l-c at a point x of M if M is aposyndetic at each point y of $M-x$ with respect to x. So we have the following:

EQUIVALENCE THEOREM. *In order that a continuum M be s-l-c it is necessary and sufficient that M be aposyndetic* [4].

As a point-wise property, the notion of semi-local-connectedness, while not yielding the boundary point theorem, is useful in another direction.

ACCESSIBILITY THEOREM. *If D is a complementary domain of a continuum M in S^2, then the boundary B of D is accessible from all sides from D at those points of B where M is s-l-c.*[8]

So the notions of a continuum being aposyndetic at a point and a continuum being s-l-c at a point are sort of complementary notions distilled from the notion of a continuous curve.

Now to return to the comparison of aposyndetic continua with continuous curves. Whyburn's main interest in aposyndetic continua was in connection with his cyclic element theory. In general, he has shown that for this theory aposyndetic continua (i.e., s-l-c continua) are just as satisfactory as continuous curves. In theorem after theorem in his book he has made this substitution [15].

A continuous curve in S^2 may be characterized by means of its complement [12]. There is an analogous theorem for aposyndetic continua.

COMPLEMENT THEOREM. *In order that a continuum M in S^2 be aposyndetic it is necessary and sufficient that its complement be non-folded* [5].

To give you an intuitive idea of what it takes to make a domain folded in the eyes of a topologist consider Example 2 again. There exists in D an infinite sequence of crossed arc-segments (open arcs) A_i+B_i whose endpoints lie on the boundary of D such that the end-points of the B_i converge to a point not in the continuum to which

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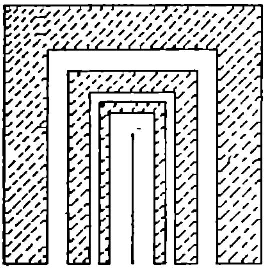
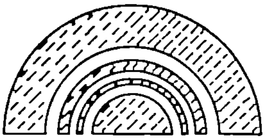
[8] Neither B nor M is necessarily *s-l-c* at every point where B is accessible from all sides.

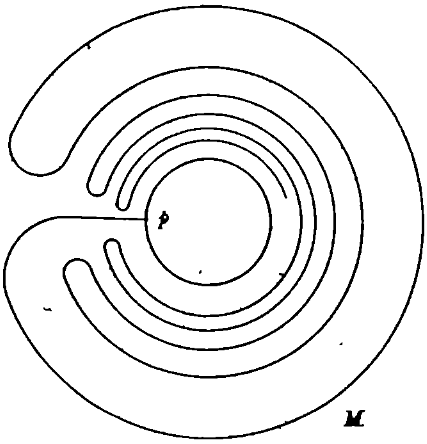


EXAMPLE 4-1



EXAMPLE 4-2

the A_i converge. Consider Example 4. The shaded areas in Example 4.1 form a folded domain, while in Example 4.2 they do not—the reason being that the latter may be straightened out (by stretching) without doing violence to the topology.



EXAMPLE 5

A *cyclic*[6] aposyndetic continuum in S^2 may be characterized as a continuum each of whose complementary domains is simple (i.e., has a simple closed curve for a boundary) such that this collection of simple domains contains no *folded subcollection* [5]. That is, the situation must be like Example 4.2, not Example 4.1. Wilder also has this theorem or something very analogous in his book [18].

In connection with the term cyclic, it is fortunate that aposyndetic continua have another characteristic in common with continuous curves:

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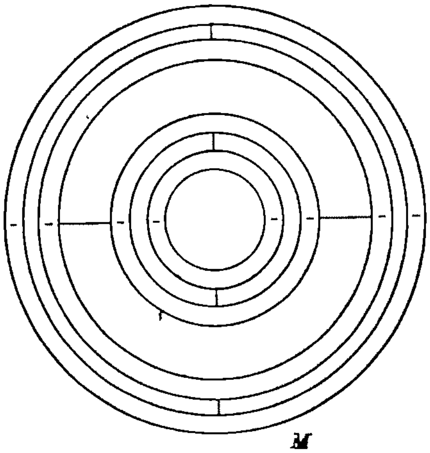
[6] A continuum is said to be *cyclic* provided that no point of the continuum disconnects it.

CUTPOINT EQUIVALENCE THEOREM. *In order that a point p of an aposyndetic continuum M be a separating point of M it is necessary and sufficient that p be a weak cutpoint of M* [14].

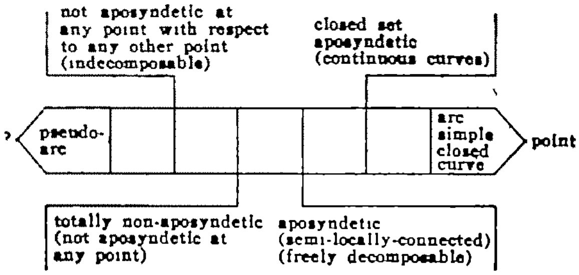
A point p is a ***separating point*** (often called cutpoint) of a continuum M provided that $M-p$ is not connected. A point p is a ***weak cutpoint*** (often called cutpoint) of a continuum M provided that $M-p$ is not strongly connected (i.e., continuum-wise connected). For continua in general the two properties are not the same. Consider Example 5. (In this example, *if* the circle were considered to be a point, M would be a simple closed curve.) If p is removed from M the remaining set is connected but not strongly connected. For aposyndetic continua Whyburn (and I would guess Wilder also) showed that this cannot happen.



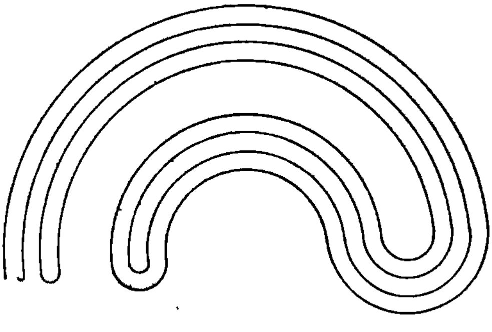
EXAMPLE 6

There are, however, considerable differences between continuous curves and aposyndetic continua. In Example 1, M is aposyndetic at every point and connected im kleinen at p but is *not* locally connected at p. In fact a cyclic continuum may be aposyndetic and not be connected im kleinen at any point whatsoever. In Example 6 the continuum M consists of all circles centered on $(1/2, 0)$ and passing through a point of the Cantor discontinuum (on the interval from 0 to 1 on the x-axis) together with the bridging arcs as indicated each of which is to be considered to be a point. Since these bridging arcs alternate their position between the vertical and horizontal (as the size of the gap bridged decreases) it is clear that M is not connected im kleinen at any point of M. A little examination, however, shows that M *is* aposyndetic at every point of M.

Classification of continua. With the meaning of aposyndetic well in mind but before proceeding further we shall try to get a view of all continua in the light of this notion. If a person were to start to classify continua he would most certainly want the well-behaved continua (like arcs, simple closed curves, and continuous curves in general) at one end of the classification and the ill-behaved or psychopathic continua (like indecomposable continua) at the other. Imagine all continua spread out in a sort of spectrum with the bad ones on the left and the good ones on the right. This is illustrated in the figure. An indecomposable continuum is one which is not the sum of two proper subcontinua. One of the simplest examples of such a continuum is the one indicated in Example 7 [8]. This continuum consists of a countable infinity of collections of semi-circles, the first such collection being concentric at (1/2, 0) on the X-axis and using up all of the points of the Cantor discontinuum as endpoints of its elements, the second such collection being concentric at (5/6, 0) and using up all of the points of the right-hand half of the Cantor set as endpoints of its elements, etc., as indicated. Notice that in this


not aposyndetic at any point with respect to any other point (indecomposable)
closed set aposyndetic (continuous curves)
pseudo-arc
arc
simple closed curve
point
totally non-aposyndetic (not aposyndetic at any point)
aposyndetic (semi-locally-connected) (freely decomposable)

Spectrum

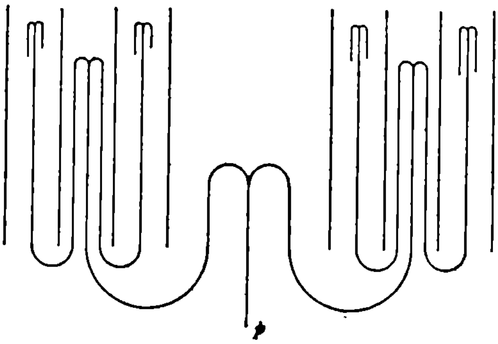


EXAMPLE 7

example no point is a separating point but every point is a weak cutpoint.

Fundamental cutpoint theorems. In the left-hand half of the spectrum the weak cutpoint notion is of considerable importance. Actually we have two fundamental theorems.

THEOREM 1. *A totally non-aposyndetic continuum contains a weak cutpoint* [6]. (*A totally non-aposyndetic continuum is one which is not aposyndetic at any one of its points.*)



EXAMPLE 8

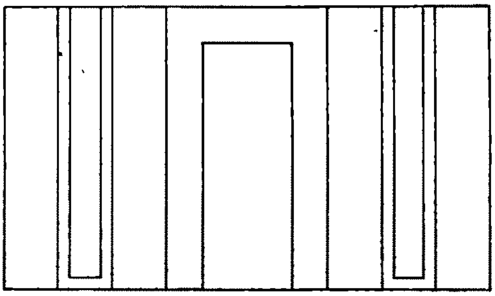
One is the largest number of weak cutpoints that a totally non-aposyndetic continuum must necessarily contain. For a simple case, consider again Example 5, and replace the ray (which starts at p and is wrapped back and forth around the circle) by a bundle of rays (with a Cantor set cross-section) all of which start at p. A little examining will show that this continuum is totally non-aposyndetic. Nevertheless, it has *no* separating point and p is its only weak cutpoint.

THEOREM 2. *A totally non-semi-locally-connected continuum possesses a dense set of weak cutpoints* [6].

Theorem 2 might lead one to suspect that a totally non-semi-locally-connected continuum is more non-aposyndetic than a totally non-aposyndetic continuum. This is not the case, for such a continuum need not be totally non-aposyndetic, in fact, such a continuum may be locally connected at one of its points. Consider Example 8. The continuum indicated in this example consists of vertical unit intervals erected at each point of the Cantor discontinuum on the interval from 0 to 1 of the X-axis (in the plane) together with the branching structure emanating from the point p. This branching structure consists of a bundle of rays (with a Cantor set cross-section) emanating from p and separating into equal branches at the places indicated in the figure. The continuum thus obtained is totally non-semi-locally-connected, each of its points is a weak cutpoint, but it is locally connected at p.

Using these two theorems, one obtains a rather interesting theorem about continua which have no weak cutpoints.

THEOREM 3. *A continuum which contains no weak cutpoint is both aposyndetic and s-l-c at all points of a dense inner limiting subset* (G_δ-*set*).



EXAMPLE 9

In spite of this theorem and Wilder's terminology (avoidability) a continuum containing no weak cutpoint may be composed largely of points at which it is neither aposyndetic nor s-l-c. To give a strikingly good example would take up too much space but the following example will suffice. In Example 9 the continuum M consists of a rectangle (in the plane) together with the vertical intervals spanning the rectangle at the points of the Cantor discontinuum on its base plus infinitely many U-shaped arcs based alternately on the top and bottom sides of the rectangle as indicated. The continuum M is cyclic but fails to be aposyndetic or semi-locally-connected at each nonendpoint of each of the vertical intervals.

I suspect these cutpoint theorems of having many applications. I want to mention one. Bing showed that if a nondegenerate continuum is cut (in the weak sense) by no one point but *is* cut by each pair of its points, then it is a simple closed curve. Using the first Fundamental Cutpoint Theorem he knew that the continuum had to be aposyndetic at a point and this gave him just enough of a hold on the problem to complete the proof [3].

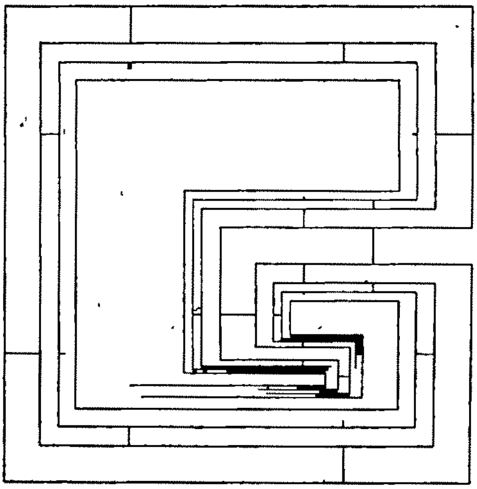
Topological products. With the discussion and examples up to this point in mind, it would not be unreasonable to suspect that the topological product of two nondegenerate continua (it contains no weak cutpoint) should be aposyndetic. While this is not self-evident in the case of indecomposable continua, we do have the following:

PRODUCT THEOREM (WHYBURN-JONES). *If H and K are nondegenerate continua their Cartesian product $H\times K$ is aposyndetic* [6; **14**].

Whyburn first discovered this theorem but required *one* of the continua to be aposyndetic. There being so many products these days, it may be well to point out that the plane is the Cartesian product of the X- and Y-axis. An annulus is the product of an arc with a circle. So a totally non-aposyndetic continuum must be a kind of thin continuum: that is, it cannot be a topological product and it must contain a weak cutpoint.

Spectrum analysis. Looking at the spectrum of continua (see the figure) let us examine continua of various kinds and various properties of continua. Consider first *arc-wise connectedness.* Using the various examples that I have given, it is easy to see that arc-wise connected continua lie in every part of the spectrum except in the indecomposable end. So there is a problem of possibly expanding that portion of the spectrum where they *must* be arc-wise connected. Let us say that a continuum M is G-aposyndetic if G is a family of sets and for each point p of M and each element g of G lying in M but not containing p, there exist a subcontinuum H of M and an open subset U of M such that $M-g\supset H\supset U\supset p$. Roughly, M is bound together at each of its points away from each element of G. If G is the collection of all closed sets, a continuum is G-aposyndetic if and only if it is locally connected and such continua are known to be arc-wise connected. If G is the collection of degenerate sets, a continuum is G-aposyndetic if and only if it is aposyndetic and even in the plane such a continuum may be far from being arc-wise connected. Whyburn pointed this out with a different (and simpler) example [14] but the reader may see this by looking at Example 6.

Now consider continua in a given 2-sphere, S^2, and suppose that we let G_1 be the family of all continua in S^2. Let G_2 be the family of all continua in S^2 that are G_1-aposyndetic. In general, for each positive integer, i, let G_{i+1} be the family of all continua in S^2 which are G_i-aposyndetic. For each $i>1$, each element of G_i is aposyndetic. Those continua in G_1 do not have to be arc-wise connected. Those in G_2 must *all* be arc-wise connected. When I first discovered this, I thought that perhaps here was an extension of the arc-wise connectedness theorem. But no—the elements of G_2 are continuous curves! What is the nature of the elements of G_3? While they are all aposyndetic, they are *not* all arc-wise connected. Consider Example 10.



EXAMPLE 10

Start with a square (in the plane) plus its interior. Using the method of Yoneyama [19] construct an indecomposable continuum by digging a canal (i.e., removing a connected open set) into this square that winds back and forth within the square so as to come arbitrarily close to every point of what is left by a sufficiently devious path. Then add to this continuum infinitely many open curves which have been drawn into the entire length of the canal and which have been bridged with arcs as indicated, the distance between such bridges being roughly equal to the width of the canal at the point of the bridge. Call this continuum M. To get from the open curve part of M to the indecomposable part of M with an arc (or continuous curve) one either has to oscillate too much because of the staggered nature of the bridges or go too far down the canal (its entire length in fact) in trying to reduce the distance between bridges. Therefore the elements of G_2 (continuous curves) lying in the continuum M must lie wholly within its indecomposable (Yoneyama) subcontinuum or entirely outside of this subcontinuum. With this in mind one can see that M is G_3-aposyndetic but *not* arc-wise connected. A simpler example than Example 10 will not suffice because from the Wilder-Whyburn Boundary Theorem those elements of G_2 possible in a given

G_3-aposyndetic continuum will consist, in addition to degenerate and possibly other sets, of the boundaries of its complementary domains. For instance in Example 6, M (which is not arc-wise connected) is not G_3-aposyndetic.

Now, how about the elements of G_4? Are they all arc-wise connected? By elementary (and facetious) induction, when i is odd the elements of G_i do not have to be arc-wise connected, and when i is even the elements of G_i are arc-wise connected. So the only problem here is in the limit: Does $\mathrm{Lim}_{n\to\infty} G_{2n+1} = \mathrm{Lim}_{n\to\infty} G_{2n}$? And are the elements of the limit collection arc-wise connected? As a matter of fact my knowledge does not extend to G_4 but perhaps this view of the problem will tempt someone else to investigate G_i, $i>3$.

Thus for arc-wise connectedness, looking at the spectrum and attempting to further refine its structure has not yet proved fruitful (except to learn that aposyndetic continua in S^2 do have to *contain* arcs and that elements of G_2 are continuous curves).

Now consider *irreducible continua*. Whyburn showed that those which are aposyndetic must be arcs [14]. If they are non-aposyndetic, they may be scattered anywhere in that portion of the spectrum.

Consider a continuum M which is *topologically equivalent to each of its nondegenerate subcontinua*. If M is aposyndetic, it is an arc because it is irreducible between some pair of its points. If M is non-aposyndetic it may lie at the other end of the spectrum (as shown by Moise's example of a pseudo-arc) [9]. But must it? This question, perhaps not quite in this form, has been unanswered for a long time, and I suspect that a good many more division lines will have to be added to the spectrum before an answer can be obtained.

Consider a *homogeneous* continuum M in S^2. If M is aposyndetic, it must be a simple closed curve [7]. If M is *not* aposyndetic (and such continua do exist [2; 10]), two cases arise: (1) If M does not separate S^2, M is indecomposable;[7] (2) if M does separate S^2, I would *guess* it may exist outside of the indecomposable end of the spectrum but I am not quite satisfied about this.

The number of problems that this scheme of spectrum analysis calls to mind, even for continua in the plane, is immense. I shall mention two more:

1. What does it take to push a totally non-aposyndetic continuum into the indecomposable end of the spectrum?
2. Is there a continuum at the extreme left end of the spectrum which if used as a "point" would decompose S^2 into itself? From an

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[7] The proof of this will appear elsewhere.

esthetic point of view, the answer to this question should be *yes*, in order that the spectrum return to its starting point.[8]

I have not touched upon dimension theory at all. I would like to point out that one cannot raise the dimension of a Knaster continuum (and keep it a Knaster continuum) by means of Cartesian products for then they become aposyndetic.[9]

As I indicated in the introduction, we have here only a beginning of a classification of continua by means of their aposyndetic properties. I have indicated one way, but not a very fruitful way, of refining this classification. And I have indicated in a rough sort of way how this crude classification may be used not only to suggest new problems about continua but also to view the class of continua as a whole. I hope that this discussion will challenge others to work on these problems, but the research worker who expects to lead a life of ease should stay out of the left end of the spectrum.

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[8] This question has been answered by R. D. Anderson in the affirmative.

[9] Bing has raised the dimension of a Knaster continuum but by another method. His paper *Higher-dimensional hereditarily indecomposable continua* appears in Trans. Amer. Math. Soc. vol. 71 (1951) pp. 267–273.

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THE UNIVERSITY OF TEXAS AND
THE UNIVERSITY OF NORTH CAROLINA

THE NOVEMBER MEETING IN NORMAN

The four hundred seventy-fifth meeting of the American Mathematical Society, was held at the University of Oklahoma on Friday and Saturday, November 23–24, 1951. There were about 50 registrations including the following 29 members of the Society:

R. V. Andree, Arthur Bernhart, J. C. Brixey, Y. W. Chen, L. A. Colquitt, N. A. Court, G. M. Ewing, Casper Goffman, I. E. Glover, A. A. Grau, E. V. Greer, O. H. Hamilton, J. O. Hassler, W. N. Huff, L. W. Johnson, A. E. Labarre, G. Q. LaFon, H. W. Linscheid, Dora McFarland, B. L. Mackin, W. C. Orthwein, C. J. Pipes, E. H. Rothe, G. W. Smith, R. G. Smith, O. S. Spears, C. E. Springer, E. B. Stouffer, J. W. T. Youngs.

By invitation of the Committee to Select Hour Speakers for Western Sectional Meetings, Professor E. H. Rothe of the University of Michigan delivered an address entitled *Gradient mappings* at 2:00 P.M. on Friday. Dean E. B. Stouffer presided at the session.

Sessions for the presentation of contributed papers were held at 3:30 P.M. on Friday and 10:00 A.M. on Saturday. Presiding officers at these sessions were Professors C. E. Springer and O. H. Hamilton.

The Society was entertained by President and Mrs. Cross of the University of Oklahoma at a tea in their residence on Friday afternoon. That same evening there was a dinner in the Union for the Society and its guests.

The meeting was characterized by a pleasant air of informality, together with active participation in the discussions following the papers by a large percentage of the audience.

The Society is indebted to the Committee on Local Arrangements for a well planned meeting, and for the obvious pains which were taken to look after the physical comfort of the visitors.

ALGEBRA AND THEORY OF NUMBERS

106. D. W. Dubois: *Partly ordered fields*. Preliminary report.

A commutative field K is called a *partly ordered field* if a relation "x is positive" (abbreviated "$x>0$") is defined for at least one $x\in K$ such that if a and b are positive, so are $a+b$, ab, and ab^{-1}. One defines $a<0$, $x>y$, $x<y$ as usual. One shows that every partly ordered field is a partly ordered set containing the rational field in its natural order and that every field of characteristic zero can be partly ordered. The partial order of K is said to be *Archimedean* if for every $x\in K$ there is an integer $n\in K$ with $n>x$. Examples are given of both Archimedean and non-Archimedean partly ordered fields which are not simply ordered by the partial orderings. The following results are proved for an arbitrary partly ordered field K: (1) K contains a subfield D already ordered by the partial order of K, and D contains every other such subfield. D consists of Dedekind cuts on the rational subfield. (2) There is a real-valued norm $\mu(x)$ on K; the completion of K by fundamental sequences yields a real Banach algebra. (3) If K is an algebraic number field, K can be Archimedean partly ordered if and only if K is formally real. (Received October 9, 1951.)

ANALYSIS

107. Y. W. Chen: *On hodograph transformations of partial differential equations with three independent variables.*

A system of three quasi-linear partial differential equations (1): $A_{ij}\cdot\partial u_i/\partial x_j=0$, $B_{ij}\cdot\partial u_i/\partial x_j=0$, and $C_{ij}\cdot\partial u_i/\partial x_j=0$ is considered for three functions $u_i(x_1, x_2, x_3)$; $i=1, 2, 3$. The coefficients A_{ij}, B_{ij}, and C_{ij} are functions of (u) and do not depend on (x). By means of the hodograph transformations nine functions $\zeta_{ij}(u_1, u_2, u_3)$ are introduced with u_i as independent variables. They satisfy the following system (2) of nine equations: (2.1) $A_{ij}(u)\cdot\zeta_{ij}(u)=0$, $B_{ij}(u)\cdot\zeta_{ij}(u)=0$, $C_{ij}(u)\zeta_{ij}(u)=0$, (2.2) $\partial\zeta_{1j}/\partial u_1+\partial\zeta_{2j}/\partial u_2+\partial\zeta_{3j}/\partial u_3=0$, $j=1, 2, 3$, (2.3) $\xi_{i3}(\partial\xi_{i1}/\partial u_2-\partial\xi_{i2}/\partial u_1)+\xi_{i1}(\partial\xi_{i2}/\partial u_3-\partial\xi_{i3}/\partial u_2)+\xi_{i2}(\partial\xi_{i3}/\partial u_1-\partial\xi_{i1}/\partial u_3)=0$, $i=1, 2, 3$, where $\xi_{\alpha j}=(\zeta_{\beta k}\zeta_{\gamma l}-\zeta_{\gamma k}\zeta_{\beta l})$ and the indices (α, β, γ), (j, k, l) are cyclic permutations of $(1, 2, 3)$. It is shown that systems (1) and (2) are equivalent for initial values problems. System (2) is also applied to study series expansions of three-dimensional steady subsonic flows. (Received October 11, 1951.)

108. E. V. Greer: *Certain relations between functions.*

A real function $g(x)$ on $[0, 1]$ will be called related to $f(x)$ in the Saks-Sierpinski sense if for every $\epsilon>0$ the inequality $|f(x)-g(x)|<\epsilon$ holds on a set of exterior measure 1. It will be called related to $f(x)$ in the Blumberg sense if $f(x)$ fully approaches $g(x)$ almost everywhere. It is shown that a measurable $g(x)$ is related to $f(x)$ in the Saks-Sierpinski sense if and only if it is related to $f(x)$ in the Blumberg sense. This fact is used to prove the following generalization of the fundamental theorem of the integral calculus: If $f(x)$ is a bounded, single-valued function defined on $[0, 1]$, then the derivative of the upper Lebesgue integral of $f(x)$ is related to $f(x)$ in the Saks-Sierpinski sense. In connection with these ideas, the following analogue of a theorem of Saks and Sierpinski is proved for planar transformations: If $f(s)$ is an arbitrary single-valued transformation of the closed unit square into the euclidean plane, then there is a single-valued transformation $g(s)$ of Baire class 2 at most which is related to $f(s)$ in the Saks-Sierpinski sense. (Received October 2, 1951.)

109. W. N. Huff and. E. D. Rainville: *On the Sheffer A-type of polynomials generated by $\phi(t)f(xt)$.*

The Sheffer A-type of the polynomial set $\{y_n(x)\}$ defined by $f(xt)\phi(t)=\sum_{n=0}^{\infty} y_n(x)t^n$ was discussed in a previous paper by Huff. The main result of the present paper is a necessary and sufficient condition that $\{y_n(x)\}$ be of Sheffer A-type k: that $f(xt)$ be a hypergeometric function with no numerator parameters, with k denominator parameters, and with argument a constant multiple of xt. (Received September 24, 1951.)

110*t*. T. S. Motzkin and W. R. Wasow: *On the approximate solution of linear elliptic differential equations by difference equations with positive coefficients.*

Let $L[u]=\sum_{i,k=1}^{n}\beta_{ik}(x)\partial^2u/\partial x_i\partial x_k+\sum_{j=1}^{n}\alpha_j(x)\partial u/\partial x_j-c(x)u$ be an elliptic differential expression. Here x denotes the vector with components $x_1, x_2, \cdots, x_n$. The finite difference expression $L_h[u]=\sum_{\nu=1}^{N}c_\nu(x, h)u(x_1+m_{1\nu}h, \cdots, x_n+m_{n\nu}h)-u(x)$ in a square lattice of mesh length h is called formally analogous to $L[u]$ if the beginning of Taylor's expansion of $L_h[u]$ about the point x is proportional to $L[u]$. The study of the relation between the solutions of boundary value problems in a bounded domain B belonging to formally analogous differential and difference expressions is greatly simplified if all $c_\nu(x, h)$ are non-negative for x in B. It is shown that, for $n>1$, this condition cannot be satisfied if the set of integers $m_{\mu\nu}$ is to be the same for all elliptic differential expressions. This is true even for the subclass of expressions $L[u]$ whose coefficients are bounded in absolute value by a given number M, for all x in B. However, if the further restriction is imposed that Det $\{\beta_{ik}(x)\}$ is to exceed in B a positive, otherwise arbitrary, number, then a set of integers $m_{\mu\nu}$, independent of $L[u]$, exists for which the $c_\nu(x, h)$ can be chosen positive in B. (Received August 17, 1951.)

111. C. J. Pipes: *Generalizations of a theorem of Sierpinski and Zygmund on continuous functions.*

Consider an abstract set property π defined by the following postulates. π is a set property such that: (1) every subset, including the empty set, of a set that has the property also has the property, (2) the union of a finite or denumerable number of sets that have the property also has the property, (3) every set consisting of a single point has the property, and (4) the continuum does not have the property. Then, for any set property π, if continuity is generalized to π-continuity by considering as negligible those sets that satisfy the property, the following result may be obtained: there is a real function that is not π-continuous on any set, relative to the set, not satisfying the property π. An analogous result may be obtained if continuity is generalized by considering as negligible all sets of exterior metric density zero at a point. (Received October 10, 1951.)

APPLIED MATHEMATICS

112*t*. Abolghassem Ghaffari: *A new approximation method for compressible subsonic flow.*

The following investigations are an attempt to find an approximation to the velocity potential ϕ and stream function ψ for compressible flow. If τ_1 is the value of $\tau=w^2/w_m^2$ (where w is the fluid speed and w_m is the maximum fluid speed) in the main stream, the Mach number in the main stream is given by $M_1=\tau_1/(1-\tau_1)$. Hence the effects of compressibility will diminish as M_1 and τ_1 tend to zero. Therefore, to pass from a compressible flow to a corresponding incompressible flow, we put $\tau=q^2\tau_1$ (where q is the velocity in incompressible flow) in the expressions of ϕ and ψ, and then let τ_1 approach zero. The simplified form of hodograph equations are (1) $P\partial\psi/\partial\tau=\partial\phi/\partial\theta$, $Q\partial\phi/\partial\tau=-\partial\psi/\partial\theta$, where $P=2\tau(1-\tau)^{-\beta}$, $Q=2\tau(1-\tau)^{\beta+1}[1-(2\beta+1)\tau]^{-1}$; $\beta=(\gamma-1)^{-1}$; γ is adiabatic index. The corresponding limiting form of (1) is (2) $q\partial\phi/\partial q=-\partial\psi/\partial\theta$, $q\partial\psi/\partial q=\partial\phi/\partial\theta$. One of the elementary solutions of (2) is (3) $\phi=C_mq^m \cos (m\theta+\epsilon_m)$, $\psi=-C_mq^m \sin (m\theta+\epsilon_m)$, where C_m, ϵ_m are constants and m is any nonzero number. It is found that the corresponding elementary solutions of (3) for $m>0$ are (4) $\psi=-C_m\psi_m(\tau)/\psi_m(\tau_1) \sin (m\theta+\epsilon_m)$, $\phi=(P/m)\partial\psi/\partial\tau$; and for

$m=-n<0$ are (5) $\psi=-C_{-n}\psi_{-n}(\tau)/\psi_{-n}(\tau_1)\sin(-n\theta+\epsilon_{-n})$, $\phi=-(P/n)\partial\psi/\partial\tau$, where $\psi_{-n}(\tau)=[\tau^{-n/2}-K\tau^{n/2}\tau_s^{-n}][1+O(\tau)]$, and K is independent of τ, τ_s being the value of τ at the sonic speed. It is shown that in the subsonic range when $m>0$ the two functions ψ_m and ϕ_m are both positive monotonically increasing functions of τ. (Received September 20, 1951.)

113*t*. Abolghassem Ghaffari: *Asymptotic behaviour of compressible subsonic flow.*

Let ψ and ϕ denote the stream function and the velocity potential of a compressible flow and let $\tau=w^2/w_m^2$ and θ be the hodograph variables, where w is the fluid speed and w_m is the maximum fluid speed. Chaplygin's formal solution (1) $\psi=B\theta+\sum_{m=1}^{\infty}A_m\psi_m(\tau)\sin(m\theta+\epsilon_m)$, $\phi=-B\phi_0(\tau)-\sum_{m=1}^{\infty}A_m\phi_m\cos(m\theta+\epsilon_m)$ are exact, but they are quite unsuitable for numerical calculations. The author endeavors to determine the asymptotic expansions of ψ and ϕ for large values of m. Considering the simplified form of hodograph equations and setting $\psi_m=\exp(\int f_m d\tau)$ where $f_m(\tau)$ is an unknown function, one finds that (2) $\psi_m\sim h(\tau)\exp[mg(\tau)-mg(\tau_1)]$, $\phi_m\sim h^{-1}(\tau)\exp[mg(\tau)-mg(\tau_1)]$, where $h(\tau)=(1-\tau)^{(2\beta+1)/4}[1-(2\beta+1)\tau]^{-1/4}$, $g(\tau)=\int_{\tau_1}^{\tau}(2\tau)^{-1}(1-\tau)^{-1/2}[1-(2\beta+1)\tau]^{1/2}d\tau$, τ_1 being the value of τ in the main stream. Taking into account (2), the formal solutions (1) become (3) $\psi\sim h(\tau)\sum_1^{\infty}A_m\exp[mg(\tau)-mg(\tau_1)]\sin(m\theta+\epsilon_m)$, $\phi\tau-h^{-1}(\tau)\sum_1^{\infty}A_m\exp[mg(\tau)-mg(\tau_1)]\cos(m\theta+\epsilon_m)$. Now let $w_m\to\infty$, then $g(\tau)-g(\tau_1)\to 1/2\log\tau/\tau_1$; therefore, the final expressions of ψ and ϕ for small τ approach (4) $\psi\sim\sum_1^{\infty}A_m(w/w_1)^m\sin(m\theta+\epsilon_m)$, $\phi\sim-\sum_1^{\infty}A_m(w/w_1)^m\cos(m\theta+\epsilon_m)$, where w_1 is the fluid speed in the main stream. As we are dealing with subsonic flow in which $0\leq\tau\leq\tau_1\leq\tau_s$, the series (3) and (4) are absolutely and uniformly convergent. (Received September 20, 1951.)

GEOMETRY

114. N. A. Court: *Isogonal points for a tetrahedron.*

Let (M), (M') be the pedal tetrahedrons for tetrahedron $(T)=ABCD$ of the points M, M' isogonal for (T), (L) the common pedal sphere of M, M' for (T), and (A), (B), (C), (D), the spheres orthogonal to (L) and having for centers the vertices of (T). The following are some of the results obtained. a. The polar planes of M for the four spheres $(A), \cdots, (D)$ coincide with the faces of the tetrahedron (M'); b. The lines joining M to the vertices of (T) meet the respective faces of (M) in four points forming a tetrahedron which is affine to the analogous tetrahedron obtained for the point M'; c. The projections of the vertices of the two tetrahedrons (M), (M') upon the line MM' are two tetrads of points having the same biratio (that is, anharmonic ratio). The value of this biratio is equal to the biratio of the pencil of planes $MM'(ABCD)$. (Received October 8, 1951.)

TOPOLOGY

115. Arthur Bernhart: *Seven-rings in minimal five-color maps.*

The simultaneous system of linear homogeneous equations in twice 91 unknowns, together with inequalities derived from the minimal nature of the map, are solved. Over 150 solutions have been obtained. Unless there are topological conditions stronger than these colorability requirements, an indirect proof of the 4-color theorem seems unlikely. (Received October 9, 1951.)

116. G. M. Ewing and W. R. Utz: *Continuous solutions of the functional equation* $f^n(x) = f(x)$.

Let $f^n(x)$ denote f iterated n times. The authors find all continuous real functions defined on the real axis such that for an integer $n \geq 2$ the stated equation is satisfied. (Received November 23, 1951.)

J. W. T. Youngs,
Associate Secretary

THE ANNUAL MEETING OF THE SOCIETY

The fifty-eighth Annual Meeting of the American Mathematical Society was held at Brown University, Providence, Rhode Island, Wednesday through Friday, December 26–28, 1951, in conjunction with the annual meeting of the Mathematical Association of America.

The registered attendance at the meeting exceeded 525 and included the following 424 members of the Society.

C. R. Adams, R. B. Adams, Jose Adem Chahin, Shmuel Agmon, R. P. Agnew, L. V. Ahlfors, M. I. Aissen, E. J. Akutowicz, R. G. Albert, R. W. Allen, C. B. Allendoerfer, Warren Ambrose, R. D. Anderson, Joseph Andrushkiw, R. C. Archibald, H. E. Arnold, L. G. Arnold, Nachman Aronszajn, Maurice Auslander, W. E. Barnes, R. B. Barrar, Iacopo Barsotti, A. F. Bartholomay, R. G. Bartle, G. E. Bates, J. D. Baum, W. R. Baum, H. P. Beard, Ralph Beatley, E. G. Begle, R. E. Bellman, A. A. Bennett, Stefan Bergman, H. G. Bergmann, S. D. Bernardi, Lipman Bers, E. W. Beth, Kurt Bing, Garrett Birkhoff, D. W. Blackett, Jerome Blackman, A. L. Blakers, A. A. Blank, J. H. Blau, I. E. Block, H. W. Bode, Volodymyr Bohun-Chudyniv, Raoul Bott, S. G. Bourne, J. W. Bower, H. E. Bowie, Laura Brant, R. H. Breusch, H. W. Brinkmann, F. E. Browder, A. R. Brown, Bailey Brown, H. D. Brunk, G. S. Bruton, D. A. Buchsbaum, J. A. Bullard, R. S. Burington, L. J. Burton, Jewell H. Bushey, E. A. Cameron, M. E. Carlen, L. V. Carlton, Evelyn Carroll-Rusk, W. B. Carver, Lamberto Cesari, R. E. Chamberlin, Sarvadaman Chowla, K. L. Chung, R. A. Clark, A. B. Clarke, G. R. Clements, E. A. Coddington, L. W. Cohen, Harvey Cohn, Nancy Cole, A. H. Copeland, A. H. Copeland, Jr., Richard Courant, R. R. Coveyou, J. B. Crabtree, H. F. Cullen, Frederic Cunningham, J. M. Danskin, Norman Davids, Robert Davies, M. D. Davis, Philip Davis, R. B. Davis, W. M. Davis, C. R. DePrima, R. J. De Vogelaere, A. H. Diamond, J. B. Diaz, G. P. Dinneen, W. F. Donoghue, J. L. Doob, H. L. Dorwart, Jim Douglas, Avron Douglis, Melvin Dresher, F. G. Dressel, G. R. Dubé, Nelson Dunford, Albert Edrei, Samuel Eilenberg, B. J. Eisenstadt, Joanne Elliott, E. S. Elyash, L. N. Enequist, Carl Engelman, Benjamin Epstein, D. H. Erkiletian, Trevor Evans, M. I. Fauth, Herbert Federer, W. E. Ferguson, A. D. Fialkow, F. A. Ficken, R. S. Finn, C. D. Firestone, L. R. Ford, Tomlinson Fort, G. A. Foyle, J. S. Frame, F. N. Frenkiel, Bernard Friedman, H. D. Friedman, Orrin Frink, F. B. Fuller, R. E. Fullerton, J. W. Gaddum, I. S. Gál, A. S. Galbraith, David Gale, H. M. Gehman, B. R. Gelbaum, H. H. Germond, Irving Gerst, Murray Gerstenhaber, J. H. Giese, J. B. Giever, S. A. Gilbert, W. M. Gilbert, R. E. Gilman, R. D. Glauz, A. M. Gleason, Kurt Gödel, H. E. Goheen, Samuel Goldberg, Oscar Goldman, W. O. Gordon, Daniel Gorenstein, W. H. Gottschalk, Arthur Grad, J. W. Green, L. W. Green, T. N. E. Greville, Emil Grosswald, V. B. Haas, Franklin Haimo, P. C. Hammer, E. E. Hammond, G. H. Handelman, Frank Harary, W. L. Hart, F. S. Hawthorne, E. V. Haynesworth, G. A. Hedlund, A. E. Heins, M. H. Heins, Melvin Henriksen, Manuel Herschdorfer, Aaron Herschfeld, I. R. Hershner, Fritz Herzog, F. B. Hildebrand, T. H. Hildebrandt, Einar Hille, W. M. Hirsch, G. P. Hochschild, S. P. Hoffman, D. L. Holl, L. A. Hostinsky, E. M. Hove, A. A. Howard, A. S. Howard, J. L. Howell, C. C. Hsiung, M. G. Humphreys, Witold Hurewicz, W. A. Hurwitz, L. C. Hutchinson, Shikao Ikehara, B. M. Ingersoll, H. G. Jacob, A. R. Jacoby, R. C. James, R. N. Johanson, L. W. Johnson, R. E. Johnson, R. F. Johnson, B. W. Jones, F. B. Jones, Bjarni Jónsson, R. V. Kadison, Shizuo Kakutani, G. K.

Kalisch, L. H. Kanter, Irving Kaplansky, S. N. Karp, R. E. Keirstead, M. E. Kellar, D. E. Kibbey, J. F. Kiefer, H. S. Kieval, J. R. Kinney, V. L. Klee, S. C. Kleene, J. S. Klein, J. R. Kline, Morris Kline, E. R. Kolchin, Marc Krasner, Wouter van der Kulk, H. W. Kuhn, Serge Lang, C. E. Langenhop, E. H. Larguier, J. A. Larrivee, P. D. Lax, E. H. Lee, B. A. Lengyel, Benjamin Lepson, F. W. Light, C. C. Lin, B. W. Lindgren, S. I. Lipsey, W. G. Lister, L. H. Loomis, Lee Lorch, Eugene Lukacs, R. C. Lyndon, V. O. McBrien, N. H. McCoy, L. A. MacColl, R. W. MacDowell, Saunders MacLane, H. M. MacNeille, H. P. Manning, Murry Mannos, R. J. Marcou, Morris Marden, L. F. Markus, M. H. Martin, W. T. Martin, A. E. Meder, Jr., D. F. Mela, Karl Menger, E. A. Michael, W. H. Mills, Don Mittleman, E. E. Moise, Deane Montgomery, T. W. Moore, R. K. Morley, G. D. Mostow, T. S. Motzkin, W. L. Murdock, G. G. Murray, E. D. Nering, P. P. Nesbeda, John von Neumann, C. V. Newson, E. N. Nilson, I. L. Novak, C. O. Oakley, G. G. O'Brien, L. F. Ollmann, Alex Orden, Oystein Ore, E. F. O'Shea, J. C. Oxtoby, D. K. Pease, W. H. Pell, J. L. Penez, A. J. Penico, P. M. Pepper, I. E. Perlin, H. P. Pettit, C. R. Phelps, H. L. Platzer, J. C. Polley, William Prager, Walter Prenowitz, G. B. Price, M. H. Protter, Tibor Radó, J. F. Randolph, G. E. Raynor, C. J. Rees, M. S. Rees, Eric Reissner, Helene Reschovsky, C. N. Reynolds, Harris Rice, H. G. Rice, D. E. Richmond, C. E. Rickart, G. de B. Robinson, S. L. Robinson, G. B. Robison, W. L. Root, I. H. Rose, Alex Rosenberg, P. C. Rosenbloom, Maxwell Rosenlicht, Arthur Rosenthal, J. C. Rothe, H. L. Royden, Herbert Ruderfer, Walter Rudin, Charles Saltzer, James Sanders, Arthur Sard, Leo Sario, S. S. Saslaw, J. A. Schatz, F. J. Scheid, M. A. Scheier, E. V. Schenkman, E. C. Schlesinger, E. R. Schneckenburger, I. J. Schoenberg, A. L. Schurrer, Abraham Schwartz, B. L. Schwartz, C. H. W. Sedgewick, I. E. Segal, Esther Seiden, Seymour Sherman, Harold Shulman, I. M. Singer, Maurice Sion, M. L. Slater, E. C. Smith, P. A. Smith, Andrew Sobczyk, E. S. Sokolnikoff, J. J. Sopka, Joseph Spear, D. E. Spencer, C. J. Standish, M. E. Stark, E. P. Starke, J. R. K. Stauffer, N. E. Steenrod, S. K. B. Stein, R. L. Sternberg, F. M. Stewart, R. W. Stokes, D. J. Struik, J. S. Stubbe, R. L. Swain, Otto Szász, Gabor Szegö, R. C. Taliaferro, J. T. Tate, Olga Taussky, J. J. Taylor, W. C. Taylor, G. B. Thomas, J. M. Thomas, G. L. Thompson, D. L. Thomsen, R. M. Thrall, W. J. Thron, D. E. van Tijn, John Todd, M. M. Torrey, J. I. Tracey, A. W. Tucker, Robert Ullman, E. P. Vance, H. E. Vansant, A. H. Van Tuyl, M. F. Vaudreuil, T. L. Wade, D. H. Wagner, R. W. Wagner, S. E. Walkley, J. L. Walsh, H. C. Wang, W. H. Warner, S. E. Warschawski, M. T. Wechsler, J. V. Wehausen, Herschel Weil, H. F. Weinberger, M. J. Weiss, B. A. Welch, David Wellinger, John Wermer, G. N. White, G. W. Whitehead, P. M. Whitman, G. T. Whyburn, D. V. Widder, A. B. Willcox, W. L. Williams, Henry Wolf, Jacob Wolfowitz, E. S. Wolk, Bertram Yood, D. M. Young, Oscar Zariski, Arthur Zeichner, Daniel Zelinsky, J. L. Zemmer, J. A. Zilber.

The twenty-fifth Josiah Willard Gibbs Lecture, entitled *Some basic theorems on the foundations of mathematics and their philosophical implications*, was delivered by Professor Kurt Gödel of the Institute for Advanced Study on Wednesday evening, December 26. Professor John von Neumann, President of the American Mathematical Society, presided.

Two addresses were presented at the invitation of the Committee to Select Hour Speakers for Annual and Summer Meetings. On

Thursday afternoon, December 27, Professor Shizuo Kakutani of Yale University spoke on *Brownian motion*. Professor J. L. Doob presided. On Friday afternoon, December 28, Professor G. P. Hochschild of the University of Illinois and Yale University spoke on *Cohomology in algebraic number fields*. Professor Saunders MacLane, President of the Mathematical Association of America, presided.

The annual Business Meeting and Election of Officers was held at 3:15 P.M., Friday, December 28, with President John von Neumann in the chair. Details of the proceedings are reported below.

The members of the Society had the opportunity of visiting the new offices of the Society in the building at 80 Waterman Street, Providence.

The members of the Society and the Association were entertained at tea by the ladies of the Department of Mathematics, of the Graduate Division of Applied Mathematics, and the Department of the History of Mathematics on Wednesday afternoon, December 26.

On Thursday afternoon, December 27, there was a conducted tour of Pendleton House, a museum maintained by the Rhode Island School of Design. In the evening a film program was presented in the Faunce House Theatre.

The dinner for members of the Society, the Association, and their guests was held in the University Refectory on Friday evening, December 28. Professor C. R. Adams, toastmaster, introduced Dr. Henry Wriston, President of Brown University, who welcomed the organizations and made an address. He was followed by Professor John von Neumann, President of the American Mathematical Society, and Professor Jewell H. Bushey, Vice President of the Mathematical Association of America. The toastmaster recognized Professor C. B. Allendoerfer of the University of Washington who introduced a resolution expressing the appreciation and thanks of the Society and Association to Brown University, the Committee on Arrangements, and all who had helped in making the meetings successful and enjoyable. This resolution was adopted by a rising vote.

The Annual Business Meeting of the Society was held on Friday afternoon, December 28, 1951. The Secretary reported on the affairs of the Society.

The President announced that the Frank Nelson Cole Prize in the Theory of Numbers had been awarded "to Paul Erdös, for his many papers in the Theory of Numbers, and in particular for his paper *On a new method in elementary number theory which leads to an elementary proof of the prime number theorem*, Proceedings of the National Academy, vol. 35, pp. 374–385, July 1949, in which he makes important contributions to the new elementary theory of primes inaugurated by A. Selberg."

At the annual election, in which 639 votes were cast, the following officers were elected.

President Elect, Professor G. T. Whyburn.

Vice Presidents, Professors Richard Brauer and Deane Montgomery.

Associate Secretaries, Professors J. W. Green, W. M. Whyburn, and J. W. T. Youngs.

Member of the Editorial Committee of the Proceedings, Professor G. A. Hedlund.

Member of the Editorial Committee of the Transactions, Professor Saunders MacLane.

Member of the Editorial Committee of the Colloquium Publications, Professor Einar Hille.

Member of the Editorial Committee of Mathematical Reviews, Professor Hassler Whitney.

Member of the Editorial Committee of Mathematical Surveys, Professor R. J. Walker.

Members-at-large of the Council, Professors L. V. Ahlfors, C. B. Allendoerfer, R. H. Bing, E. R. Lorch, and J. C. Oxtoby.

Members of the Board of Trustees, Professors Einar Hille and P. A. Smith.

The Council met on Thursday evening, December 27, 1951.

The Secretary announced the election of the following twenty-three persons to ordinary membership in the Society.

Miss Lenora Yates Amos, Jarvis Christian College, Hawkins, Texas;
Mr. Randall Murray Conkling, Assistant, University of Florida;
Mr. John Edward Dutt, Columbia University;
Mr. Franz Edelman, Research Engineer, R.C.A. Victor Division, Indianapolis, Indiana;
Mr. Irving Leonard Glicksberg, Assistant Mathematician, Rand Corporation, Santa Monica, California;
Mr. Harold A. Goldberger, Brown University;
Mr. Louis Grossberg, Computer, New York University;
Mr. Isidore Heller, Research Associate, George Washington University;
Mr. Chaim Samuel Honig, Assistant, University of São Paulo, São Paulo, Brazil;
Assistant Professor Chen-Jung Hsu, Mathematical Institute, National Taiwan University, Taipei, Formosa;
Mr. Martin Morris Lipschutz, Evans Signal Laboratory, Belmar, New Jersey;
Mr. Geoffrey Stuart Stephen Ludford, University of Maryland;
Mr. LeRoy Frederick Meyers, Stanford University;
Mr. John David Neff, Purdue University;
Mr. Wallace Eldon Parr, Naval Ordnance Laboratory, White Oak, Maryland;
Mr. Robert Benedict Reisel, Northwestern University;
Mr. Wimberly C. Royster, University of Kentucky;

Mr. Douglas Howerth Shaffer, Carnegie Institute of Technology;
Associate Professor Jesse William Smith, Department of Physics, Roosevelt College of Chicago, Chicago, Illinois;
Mr. William Cady Stone, Union College, Schenectady, New York;
Mr. Nick Hampton Vaughan, Mathematician, Indianapolis Naval Ordnance Plant, Indianapolis, Indiana;
Mr. Robert Weinstock, Stanford University;
Mr. Emmet Finlay Whittlesey, Bates College.

It was reported that the following ten persons had been elected to membership on nomination of institutional members as indicated:

Lehigh University: Mr. Robert A. Chisholm Lane.
University of Maryland: Miss Ruth Margaret Davis and Mr. Donald Greenspan.
Michigan State College: Mr. Dale Marsh Mesner, Mr. Edward Stafford Northam, and Mr. James Henry Powell
University of Minnesota: Mr. Robert Edwin Zink.
Oregon State College: Mr. Verner Emil Hoggatt, Jr.
Swarthmore College: Mr. Louis Marshall Winer.
Wellesley College: Miss Jacqueline Pascal Evans.

The Secretary announced that the following had been admitted to the Society in accordance with reciprocity agreements with various mathematical organizations: London Mathematical Society: Albert Edrei, Assistant Professor, University of Colorado, and John Arthur Jacobs, Associate Professor, University of Toronto; Matematisk Forening, Copenhagen: Dr. Thøger S. V. Bang, Amanuensis, Copenhagen University; Unione Matematica Italiana: Dr. Luigi Gatteschi, Visiting Research Associate, Stanford University; Wiskundig Genootschap te Amsterdam: Evert Willen Beth, Professor, University of California, Berkeley.

The Alabama Polytechnic Institute, Auburn, Alabama; Florida State University, Tallahassee, Florida; The University of Mississippi, Oxford, Mississippi; and the State College of Washington, Pullman, Washington, were elected to institutional membership.

The Secretary is pleased to report at this time that the ordinary membership of the Society is now 4458, including 353 nominees of institutional members and 44 life members. There are also 111 institutional members. The total attendance at all meetings in 1951 was 2165; the number of papers read was 636; there were 19 hour addresses, 1 Gibbs Lecture, 4 Colloquium Lectures, and 15 papers at the Applied Mathematics Symposium. The number of members attending at least one meeting was 1377.

The following appointments by the President were reported: as a committee to determine the facts concerning the loyalty oath requirements at the Oklahoma Agricultural and Mechanical College and the University of Oklahoma and their effects on mathematics and mathematicians: W. E. Duren, Chairman, G. M. Ewing, and J. F. Randolph; as a member of the Committee on Printing Contracts for the period 1952–1954: Professor J. R. Kline (committee now consists of Professors C. J. Rees, Chairman, R. A. Johnson, J. R. Kline, and H. M. MacNeille, ex-officio); as a member of the Committee on Places of Meetings for the period 1952–1954: Professor M. F. Smiley (committee now consists of Professor J. M. Thomas, Chairman, Orrin Frink, and M. F. Smiley); as members of the Committee on Applied Mathematics (for a period of three years beginning January 1, 1952): Professors F. J. Murray and Eric Reissner (committee now consists of Professors M. H. Martin, Chairman, R. V. Churchill, K. O. Friedrichs, John von Neumann, F. J. Murray, and Eric Reissner); as tellers for the 1951 annual election: Professors A. A. Bennett and Rohn Truell; as a member of the Committee to Select Hour Speakers for Summer and Annual Meetings for the period 1952–1953: Professor Tibor Radó (committee now consists of Professors E. G. Begle, Chairman, Deane Montgomery, and Tibor Radó); as a Committee to Select Hour Speakers for Eastern Sectional Meetings for the period 1952–1953: Professor R. D. Schafer (committee now consists of Professors L. W. Cohen, Chairman, M. H. Heins, and R. D. Schafer); as a Committee to Select Hour Speakers for Southeastern Sectional Meetings for the period 1952–1953: Professor Wallace Givens (committee now consists of Professors W. M. Whyburn, Chairman, B. J. Pettis, and Wallace Givens); as a Committee to Select Hour Speakers for Far Western Sectional Meetings for the period 1952–1953: Professor R. P. Dilworth (committee now consists of Professors W. T. Puckett, Chairman, A. P. Morse, and R. P. Dilworth); as Editor of Volume V of the Proceedings of the Symposium on Applied Mathematics: Professor A. E. Heins.

In an appendix to this report are excerpts from the report of the Treasurer for the fiscal year 1951 as verified by the auditors. A copy of the complete report will be sent, on request, to any member of the Society.

The Secretary reported that the Board of Trustees had appointed Dr. H. M. MacNeille (Chairman), Professor E. G. Begle, Professor Jekuthiel Ginsburg, and Dean A. E. Meder, Jr., a committee to advise the Council and the Board of Trustees concerning the problem of exchanges of the Society's publications. The Board of Trustees also called to the attention of the Council the fact that its action on September 4 in Minneapolis in placing the Executive Editor of Mathematical Reviews in charge of exchanges was in violation of Section 3 of Article 1 of the by-laws of the Society.

The Secretary reported that the following had accepted invitations to deliver addresses: Professor Alexander Weinstein at the meeting in New York City, February 23, 1952; Professors E. R. Kolchin and Marston Morse at the meeting in New York City on April 25–26, 1952; Professor S. M. Ulam at the meeting in Chicago on April 25–26, 1952; and Professor I. J. Schoenberg at the meeting in Fresno on May 3, 1952.

The Council voted to approve the following dates of meetings of the Society in 1952: October 25, 1952, at Yale University; November 29, 1952, at the University of Southern California.

After reviewing its actions concerning the University of California, the Council voted to request the President to appoint a committee of five to consider problems of controversial questions both as to procedures of the Council and as to the membership of the Society.

The Bulletin Editorial Committee reported that the journal used 520 of the 600 pages authorized in 1951, but that there had been a large influx of manuscripts of invited addresses at the end of 1951. The Council voted to recommend to the Board of Trustees that a total of 680 pages be authorized for the 1952 Bulletin.

The Transactions and Memoirs Editorial Committee reported that the Transactions published just a few pages less than the 1100 pages authorized for 1951. The interval between receipt of manuscripts and publications is approximately nine months. The Council voted to recommend to the Board of Trustees that 1100 pages be published in the Transactions for 1952.

The Mathematical Reviews Editorial Committee reported that 892 pages had been published in 1951. The subscription list as of November 1951 was 2125.

The Proceedings Editorial Committee reported that the interval between receipt of manuscripts and publication is now approximately one year. The Council voted to recommend to the Board of Trustees that 1000 pages be authorized for the 1952 Proceedings. Professors Bjarni Jónsson and A. E. Taylor were reported as new Assistant Editors for the Proceedings.

The Colloquium Editorial Committee recommended that Professor Antoni Zygmund be invited to deliver the Colloquium Lectures at the 1953 summer meeting. The Council voted to approve this recommendation.

The Council voted to invite Professor Marston Morse to deliver the Josiah Willard Gibbs Lecture at the annual meeting in 1952.

The Council voted to direct its Secretary to obtain, as a condition of holding a meeting, assurances that at any event scheduled in the program there will be no discrimination as to race, color, religion, or nationality, and that when accommodations and other facilities are provided these shall be provided to all attending the meeting.

The following resolution was read:

> It is with great regret that the Council of the American Mathematical Society finds that the transfer of the office from New York to Providence involves the loss to the Society of the services of Miss Evelyn Hull. For seventeen years she has served the Society as Chief Clerk of the New York Office. In this capacity she has done extraordinary work in all branches of the Society's activity. Her work for the Librarian, for the Secretary, and for the Treasurer has been an invaluable aid in the performance of the duties of their positions; without her efficient and willing attention to details, no one of these officers could have performed the functions of his office in addition to his full-time academic duties. Her negotiations with the publishers of our journals and books have been carried on with great skill and never failing courtesy. She has given freely of her time and energy, devoting to the work of the Society many evening and week-end hours. She has won the respect and high regard of all with whom she was associated in the work of American mathematics.
>
> The Council of the Society hereby expresses to Miss Hull its deep sense of loss over her resignation and its wholehearted appreciation for her work for the advancement of mathematics. The Council wishes her every success and happiness in her future career.

The Council unanimously voted to adopt this resolution.

Presiding officers at the sessions for contributed papers were Professor Gabor Szegö, Dr. R. S. Burington, Professors Morris Marden, J. S. Frame, Einar Hille, Samuel Eilenberg, W. A. Hurwitz, G. H. Handleman, F. J. Randolph, R. E. Johnson, C. E. Rickart, J. L. Walsh, J. W. Green.

Abstracts of the papers read follow. Those having the letter "*t*" after their numbers were read by title. Paper number 128 was presented by Professor Copeland, 131 by Dr. Harary, 150 by Dr. Klee, 152 by Professor Edrei, 157 by Professor Marden, 164 by Dr. Lindgren, 166 by Dr. Danskin, 185 and 186 by Professor Schoenberg, 205 by Mr. Rosenfeld, 206 by Professor Walsh, 209 by Professor Clark, 211 by Professor Diaz, 212 by Dr. Gerst, 232 by Dr. Hammer, 234 by Dr. Sobczyk.

Professor Finzi and Mr. Ankeny were introduced by Professor I. E. Segal, Professor Hopkins by Professor Rohn Truell, Miss Schuepbach by Professor Sarvadaman Chowla, Dr. Peters by Professor J. R. Kline, and Mr. Kokoris by Professor A. A. Albert.

ALGEBRA AND THEORY OF NUMBERS

117. N. C. Ankeny: *Euclidean algorithm in Abelian extensions of the rationals.*

H. Davenport has shown that there are only a finite number of Euclidean fields, i.e. which have Euclidean algorithm, which are quadratic, cubic which are not totally real, or totally imaginary quartic extensions of the rationals. H. Heilbronn extended this to normal cubic fields. These results are generalized to Abelian extensions of the rationals and to certain types of normal extensions of the rationals. Also analogous results can be obtained if ground fields other than the rationals are taken. (Received November 13, 1951.)

118*t*. H. W. Becker: *Words isomorphic with compositions.*

These are subsets of rhyme schemes and permutations of n letters enumerated by $c_n=2^{n-1}$ (except $c_0=1$). Most of the breakdowns $c_{n,m}=(n-1, m-1)$, c_{n-m}, or $(n, 2m)$. Two of them involve the Fibonacci series with initial entries $\phi_0=1$, $\phi_1=0$ whose generating function is $y_0=1/(1-t\phi)=(1-t)/(1-t-t^2)$. If m is the number of rhyme scheme singletons, $c_{n+1,m}=c_{n,m}+c_{n-1,m}+c_{n,m-1}-c_{n-1,m-1}=\sum_0^m \phi_i\cdot c_{n-i,m-1}$, which has the generating function $y_m=1/(1-tc_{n,m})=t^m y_0^m+1$. If m is the position of the last singleton, then $c_{n,m}=\phi_{n-m}\cdot c_{m-1}$. Interpreting these rhyme schemes as products, the sum of all c_{n+1} such products is $\Psi_{n+1}=\sum_0^n a^{i+1}\cdot\Psi'_{n-i}=\Psi_n * \sum_0^n k$. Here Ψ' denotes transformation of all letters $a\to b$, $b\to c$, etc., and $*$ a selective multiplication in which k multiplies only those terms of Ψ_n whose alphabetically last factors are powers of k or $k-1$. There is a subset of the exponential polynomial Y_n of E. T. Bell (Ann. of Math. vol. 35 (1934) pp. 258–277) in whose metrical interpretation there remains an unsolved problem: to find the number of n line stanzas wherein each line of m syllables is balanced by at least one other line of the same length. The inverse lexicon theorem, for finding the word of given rank, employs binary digits. A word of odd rank ends with a couplet. (Received November 13, 1951.)

119. Volodymyr Bohun-Chudyniv: *On a method and general scheme for solution of the Euler problem.*

We consider a problem Euler proposed in a letter to Lagrange (L. Euler, *Opera Posthuma*, Acad. Petrop. 1862, vol. I, p. 576). The scheme for solution offered in this paper depends on the product of four quaternions: $(x_0+\sum_{\beta=1}^3 x_\beta i_\beta)k_\alpha(x'+\sum_{\beta=1}^3 x'_\beta i_\beta)=A'_{\alpha,1}+\sum_{\beta=1}^3 A'_{\alpha,\beta+1}i_\beta$; $(y_0+\sum_{\beta=1}^3 y_\beta i_\beta)k'_\alpha(y'_0+\sum_{\beta=1}^3 y'_\beta i_\beta)=A''_{\alpha,1}+\sum_{\beta=1}^3 A''_{\alpha,\beta+1}i_\beta$; $\alpha=1, 2$. The quantities sought are the $A'_{\alpha,j}$, $A''_{\alpha,j}$ ($\alpha=1, 2$; $j=1, 2, 3, 4$). By suitable choices of the x_j, x'_j, y_j, y'_j ($j=0, 1, 2, 3$) and k_α, k'_α two solution schemes due to Euler (Comm. Arith. I, pp. 427–443, Acad. Petrop. 1849), one due to C. Avery (Math. Miscellany, New York, p. 101, 1839), two due to G. K. Perkins (Ibid. pp. 102–105) are obtained as well as a scheme previously given by the author (V. Chudyniv-Bohun, *Solution for Euler's problem*, Ukrainian Free Acad. of Sci., Regensburg, 1947). (Received November 13, 1951.)

120. Samuel Bourne: *On the homomorphism theorem for semirings.*

A semiring is a system consisting of a set S together with two binary operations, called addition and multiplication, which forms a semigroup relative to addition, a semigroup relative to multiplication, and in which the right and left distributive laws hold. A semiring S is said to be semi-isomorphic to the semiring S', if S is homomorphic to S' and the kernel of this homomorphism is (0). *If the semiring S is homomorphic to the semiring S', then the difference semiring $S-I$ is semi-isomorphic to S', where I is the ideal of elements mapped onto* $0'$. The correspondence $\bar{s}\to s'$, where s' is the image of an element of the coset $\bar{s}$, is a semi-isomorphism of $\bar{S}=S-I$ into S'. This correspondence is certainly a homomorphism, for if $\bar{s}=(s_1, s_2, \cdots)$, then $s_1+i_1=s_2+i_2$, i_1 and i_2 in I, and $(s_1+i_1)'=s_1'=(s_2+i_2)'=s_2'$. $\bar{s}+\bar{t}\to s'+t'$ and $\bar{s}\bar{t}\to s't'$ follow from coset addition and multiplication. If $s \to 0'$, then $s_1+i_1=s_2+i_2$, i_1 and i_2 in I, and $s_1'=s_2'=0'$. Thus s_1 and s_2 are contained in I. This implies that the kernel of the homomorphism $s \to s'$ is $\bar{0}$ and this homomorphism is a semi-isomorphism. A previous result of the author (Proc. Nat. Acad. Sci. U.S.A. vol. 37 (1951) p. 164) is the converse of this theorem. (Received November 2, 1951.)

121*t*. Leonard Carlitz: *Congruences for the coefficients of hyperelliptic functions.* II.

We use the notation of the abstract of the first part of this paper (Bull. Amer. Math. Soc. Abstract 58-1-37). The main new result is the following. Put $(x/g(x))^\lambda = \sum_0^\infty \beta_m^{(\lambda)} x^m/m!$. Then $\sum_{i=0}^r (-1)^{r-i} C_{r,i} C_p^{r-i} \beta_{m+i(p-1)}^{(\lambda)} \equiv 0 \pmod{p^s}$, where $\lambda \geqq 2$; $m \geqq r \geqq r_0(p, \lambda)$, $s=[2^{-1}(r+1)]-\lambda_1$ and $1 \leqq \lambda_1 < \lambda$. In particular this result holds for the Bernoulli numbers of higher order as well as for the coefficients in $(x/\text{sn } x)^\lambda$, where sn x is the Jacobi elliptic function. (Received October 31, 1951.)

122*t*. Leonard Carlitz: *Some congruences for the Bernoulli numbers.*

The following theorem is proved. Let $(p-1)p^{e-1}|b$, $c=(p-1)n>re$; then $B^c(B^b-1)^r \equiv 0 \pmod{p^{re-k}}$, where $k=e$ for $r \leqq p-1$ or $r=p-1$, $e>1$; $k=2$ for $r=p-1$, $e=1$; k = least integer $\geqq (re+1)/p$ for $r \geqq p$. A similar result is proved for $\sigma^c(\sigma^b-1)^r$, where $\sigma_m=(1/m)(B_m+1/p-1)$ for $p-1|m$. (Received November 13, 1951.)

123*t*. Leonard Carlitz: *Some theorems on the Schur derivative.*

Schur (Preuss. Akad. Wiss. Sitzungsber. (1933) pp. 145–151) defined the derivative of a sequence $\{a_m\}$ by means of $\Delta a_m=(a_{m+1}-a_m)/p^{m+1}$; higher derivatives are defined in the obvious way. He proved that if $p \nmid a$, then the derivatives $\Delta^1 a^{p^m}, \cdots, \Delta^{p-1} a^{p^m}$ are all integral, while $\Delta^p a^{p^m}$ has the denominator p. A. Brauer gave another proof of these results. M. Zorn proved Schur's results by p-adic methods (Ann. of Math. (2) vol. 38 (1937) pp. 451–464) and indeed found the residue of $\Delta^r X_m \pmod{p^m}$, where $X_m=(x^{p^m}-1)/p^{m+1}$, $x \equiv 1 \pmod{p}$. In the present paper a simple elementary proof is given of Zorn's results as well as a number of similar results. For example, the residue of $\Delta^r a^{p^m}$ is determined. In addition several generalizations of Schur's theorem are proved, one being a generalization to algebraic numbers. Finally some applications are indicated to the Euler and Bernoulli numbers; for example, $\Delta^r E_{k+p^m}$ is integral for $p>2$, $r<p$, $r \leqq m$. (Received October 31, 1951.)

124*t*. K. T. Chen: *Iterated integrals of paths and exponential mappings.*

In a euclidean space E with coordinates $(x_1, \cdots, x_n)$, all piecewise smooth paths initiating from the origin form a semigroup Ω. Introduce α^{-1} and the relations $\alpha\alpha^{-1}=1$, $\alpha \in \Omega$; a group G is thus obtained from Ω. Let α be given by $\{x_1(t), \cdots, x_n(t)\}$, $0 \leqq t \leqq 1$. Define $I_{i_1 \ldots i_p}(\alpha)=\int_0^1\int_0^{t_p} \cdots \int_0^{t_2} dx_{i_1}(t_1) \cdots dx_{i_{p-1}}(t_{p-1})dx_{i_p}(t_p)$. Then $I_{i_1 \ldots i_p}(\beta\alpha\alpha^{-1}\gamma)=I_{i_1 \ldots i_p}(\beta\gamma)$, α, β, $\gamma \in \Omega$. Consequently $I_{i_1 \ldots i_p}(\alpha)$ is defined for α taken as an element of G. Let $X_1, \cdots, X_n$ be nonabelian indeterminates. The exponential mapping θ is a homomorphism with $\theta(\alpha)=1+\sum_{p=1}^\infty \psi_p(\alpha)$, $\psi_p(\alpha)=\sum_{i_1 \ldots i_p} I_{i_1 \ldots i_p}(\alpha)X_{i_1} \cdots X_{i_p}$. Let G_d consist of all $\alpha \in G$ with $\psi_1(\alpha)= \cdots =\psi_{d-1}(\alpha)=0$, i.e. $I_{i_1 \ldots i_p}(\alpha)=0$, $p<d$. Then G_d/G_{d+1}, $d \geqq 1$, is a vector space over the reals. All straight paths along the coordinate axes generate a subgroup H in G. H is a free product $R_1 * \cdots * R_n$, where each R_i is the additive group of the reals. For $a \in R_i$, $\theta(a)=1+aX_i+(aX_i)^2/2!+ \cdots$. Moreover $\bigcap_{d=1}^\infty H_d$

$=\{1\}$ ($H_d = H \cap G_d$), and each H_d/H_{d+1} is also a vector space over the reals. There is a free group $F \subset G$. By using results of Magnus and Witt, $F_d = F \cap G_d$ is identified with the dth lower central commutator subgroup of F. (Received November 13, 1951.)

125. Sarvadaman Chowla: *The rational points on a cubic curve.*

This paper contains remarks on the problem of determining the "rank" of a given cubic curve (see A. Weil, Bull. Soc. Math. France vol. 54 (1930) p. 182). (Received November 13, 1951.)

126*t*. Eckford Cohen: *Rings of arithmetic functions.*

An arithmetic function f relative to a positive integer r and a field F of characteristic 0 is a single-valued function such that $f(a) \in F$ if a is an integer and $f(a) = f(a')$ if $a \equiv a'$ (mod r). Under ordinary addition and Cauchy multiplication the set of all such functions forms a commutative algebra over F. Structure properties of this algebra are determined and a principle relating to additive representations is deduced. Corresponding results for $GF[p^n, x]$ are also obtained. (Received November 14, 1951.)

127*t*. Harvey Cohn: *Some remarks on fields of small discriminant.* Preliminary report.

It has been conjectured that for totally real fields of degree $n = (p-1)/2$ (p prime), the field of smallest discriminant is $R(\cos 2\pi/p)$. This is known to be true for $n = 2, 3$. It is here shown that the conjecture is false for $n = 6$ and $n = 18$, the counterexamples being $R(\cos 2\pi/7, 5^{1/2})$ and $R((6-6 \cos 2\pi/19)^{1/2})$ $R((7+2(7+2\cdot 5^{1/2})^{1/2})^{1/2})$. (Received October 10, 1951.)

128. A. H. Copeland and Frank Harary: *The extension of an arbitrary Boolean algebra to an implicative Boolean algebra.*

An implicative Boolean algebra is one which contains an implication or conditional which is appropriate to the theory of probability. The conditional has been defined as an inverse of a cross product operation which is associative, noncommutative, distributive with respect to the binary Boolean operations, satisfies left cancellation, and has the unit element as its unit. It is known that not all Boolean algebras are implicative, in fact it is shown here that any implicative Boolean algebra is atomless. The question then arises whether there is an essential restriction imposed by introducing such a conditional into an arbitrary Boolean algebra. The question is answered in the negative by the fact that the Boolean algebra can be enlarged so as to include the additional elements produced by the operations of the cross product and the conditional. To accomplish the extension, the Stone representation is modified to one in which all nonzero elements have the same cardinality. The cross product operation and its inverse are then defined in terms of mappings with respect to the points of the representation space. These mappings produce the desired new elements. (Received November 13, 1951.)

129. Trevor Evans: *Embedding theorems for multiplicative systems and projective geometries.*

The following theorems are proved. Any countable loop (quasigroup, groupoid) can be embedded in a loop (quasigroup, groupoid) generated by one element. Any countable semigroup can be embedded in a semigroup generated by two elements. Any finitely generated projective plane can be embedded in a projective plane generated by four points. These results correspond to recent theorems for groups, Graham Higman, B. H. Neumann, and Hanna Neumann, *Embedding theorems for groups*, J. London Math. Soc. vol. 24 (1949) pp. 247–254, and for nonassociative algebras, A. I. Zukov, *Reduced systems of defining relations in non-associative algebras*, Mat. Sbornik N.S. vol. 27 (1950) pp. 267–280. For all such theorems the author has the following interpretation, which, for simplicity, is stated for the semigroup case. Let F be the free semigroup on two generators. A suitable free subsemigroup on a countable set of generators is chosen and on these generators are imposed the defining relations of the semigroup to be embedded. Then these relations considered as relations in F do not imply any new relations in the subsemigroup and so F with these relations added is the required containing semigroup. (Received November 13, 1951.)

130. Franklin Haimo: *The FC-chain of a group*. Preliminary report.

Let G be a group and let subgroups $H(i)$ be defined, $i=0, 1, 2, \cdots$, where $H(0)=(e)$ and if $H(i)$ is defined then $H(i+1)$ is the complete inverse image of that subgroup in $G/H(i)$ which consists of the elements with only a finite number of conjugates. (See Baer, Duke Math. J. vol. 15 (1948) pp. 1021–1032.) The $H(i)$ turn out to be an ascending chain of strictly characteristic subgroups. A recent result due to B. H. Neumann (Proc. London, Math. Soc. (3) vol. 1 (1951) pp. 178–187) is extended to this chain. Questions involving nilpotency are also discussed. (Received November 13, 1951.)

131. Frank Harary and G. E. Uhlenbeck: *On some generalizations of rooted trees*.

The problem of counting linear graphs satisfying certain conditions has come up in certain problems in statistical mechanics, especially connected to the theory of condensation. The appropriate kind of graph is a "generalized Husimi tree." A partial solution of this problem has been secured. A Husimi tree of type n_2, n_3 is a connected linear graph with n_2 segments (cycles of length 2), n_3 triangles (cycles of length 3), and no other cycles, in which two different cycles have at most one common point. Husimi trees of type $n_2, n_3, \cdots, n_k$ are defined similarly. A rooted Husimi tree is one in which there is a preferred point. If $H(n_2, n_3)$ is the number of different rooted Husimi trees of type n_2, n_3 and $h(x, y)=\sum_{n_2,n_3=0}^{\infty} H(n_2, n_3)x^{n_2}y^{n_3}$, then an explicit functional equation for $h(x, y)$ has been found. An analogous functional equation has also been obtained for rooted Husimi trees of type $n_2, n_3, \cdots, n_k$. The methods used are due to Pólya, Acta Math. vol. 68 (1937) pp. 145–254. A Cayley tree is a linear graph with no cycles. A Cayley tree with n points is said to have r roots, if r of the points are distinguished from the remaining points and also from each other. Functional equations for these multiply rooted Cayley trees have been found which are generalizations of Cayley's equation for the number of singly rooted trees. (Received November 13, 1951.)

132. Melvin Henriksen: *On the free ideals of the ring of entire functions*. Preliminary report.

Let R denote the ring of entire functions. An ideal of R is called *fixed* if all the functions in it vanish at at least one common point. Otherwise it is called *free*. The maximal free ideals and their residue class fields have previously been investigaged by the author. (See Bull. Amer. Math. Soc. Abstract 51-4-325.—The contents of that abstract will appear in a subsequent issue of the Pacific Journal of Mathematics.) Further properties of free ideals are obtained. In particular it is shown that if f is any function with simple zeros only in a maximal free ideal M, then M contains uncountably many proper divisors of f with simple zeros only. The principal tools used are the two-valued measure functions of S. Ulam (Fund. Math. vol. 14 (1929) pp. 231–233) and A. Tarski (Fund. Math. vol. 15 (1930) pp. 42–50). (Received November 15, 1951.)

133. B. W. Jones: Automorphs of symmetric matrices.

The following theorem is proved: If S is an automorph of a nonsingular symmetric matrix A such that the matrices $S-I$ and $S+I$ are both singular and have ranks whose sum is not greater than n, the order of A, then $S^2=I$ and, if r is the rank of $S-I$, there is an n by r matrix U of rank r such that U^TAU is nonsingular and $S=I-2U(U^TAU)^{-1}U^TA$. This theorem holds in any field in which $2\neq 0$. For any n by r matrix U for which U^TAU is nonsingular, an S defined as above is an automorph. (Received November 13, 1951.)

134. Irving Kaplansky: *Representations of separable algebras.*

The author presents two results, the first of which is a new proof of a theorem of Johnson and Kiokemeister (Trans. Amer. Math. Soc. vol. 62 (1947) pp. 404–430), while the second is the Hilbert space analogue. (1) Let A be the ring of all linear transformations on a vector space of countable dimension. Then any representation of A on a vector space of countable dimension is faithful, a direct sum of irreducible representations, and continuous in the weak topology. (2) Let B be the algebra of all bounded operators on a separable Hilbert space. Then any *-representation of A on a separable Hilbert space is faithful, a direct sum of irreducible representations, and continuous in the strongest topology. (Received November 7, 1951.)

135. L. A. Kakoris: *New results on power-associative algebras.*

We call a commutative algebra A over F power-associative if the algebra $K[x]$ generated by every x of A_K is associative for every scalar extension K of F. With this definition the known structure theory for commutative power-associative algebras of characteristic prime to 30 (see A. A. Albert, *A theory of power-associative commutative algebras*, Trans. Amer. Math. Soc. vol. 69 (1950) pp. 503–527) is extended to include the cases of algebras of characteristic 3 and 5. Proofs of the results require the associativity of fourth and fifth powers in the case of characteristic 3 and the associativity of fourth and sixth powers in case the characteristic is 5. Albert's results indicated the possibility of the existence of simple commutative power-associative algebras of degree two not Jordan algebras. We construct two types of such algebras. The algebras have characteristic p. One of them has dimension $4p$ and is stable. The other has dimension $3p$ and provides the first known example of a simple nonstable commutative power-associative algebra. (Received December 24, 1951.)

136. W. G. Lister: *A structure theory of Lie triple systems.*

In a recent paper (Trans. Amer. Math. Soc. vol. 70 (1951) pp. 141–169) Jacobson has characterized subspaces of Lie algebras closed under triple products, called them Lie triple systems, and shown that every Jordan algebra can be regarded as a Lie triple system. In the present paper a structure theory similar to that of Lie algebras is developed for Lie triple systems. A radical is defined, the relations between it and the radical of an enveloping Lie algebra given, and the existence of a Levi decomposition is established. The class of simple Lie triple systems is shown to determine in the usual way all semi-simple systems, and the simple systems are classified. Finally, results similar to those of Jacobson for Lie algebras (Proceedings of the American Mathematical Society vol. 2 (1951) pp. 105–113) on the complete reducibility of Lie triple systems of linear transformations are obtained. (Received November 13, 1951.)

137. W. H. Mills: *On the nonisomorphism of certain holomorphs.*

Let G be a finite abelian group, H the holomorph of G, and N an arbitrary group with holomorph isomorphic to H. Write G as the direct product of its Sylow subgroups: $G=P_1\times P_2\times\cdots\times P_t$. Then N can be written as a direct product $N=J_1\times J_2\times\cdots\times J_t$, where the holomorphs of P_i and J_i are isomorphic. It is shown that a finite abelian group P_i of prime power order and a group that is not abelian cannot have isomorphic holomorphs. Hence N is abelian. In a previous paper (*Multiple holomorphs of finitely generated abelian groups*, Trans. Amer. Math. Soc. vol. 71 (1951) pp. 379–392) it was shown that two finitely generated abelian groups have isomorphic holomorphs if and only if they are isomorphic. It follows that N is isomorphic to G. Thus a finite abelian group and an arbitrary group have isomorphic holomorphs if and only if they are isomorphic. (Received October 25, 1951.)

138. P. M. Pepper: *An expected value of the number of pairs of twin primes $\leqq x$.*

For a given integer k, at least one of the integers $6k\pm 1$ is composite if and only if there exist a prime $p\geqq 5$ satisfying $p^2\leqq 6k+1$, an integer $b\geqq[(p+1)/6]$, and a unit $\epsilon=\pm 1$, such that $k=bp+\epsilon[(p+1)/6]$. This characterization describes a generalized sieve of Eratosthenes, G, which screens from the class of all positive integers precisely those integers k for which both of the numbers $6k\pm 1$ are primes $\geqq 5$. A study of sieves formed by superposing several general periodic sieves having relatively prime minimal periods leads to a generalization of the Euler ϕ function, which (a), under suitable specialization, shows that the Euler ϕ function has a much broader scope of applicability than is usually attributed to it, and (b) leads to certain invariants of composite sieves which implement the definition of a probability function by means of which is derived an expected value $E(x)=3+\sum_{i=1}^{m-1} D_iP_i+P_m(x-p_m^2)/6$ for the number of prime pairs $\leqq x$ wherein p_i is the ith prime $\geqq 5$, $D_i=(p_{i+1}^2-p_i^2)/6$, $P_i=\prod_{j=1}^{i}(1-(2/p_j))$, and m is the greatest integer for which $p_m^2\leqq x$. Moreover, $E(p_{i+1}^2)-E(p_i^2)\geqq p_{i+1}-p_i$, whereas, for any real N, there exist infinitely many values of i for which $p_{i+1}-p_i>N$. Finally, $E(p)_m^2\geqq p_m-2$, whence $E(x)>p_m-2$ for $x>p_m^2$. (Received December 26, 1951.)

139. H. J. Reiter: *Investigations in harmonic analysis.* I. *A generalisation of Wiener's theorem.*

Let G be a locally compact abelian group, with a denumerable fundamental system of neighborhoods of the identity, and let $\widehat{G}$ be the dual group. Let I be a closed ideal in L^1 and $\widehat{Z}_I$ the set of elements of $\widehat{G}$, where all Fourier transforms of functions in I are zero. The following is proved: Let $k(x)$ be in L^1 and $\widehat{Z}_k$ the set of zeros of its Fourier transform. If (i) $\widehat{Z}_I\subset\widehat{Z}_k$ and (ii) the intersection of the frontiers of $\widehat{Z}_I$ and $\widehat{Z}_k$ is *denumerable*, then $k(x)$ belongs to I. The proof is based on the theorem of Plancherel-Weil and the theory of bounded linear functionals on L^1; it is connected with that given by S. Mandelbrojt and S. Agmon (*Une généralisation du théorème* *tauberien de Wiener*, Acta Math. Szeged. vol. 12 (1950) part B, pp. 167–176) in the case of the group of the real numbers. (Received November 13, 1951.)

140t. H. J. Reiter: *Investigations in harmonic analysis.* II. *On functions orthogonal to closed ideals in* L^1.

Let G, I, $\mathfrak{Z}_I$ be defined as in part I. A bounded, measurable function ϕ is called orthogonal to I if $\int f(xy)\overline{\phi(y)}dy=0$ for all $f(x)$ in I. The following results are proved: Let ϕ be orthogonal to I. If $\mathfrak{Z}_I$ is finite, ϕ is a linear combination, with constant coefficients, of characters of G. If $\mathfrak{Z}_I$ is discrete, any uniformly continuous ϕ is almost periodic; moreover, *any* ϕ has a "Fourier series." The uniqueness theorem and "Bessel's inequality" hold. "Parseval's equality" is proved under certain restrictions on the "density" of $\mathfrak{Z}_I$. If G is the group of real numbers and $\mathfrak{Z}_I$ is discrete, any ϕ is the limit (almost everywhere) of uniformly bounded trigonometric polynomials. If the distance between any two elements of $\mathfrak{Z}_I$ is larger than some positive constant (or if $\mathfrak{Z}_I$ is the union of a finite number of sets satisfying that condition), then any ϕ is almost periodic B^2 (Besicovitch). The proofs are based mainly on the results in part I. (Received November 13, 1951.)

141t. H. J. Reiter: *Investigations in harmonic analysis.* III. *Closed ideals in* L^1 *and their homomorphisms.*

Let G be a locally compact abelian group. For any function $f(x)$ in L^1 the following is proved: $\inf \int \left| f(x) - \sum_1^N \lambda_n f(xy_n) \right| dx = \left| \int f(x)dx \right|$, where N ranges over all positive integers, $\lambda_1, \lambda_2, \cdots, \lambda_N$ range over all complex numbers satisfying $\sum_1^N \lambda_n = 0$, and $y_1, y_2, \cdots, y_N$ over the elements of G. This leads to the proof of the following theorem: Let I be any closed ideal in L^1. Any continuous homomorphism of I upon C, the field of complex numbers, is given by a Fourier transform; moreover, the isomorphism $I/I_0 \cong C$ is isometric (I_0 is the kernel of the given homomorphism and the Banach algebra I/I_0 is normed in the usual way). Another result is the following: If $\overline{G}$ is a homomorphic image of G, then the L^1-algebra on $\overline{G}$ is a homomorphic image of the L^1-algebra on G and the isomorphism $L^1(G)/I_0 \cong L^1(\overline{G})$ is isometric. The proofs are based on the Hahn-Banach theorem (complex case) and several lemmas, e.g.: If $\int f(x)dx=0$, then the integral equation $\int f(xy)\phi(x)dx=1$ (for all y in G) has no bounded, measurable solution ϕ. (Received November 13, 1951.)

142. G. de B. Robinson: *On a conjecture by J. H. Chung.*

The modular representation theory of the symmetric group is far more explicit than in the general case as a result of its relationship with the Young diagrams. Recently (Canadian Journal of Mathematics vol. 3 (1951) pp. 309–327), J. H. Chung conjectured that the number of ordinary (modular) irreducible representations of S_n belonging to a given p-block is independent of the p-core, and is determined only by the weight of the block for a given prime p. A proof of this statement can be given which utilizes the notion of *integration* (Robinson, Canadian Journal of Mathematics vol. 2 (1950) pp. 334–343; Murnaghan, Proc. Nat. Acad. Sci. U.S.A. vol. 37 (1951) pp. 55–58) as applied to the star diagram (Proc. Nat. Acad. Sci. U.S.A. vol. 37 (1951) pp. 694–696). (Received November 13, 1951.)

143. Eugene Schenkman: *On infinite Lie algebras.*

The object of this paper is to study Lie algebras whose underlying vector spaces need not be finite-dimensional. We assume, however, that the Lie algebra L under consideration has a nilpotent ideal M (i.e. $M^k=0$ for some k) such that (a) L/M is finite-dimensional and (b) if M' stands for $[M, M]$, then M/M' is a direct sum of ideals $P_{\alpha M'}$ of L/M' each of dimension less than j. Our results are then as follows: (1) an extension of Engel's theorem is true, namely, that L is nilpotent if all its elements are nilpotent. (2) L is nilpotent if each of its maximal subalgebras is an ideal or equivalently if the meet of all the maximal subalgebras contains $L'=[L, L]$. (3) An extension of the tower theorem is valid as follows: Let L have zero center and let Z be the center of $L^\omega=\bigcap_{i=1}^{\omega} L^i$. Let D_i be any algebra of the tower of derivation algebras $L=D_0, D_1, \cdots, D_n, \cdots$ of L, then $D_{i/Z}$ is isomorphic to a subalgebra of $D(L^\omega)$ the algebra of derivations of L^ω. Our result (2) is analogous to a result of Hirsch for infinite solvable groups in Proc. London Math. Soc. vol. 49 (1945) and (3) is an extension of our earlier result in Amer. J. Math. vol. 73 (1951), (Received December 26, 1951.)

144t. Dorothy Schuepbach: *Integer solutions of* $y^2=(x+a)(x+b)\cdot(x+c)$.

Nagell and Delaunay have found that all of the integer solutions of (1) $y^2=x^3+17$ are $(x, y)=(-2, \pm 3), (-1, \pm 4), (2, \pm 5), (4, \pm 9), (8, \pm 23), (43, \pm 282), (52, \pm 375), (5234, \pm 378{,}661)$. Hence, there are 16 solutions in integers. The author has found an equation of the type mentioned in the title, namely (2) $y^2=(x+9)(x-3)(x-6)$ which has at least 19 solutions in integers. They are: $(x, y)=(-9, 0), (-6, \pm 18), (-3, \pm 18), (1, \pm 10), (3, 0), (6, 0), (7, \pm 8), (9, \pm 18), (21, \pm 90), (51, \pm 360), (153, \pm 1890)$. Mordell has shown that (2) has a finite number of solutions in integers. (Received November 13, 1951.)

145t. Alfred Tarski: *On representable relation algebras*. Preliminary report.

Lyndon showed (Ann. of Math. vol. 51 (1950) pp. 707–729) that the class R of all relation algebras which are representable, i.e., isomorphic to proper relation algebras, cannot be axiomatically characterized by means of any—finite or infinite—system of algebraic identities. He raised the problem whether this class can be characterized by means of any system of arithmetical axioms, i.e., statements formalized within the lower predicate calculus; in other words, whether R is an arithmetical class or an intersection of arithmetical classes (cf. Bull. Amer. Math. Soc. Abstract 55-1-74). The solution is negative. This is an obvious consequence of the following results: (I) Let K be any class of algebras which is an intersection of arithmetical classes. If every finite set of elements of a given algebra $\mathfrak{A}$ can be (isomorphically) imbedded in an algebra belonging to K, then $\mathfrak{A}$ itself can be imbedded in an algebra belonging to K. (II) There is a non-representable relation algebra all finitely-generated subalgebras of which are representable. (III) Every subalgebra of a representable relation algebra is representable. (I) is a somewhat stronger formulation of a result contained in the Princeton University doctoral dissertation of Henkin (1947); (II) was found by Lyndon (op. cit.); (III) is obvious. (Received November 14, 1951.)

146t. N. A. Wiegmann: *Pairs of normal matrices with property* L.

A generalization of a theorem (due to Motzkin and Taussky) concerning pairs of hermitian matrices with property L is obtained. The theorem states that if two normal matrices A and B have property L, then they commute. The proof depends on a theorem in the earlier paper, and on the fact that if a normal matrix has its characteristic roots down the main diagonal, then it must be in diagonal form. (Received October 29, 1951.)

147. Daniel Zelinsky: *Operator-compact groups*. Preliminary report.

The concept of linear compactness for topological vector spaces [Lefschetz, *Algebraic topology*, p. 78] can be generalized to a concept of operator-compactness for groups with operators. Theorems analogous to the standard elementary theorems for compact spaces can be proved. These ideas are applied to prove a conjecture of the author's [Duke Math. J. vol. 18 (1951) p. 440]. In terms of these same ideas, a sufficient condition is given for the possibility of raising infinite collections of orthogonal idempotents from a ring modulo its radical to the ring itself. An example is given to show that some condition like operator-compactness is needed and that Wedderburn's theorem $A = S+N$ is not always extendible to infinite-dimensional algebras. (Received November 13, 1951.)

148. J. L. Zemmer: *Ordered algebras which contain divisors of zero*. Preliminary report.

Using the definition of ordered ring given by G. Birkhoff (*Lattice theory*, rev. ed., Amer. Math. Soc. Colloquium Publications, vol. 25, New York, 1948) one can construct ordered rings containing proper divisors of zero. In this paper ordered linear algebras are investigated. The main results are concerned with irreducible algebras of finite dimension which are neither nilpotent nor fields. Using a result of A. A. Albert (*On ordered algebras*, Bull. Amer. Math. Soc. vol. 46 (1940) pp. 521–522) and the Wedderburn principal theorem it is shown that if such an algebra A is ordered then (i) $A = B + eNe + U$, where B is a subfield of A, e the identity of B, N the radical of A, and U is either the set of all $x \in A$ such that $ex = 0$ or the set of all $x \in A$ such that $xe = 0$, (ii) one, but not both, of the subalgebras eNe, U may be zero, (iii) if $eU \neq 0$ $(Ue \neq 0)$ then e is a left (right) identity for U, (iv) U has either a right or a left basis over B, (v) eNe has both a right and a left basis over B, (vi) if x, y, z are nonzero elements of B, eNe, U respectively, then $|x| > |y| > |z|$, (vii) if the base field is a subfield of the real field then the elements of B commute with those of eNe. (Received November 13, 1951.)

ANALYSIS

149. Shmuel Agmon: *Complex variable Tauberians*. Preliminary report.

Let $f(s) = \int_0^\infty e^{-su} d\alpha(u)$ with $\alpha(u) \uparrow$ converge for Re $(s) > 0$. Suppose that $f(s)$ has $s = 0$ as its only singularity on the imaginary axis. Let $\phi(x) = \sum a_n x^n$ be a Taylor series converging for $|x| < 1$ and having $x = 1$ as its only singularity on the unit circle. Suppose, furthermore, that $f(s) - \phi(e^{-s})$ is regular at $s = 0$. Then, if the a_n's satisfy a very general regularity condition, the following holds: $\alpha(u) = \sum_{n \leq u} a_n + o(a_{[u]})$. In the above the condition that the origin is the only singularity of $f(s)$ on the imaginary axis can be replaced by: $s = 0$ is the dominant singularity on the imaginary axis in a certain sense. The condition on $\alpha(u)$ to be increasing can also be relaxed. The general result contains Ikehara's Tauberian as a very particular case. (Received November 9, 1951.)

150. R. D. Anderson and V. L. Klee: *Convex functions and upper semi-continuous collections.*

A real-valued function f is called *convex* if its domain D_f is an open convex subset of E^n and $f(tx+(1-t)y)\leqq tf(x)+(1-t)f(y)$ whenever $0\leqq t\leqq 1$ and $\{x, y\}\subset D_f$. A *C-collection* is an upper semicontinuous collection of compact convex sets in E^n. A theorem is proved concerning the dimensionality of certain subsets of C-collections, and with a result of Wilder [Bull. Amer. Math. Soc. Abstract 54-11-551] it shows that *no open set in E^n can be filled by a nontrivial* continuous *C-collection*. With each positive convex f there is associated a C-collection which is used to prove "smoothness" theorems for f. For each $x\in D_f$ there is a unique linear manifold $L_x\ni x$ maximal relative to having $f|D_f\cap L_x$ differentiable at x. (The dimension of L_x is k if and only if the hyperplanes supporting f at $(x, f(x))$ have $n-k$ "degrees of freedom.") It is proved that if $0\leqq k\leqq n$ and S_k is the set of all $x\in D_f$ for which dim $L_x\leqq k$, then *S_k is the union of countably many closed sets of finite k-dimensional Hausdorff measure.* (Carathéodory [p. 83 in Blaschke's *Kreis und Kugel*] and Reidemeister [Math. Ann. vol. 83 (1921) pp. 116–118] showed that if $n=2$ and $k=1$, then S_k has zero 2-dimensional measure.) Also included are some remarks on antipodal points of convex bodies. (Received November 13, 1951.)

151. Joseph Andrushkiw: *The reality interval of a polynomial.*

Let $f(z)$ be a polynomial with real coefficients. The problem is: To find necessary and sufficient conditions that all the zeros of $F(z)=f(z)+xz^{n+1}$, $x\neq 0$, be real. Without loss of generality let $f(z)=1+a_1z+\cdots+a_nz^n$. Denoting by $f_1(z)$ and $F_1(z)$ the reciprocal polynomials of $f(z)$ and $F(z)$ respectively, one obtains the equivalent problem: To find necessary and sufficient conditions that $F_1(z)=zf_1(z)+x$, $x\neq 0$, have all its zeros real. They are: 1. The derivative $(zf_1(z))'$ must have all its zeros real. 2. Maximum minimorum of $zf_1(z)$, N, must be not greater than the minimum maximorum of $zf_1(z)$, P, and, if $(zf_1(z))'$ has some multiple zeros $r_1, r_2, \cdots, r_k$, also 3. $r_1f_1(r_1)=r_2f_1(r_2)=\cdots=r_kf_1(r_k)$. For any $x\neq 0$ of the open interval $(-P, -N)$, which is termed "the reality interval of $f(z)$," the polynomial $F(z)$ has all its zeros real and distinct. If $x=-P$ or $x=-N$, $F(z)$ has some multiple zeros. These conditions are satisfied if $f(z)$ has all its zeros real and distinct. In this case $N<0$, $P>0$ and it can be shown that $-N$ is the smallest positive and $-P$ the absolutely smallest negative zero of the discriminant of $F(z)$. (Received November 6, 1951.)

152. R. J. Arms and Albert Edrei: *On the Padé table and continued fraction associated with certain mermorphic functions.*

The Padé table of the exponential function has been thoroughly investigated; its convergence properties are remarkably simple but it does not seem to have been noticed that other functions display a similar behaviour. The authors show that this is the case for functions of the form $\phi(z)=\prod_{\nu=1}^{\infty}\{(1+\alpha_\nu z)/(1-\beta_\nu z)\}$ $[\alpha_\nu\geqq 0, \beta_\nu\geqq 0, \sum(\alpha_\nu+\beta_\nu)<+\infty]$. For functions of the more general form $f(z)=\exp(\gamma z)\,\phi(z)$ $[\gamma\geqq 0]$, their assertions are slightly less precise but of the same general character. Their methods also yield information on the asymptotic behaviour of the elements k_n [which are necessarily positive] and l_n of the continued fraction associated with $f(z)$ [Perron, *Die Lehre von den Kettenbrüchen*, p. 322]. For instance, in the special case $\gamma=0$, they show that: $\sum l_\nu$ converges; $k_\nu\to 0$; $(k_1k_2\cdots k_n)^{1/n}=o(1/n)$. (Received November 8, 1951.)

153. Nachman Aronszajn: *Reproducing kernels in Banach spaces.*

Until now reproducing kernels were considered only for functional Hilbert spaces. An attempt is made here to extend this concept to Banach spaces. (Received November 23, 1951.)

154. R. G. Bartle: *Singular points of functional equations.* Preliminary report.

Let $\mathfrak{X}$ and $\mathfrak{Y}$ be real or complex Banach spaces and Φ be a continuous function on a neighborhood of the origin in $\mathfrak{X}\times\mathfrak{Y}$ to $\mathfrak{X}$. If $\Phi(0, 0)=0$ and if the partial differential $d_x\Phi(0, 0; dx)$ has a continuous inverse, Hildebrandt and Graves have shown that the equation $\Phi(x, y)=0$ can be solved explicitly for x in a neighborhood of the origin in $\mathfrak{Y}$. The author studies the case that $\Phi(x, y)$ has the form $x-K(x)+F(x, y)$ where K is a completely continuous linear transformation and F satisfies a Lipschitz-type condition assuring that it contains no linear terms in x. The Banach space problem is reduced to a system of implicit equations in y and n scalar variables, where n is the number of linearly independent eigenfunctions of K corresponding to the eigenvalue 1. This is similar to some recent work of J. Cronin. With the additional hypothesis that F can be written in a finite Taylor expansion with remainder and that $n=1$, it is possible to examine the existence, uniqueness, and reality of the solutions. This generalizes and extends some work of Erhard Schmidt and L. Lichtenstein on nonlinear integral equations. (Received November 13, 1951.)

155. Lipman Bers: *Mildly nonlinear partial difference equations of elliptic type.*

Let $F(x, y, z, p, q)$ be a smooth function, $|F_p|, |F_q| \leq A < +\infty$, $F_z \geq 0$, and set $\mathcal{L}[\phi]=\phi_{xx}+\phi_{yy}-F(x, y, \phi, \phi_x, \phi_y)$. A difference operator $\mathcal{L}_h$ is obtained by replacing the partial derivatives in $\mathcal{L}$ by differences. It is shown that the first boundary value problem for the equation $\mathcal{L}_h[\phi]=0$ has a unique solution, that this solution can be obtained by a Liebmann iteration method, and that it approximates the solution of the boundary value problem for $\mathcal{L}[\phi]=0$. These results are extended to the operator $\mathcal{L}[\phi]=A(x, y)\phi_{xx}+2B(x, y)\phi_{xy}+C(x, y)\phi_{yy}-F(x, y, \phi, \phi_x, \phi_y)$, $AC-B^2>0$. (Received August 23, 1951.)

156*t*. R. P. Boas: *Growth of analytic functions along a line.* Preliminary report.

If $\{\lambda_n\}$ is an increasing sequence of positive numbers of unit density, $|\lambda_n-n| \leq o(n)$. We may characterize the distribution of the sequence more precisely by requiring that $|\lambda_n-n| \leq \epsilon(n)$ with a given $\epsilon(n)$. It is shown that if $f(z)$ is regular and of exponential type in the right half-plane, and if $\limsup_{n\to\infty} \{\log |f(\lambda_n)|\}/\epsilon(\lambda_n) < \infty$, then $\limsup_{x\to\infty} \{\log|f(x)|\}/\epsilon(x) < \infty$ provided that $\lambda_n-\lambda_{n-1} \geq \delta>0$, $\log x=o(\epsilon(x))$, $\int^\infty t^{-1}\epsilon(t)dt$ converges, and $\epsilon(t)$ satisfies some monotonicity and regularity conditions (for example, $\epsilon(t)=t^a$, $0<a<1$). This result is intermediate between the cases $\limsup \lambda_n^{-1} \log |f(\lambda_n)| < \infty$ (V. Bernstein) and $f(\lambda_n)$ bounded (Duffin and Schaeffer), neither of which is included. Some applications will be given later. (Received November 9, 1951.)

157. F. F. Bonsall and Morris Marden: *Zeros of rational functions with self-inversive polynomial factors.*

The authors prove the following theorem concerning the rational function $\phi(z) = (fgh)/(FGH)$ where f, g, h, F, G, H are polynomials of degrees m, n, p, M, N, P respectively. Let C denote the unit circle $|z| = 1$; let γ and Γ be the circles of radii r and R tangent internally to C at a point ζ, and let δ and Δ be the circles of radii s and S tangent externally to C at ζ. Let f and F each have its zeros symmetric in C; let all the zeros of g lie inside γ, those of h outside δ, those of G outside Γ, and those of H inside Δ. Let $m+(n/r)-(p/s) > M+(N/R)-(P/S)$. Then $\phi'(\zeta) = 0$ if and only if ζ is a multiple zero of $\phi(z)$. The inequality may be replaced by an equality in certain cases. The use of this theorem and continuity leads to the following result. Let the above f and F have respectively q and Q zeros inside C and let all the zeros of g lie inside C and all those of G outside C. Let F have no multiple zeros and FG no zeros common with fg. Then, if $M+N < m+n$ and $Q \leqq n+q$, the derivative of $\phi(z) = (fg)/(FG)$ has exactly $m+n-q+Q-1$ zeros inside or on C. When $n = M = Q = 0$, this result reduces to a theorem due to A. Cohn. (Received November 13, 1951.)

158*t*. F. E. Browder: *On the existence, uniqueness, and multiplicity of solutions of nonlinear functional equations.* I.

Let X, Y be Banach spaces, T a mapping of X into Y such that $T = J+C+S$ where J is an isometry of X with Y, C completely continuous, and S satisfies a Lipschitz condition with constant less than one. Suppose that T is locally one-to-one and satisfies the following generalized a priori bound: For every pair y_1, y_2 in Y there exists a constant N such that if $Tx = ty_1+(1-t)y_2$ for any t, $0 \leqq t \leqq 1$, then $\|x\| \leqq N$. Then T is a homeomorphism of X onto Y. More generally if T is a local homeomorphism of the connected topological space X into a locally convex linear metric space and if the inverse image of every line segment in the image space is compact, then T is a homeomorphism onto. The a priori bound of the first result may be replaced by a local a priori inequality. These results are a generalization of local results obtained by Schauder to theorems in the large. (Received November 13, 1951.)

159*t*. F. E. Browder: *On the existence, uniqueness, and multiplicity of solutions of nonlinear functional equations.* II.

Let X, Y be Banach spaces, T a mapping of X into Y satisfying the conditions of I including the a priori bound except that T need be no longer locally one-to-one. Let R be the set of points in X in some neighborhood of which T is locally one-to-one, $S = Y-T(X-R)$. Suppose that $T(R)-T(X-R)$ is nonempty and either (a) R is connected or (b) S is connected. Then T is onto and a finite covering mapping of $R \cap T^{-1}(S)$ onto S. In particular the hypotheses on R, S are satisfied if $X-R$ is compact, $\dim(X) = \infty$. Analogous results can be formulated for locally convex linear metric spaces as in I. These theorems are applied to obtain results on the multiplicity of solutions of functional equations when local uniqueness is no longer present. (Received November 13, 1951.)

160*t*. F. E. Browder: *On the existence, uniqueness, and multiplicity of solutions of nonlinear functional equations.* III. *The generalized Dirichlet problem.*

Let $f(x_1, x_2, \cdots, x_n; z; \partial z/\partial x_1, \cdots, \partial z/\partial x_n; \partial^2 z/\partial x_1^2, \partial^2 z/\partial x_1 \partial x_2, \cdots, \partial^2 z/\partial x_n^2)$ be the general nonlinear elliptic partial differential operator of the second order on a domain D of E^n. Using the theorems of I and II, results are established on the existence, uniqueness, and multiplicity of solutions of the first boundary value problem for $f=0$. Assumptions are made on a priori bounds for the Hölder norms of second derivatives of solutions as well as on the local uniqueness of solutions. Theorems on the actual multiplicity of solutions are established in cases for which ramification may occur. The assumed a priori bounds are shown to hold for solutions of a general class of quasi-linear equations in the plane including the homogeneous quasi-linear equations treated by Leray and Schauder. (Received November 13, 1951.)

161. F. E. Browder: *The Dirichlet problem for self-adjoint linear elliptic equations of arbitrary even order with variable coefficients.*

Let L be a self-adjoint linear elliptic partial differential operator of order $2m$, $m \geqq 1$, on a bounded domain D in E^n. By the Dirichlet problem for L on D is meant the following: Given $g \in C^m(D)$, all of whose mth derivatives are square-integrable of D, to find $u \in C^{2m}(D)$, a solution of the equation $Lu=0$, such that the derivatives of $g-u$ of order less than m tend to zero in a suitable generalized sense on the boundary of D. It is shown that if L satisfies a certain positive definiteness condition and if the coefficients of the differential operators in L of order j have continuous derivatives up to orders $2m+j+2$, then the Dirichlet problem always has a solution and the solution is unique. The class of equations considered includes as a special case the equations with constant coefficients treated by L. Gårding. The methods used in the proof are an extension of the method of orthogonal projection as applied by Vischik and Gårding. (Received November 13, 1951.)

162*t*. F. E. Browder: *Weak and strong solutions of linear elliptic equations with variable coefficients.*

Let L be a linear elliptic differential operator defined on an arbitrary domain D in E^n. The function $v\,\mathfrak{L}^2(D)$ is said to be a weak solution of the equation $Lu=0$ if $\int v(x)\bar{L}(f(x))dx=0$ for all $f \in C^{\infty}(D)$ ($2m=$ the order of L) such that f vanishes outside a compact subset of D and $\bar{L}$ is the adjoint differential operator of L. It is shown that if the coefficients of the jth differential operators in L have continuous partial derivatives of all orders up to $j+2m+2$, then every weak solution v of $Lu=0$ is equal almost everywhere to a strong solution $u \in C^{2m}(D)$. This result is a generalization of the corresponding theorem for the case of constant coefficients proved by L. Gårding and L. Schwartz. Extensive use is made of the results of F. John on the fundamental solution of equations with analytic coefficients. (Received November 13, 1951.)

163. H. D. Brunk: *A convergence property of certain generalizations of the interpolatory cardinal series.*

The cardinal series as an interpolatory function may readily be generalized to give a function which assumes given values at equally spaced points on a line, and which satisfies one of a wide class of partial differential equations. The role of the Fourier transform of the cardinal series in obtaining conditions for approximation to a solution of the partial differential equation coinciding with a given function on a line is studied, and applications are made to special partial differential equations. (Received November 14, 1951.)

164. R. H. Cameron, B. W. Lindgren, and W. T. Martin: *Linearization of certain nonlinear functional equations.*

Relating to papers of Cameron and Martin (Trans. Amer. Math. Soc. vol. 66

(1949) pp. 253–283, and Ann. of Math. vol. 51 (1950) pp. 385–392) the present theorem shows that a minimizing process involving solutions of linear algebraic equations yields approximate solution of the equation $y(t)=x(t)+\Lambda(x|t)$, where $\Lambda(x|t)$ is in general a nonlinear functional satisfying certain smoothness conditions. These approximations can be made arbitrarily close to the true solution in the $L_1(C)$ sense. (Received September 27, 1951.)

165. Lamberto Cesari: *On the characterisation of Fréchet surfaces of finite Lebesgue area.*

Let $S\subset E_3$ be any continuous Fréchet surface, $T: x=T(w)$, $w\in Q$, any representation of S (Q simple Jordan region of the w-plane), $L(S)$ the Lebesgue area of S, T_1, T_2, T_3 the three plane mapping projections of T on the coordinate planes E_{23}, E_{31}, E_{12}, $W(T_s)$ the total variation and $N(x, y; T_s)$, $(x, y)\in E_{s}$, the characteristic function of T_s, $s=1, 2, 3$, $[S]$ the set of the points of S, C the boundary curve of S, R the set of the coordinate axes X, Y, Z. The inequality (A) $W(T_s)\leq L(S)\leq W(T_1)+W(T_2)+W(T_3)$, $s=1, 2, 3$, was proved by the author first with elementary methods for regular surfaces with $|[S]|=0$ (Mem. Acad. Italia vol. 12 (1941) pp. 1305–1397), then for all surfaces S (Annali della Scuola Normale Superiore di Pisa vol. 2 (1942p) p. 10–11, 253–294, 1–42) by using a lemma of combinatorial topology (L. Cesari, Rend. Ist. Lomb. vol. 75 (1941) pp. 267–291; S. Eilenberg, Bull. Amer. Math. Soc. vol. 53 (1947) pp. 1192–1195. The author has given now a new proof of (A) based on the following lemma: Let I_s be the set of points $(x, y)\in E_s$ such that $N(x, y; T_s)=0$ and such that the components of $T^{-1}(x, y)$ are continua of constancy for T. If $CR=0$, if $(0, 0)\in I_s$ and $(0, 0)$ is a point of density for I_s, $s=1, 2, 3$, then C is nullhomotop in E_3-R. This lemma can be proved in an elementary way. (Received November 13, 1951.)

166. J. M. Danskin and Leonard Gillman: *Explicit solution of a game over function space.*

The authors consider the zero-sum two-person game determined by the payoff $\pi(x, y)=\int_0^\infty Q_y(t)dQ_x(t)$ where $Q_x(t)=\exp\left[-\int_t^\infty x(\tau)\xi(\tau)d\tau\right]$ and $Q_y(\tau)=\exp\left[-\int_t^\infty y(\tau)\eta(\tau)d\tau\right]$, ξ and η being fixed continuous, strictly decreasing, positive, summable functions of t. The x-strategies are chosen from the space $\mathfrak{X}$ of measurable functions satisfying for a given positive X the conditions (a) $0\leq x(t)\leq 1$ and (b) $\int_0^\infty x(t)dt\leq X$. The y-strategies are chosen from the space $\mathfrak{Y}$ of measurable functions satisfying for a given positive Y the conditions (c) $0\leq y(t)\leq 1$ and (d) $\int_0^\infty y(t)dt\leq Y$. They first prove the existence of, and then obtain an explicit formula for, a (unique) saddle-point—i.e., a pair (x_0, y_0) with $x_0\in\mathfrak{X}$ and $y_0\in\mathfrak{Y}$ satisfying the condition (e) $\pi(x_0, y)\geq\pi(x_0, y_0)\geq\pi(x, y_0)$ for all $x\in\mathfrak{X}$ and $y\in\mathfrak{Y}$. (Received November 8, 1951.)

167. Melvin Dresher: *Moment spaces and inequalities.*

An integral inequality is interpreted geometrically as a condition that a given point lie in a space defined by the convex hull of a given curve. By characterizing the boundaries of various dimensions, integral inequalities are derived from the requirement that points of the space lie within elements of the boundaries. The Hölder, Minkowski, and Jensen inequalities are examples of the application of "lower boundaries" of a 3-dimensional general moment space. Using higher-dimensional moment spaces, addition inequalities are obtained, such as $\left[\int_0^1(f+g)^p d\phi/\int_0^1(f+g)^r d\phi\right]^{1/(p-r)}\leq\left[\int_0^1 f^p d\phi/\int_0^1 f^r d\phi\right]^{1/(p-r)}+\left[\int_0^1 g^p d\phi/\int_0^1 g^r d\phi\right]^{1/(p-r)}$ where $p\geq 1$, $1>r>0$, f and g are continuous functions, and ϕ is a distribution function over the interval (0, 1). The "upper boundaries" of the space yield such inequalities as the following closure of the Liapounoff inequality $(1-t^c)(1-\int_0^1 x^a d\phi) \leqq (1-t^a)(1-\int_0^1 x^c d\phi)$ where $a>b>c$ and t satisfies the equation $(1-t^b)(1-\int_0^1 x^c d\phi) = (1-t^c)(1-\int_0^1 x^b d\phi)$. (Received November 13, 1951.)

168. Joanne Elliott: *On a class of integral equations.*

The author discusses the integral equation $g(x) = \text{P.V. } \pi^{-1}\int_{-b}^{+b} f(t)(t-x)^{-1}dt + \lambda\int_{-b}^{+b} k(x, t)f(t)dt$, where g and k are given functions and f is unknown. The first integral is taken in the sense of a Cauchy principal value, and b may be infinite. It is shown that if the equation is modified formally by introducing a suitable weight function, it may be reduced in a natural way to a Fredholm equation. If b is finite, the transformation $Tg(x) = \text{P.V. } \pi^{-1}\int_{-b}^{+b} g(t)(b-t)^{-1/2}(b+t)^{1/2}(t-x)^{-1}dt$ maps the Hilbert space of functions which are square integrable with respect to $w(t) = (b-t)^{-1/2}(b+t)^{1/2}$ isometrically onto the Hilbert space of functions square integrable with respect to $1/w(x)$. With the aid of this transformation and expansions in Jacobi polynomials the desired reduction is obtained. In the case $b = \infty$, the weight function is identically 1, the transformation Tg becomes the Hilbert transform of g, and the polynomial expansions are replaced by the Fourier transform in $L_2(-\infty, +\infty)$. (Received November 5, 1951.)

169t. William Feller: *On positivity preserving semigroups of transformations on $C[r_1, r_2]$.*

The purpose of the paper is to characterize differential operators which generate positivity preserving semigroups on $C[r_1, r_2]$. It is shown that such an operator coincides (in a certain sense) almost everywhere with $a(x)d^2/dx^2+b(x)d/dx+c(x)$, but on an exceptional set the operator may be of quite different form. The theorem serves to characterize the parabolic differential equation and represents a new justification of the Fokker-Planck equation. [The paper is to appear in Annales Soc. Polonaise Mathématiques.] (Received November 13, 1951.)

170t. William Feller: *The parabolic differential equation and the associated semigroups of transformations.*

It is shown that an operator Φ can give rise to infinitely many semigroups of transformations in the same Banach space $\mathfrak{X}$. Each infinitesimal generator is a contraction of Φ to a set $\mathfrak{Z}$ dense in $\mathfrak{X}$. For a given Φ one can derive all sets $\mathfrak{Z}$ leading to continuous semigroups. This theory leads to an algebrization of the initial value problem for differential equations and to a definition of boundary conditions in terms of functional analysis. For the parabolic equation $u_t = \Phi(u)$ with $\Phi = a(x)d^2/dx^2 + b(x)d/dx+c(x)$ in $C[r_1, r_2]$ $(-\infty \leqq r_1 < r_2 \leqq \infty)$ n.a.s. conditions are given for uniqueness ("natural boundaries") and the totality of all possible homogeneous boundary conditions are derived in all other cases. They involve global functionals and differential operators of second order. In each case the true adjoint is derived in the conjugate space. In special cases this leads to the classical "adjoint equation" $v_t = \Phi^*(v)$ with $\Phi^* = d[ad/dx-b]/dx+c$, but in general this differential equation is replaced by a more general functional equation. All possible boundary conditions are again derived and they are found to be of quite different a form for Φ and Φ^*. The theory applies without distinction to so-called singular equations and answers several outstanding problems of diffusion theory. (Received November 13, 1951.)

171t. D. T. Finkbeiner: *On linear closures of convex sets*. Preliminary report.

This note investigates the relation between two notions of linear closure for convex sets in a linear space $\mathfrak{L}$ for which no topology is assumed. Definition 1 (Berg-Nikodým). A convex set A is *linearly closed* if and only if each line of $\mathfrak{L}$ intersects A in a set which is closed in the natural topology of the line. The *linear closure* (cl A) of A is the intersection of all linearly closed sets containing A. Definition 2 (Klee). The *weak linear closure* (lin A) of A is the set of all points x such that $[y, x)\subseteq A$ for some $y=y(x)\in A$. *Theorem.* $A\subseteq \text{lin } A\subseteq \text{lin}(\text{lin } A)\subseteq \cdots \subseteq \text{lin}^\nu A\subseteq \cdots \subseteq \text{cl } A$, and if equality holds somewhere in the chain, it holds thereafter. *Corollary.* A is linearly closed if and only if A is weakly linearly closed. *Theorem.* The following are equivalent: (1) $\mathfrak{L}$ is finite-dimensional, (2) $\text{lin } A=\text{lin}^2 A$ for every convex $A\subseteq\mathfrak{L}$, (3) $\text{lin } A=\text{cl } A$ for every convex $A\subseteq\mathfrak{L}$, (4) $\text{lin hull } (A, B)=\text{hull } (\text{lin } A, \text{lin } B)$ for all convex $A, B\subseteq\mathfrak{L}$, (5) $\text{lin } (A+B)=\text{lin } A+\text{lin } B$ for all convex $A, B\subseteq\mathfrak{L}$. Klee proved (Duke Math. J. vol. 18, pp. 443–466) the equivalence of (1) and (2). (Received November 13, 1951.)

172. R. E. Fullerton: *On the subdivision of surfaces into pieces with rectifiable boundaries.*

Let S be a nondegenerate Fréchet surface of finite Lebesgue area defined over a closed 2-cell. By utilizing a recent result of Cesari [Bull. Amer. Math. Soc. Abstract 57-3-188] it is shown that there exists a representation T of S defined over a unit square Q with the following property. There exists a nested sequence of partitions $\{P_k\}$ of Q into rectangles R_{ik} such that (i) the ratios of the shorter to the longer sides of the R_{ik} are bounded away from zero, (ii) for each $\epsilon>0$ there exists a k_ϵ such that the mesh of the partition P_k is less than ϵ for $k\geqq k_\epsilon$, (iii) $T(R^*_{jk})$ is a rectifiable continuous curve for each R_{ik}. (Received November 13, 1951.)

173. I. S. Gál: *On the resonance-method and its application to the principle of uniform boundedness.*

A great many results in various fields of analysis can be obtained from the well known principle of uniform boundedness of linear operations in Banach spaces. The usual proofs of these results are based on a powerful method due to H. Lebesgue (*resonance-method*) [Annales de Toulouse (3) vol. 1 (1909) p. 25], which can also be used to prove theorems which do not follow from the principle of uniform boundedness. This fact suggests the application of the resonance-method directly to the sequence of linear operations instead of using the category principle. This gives the following result: An operation $u(x)$ defined over a Banach space E to a normed vector-space E' is called *bounded* if $\|u(x)\|\leqq M\|x\|$, *homogeneous* if $\|u(\lambda x)\|=|\lambda|\cdot\|u(x)\|$ for every $x\in E$ and λ real. Thus the norm of the operation $u(x)$ can be defined as $|u|=\sup_{\|x\|\leqq 1}\|u(x)\|$. A sequence of bounded, homogeneous operations $u_n(x)$; $n=1, 2, \cdots$, is said to be *asymptotically subadditive*, if $\|u_n(x+y)\|\leqq\|u_n(x)\|+O(|u_n|\ \|y\|)$ uniformly in x, $y\in E$ and if $\inf_{y\in E}(\|u_n(x+y)\|+\|u_n(x)\|-\|u_n(y)\|)=o(|u_n|)$ for every fixed $x\in E$, but not necessarily uniformly in $x\in E$. Using the method of resonance one obtains the following: If the sequence $\{u_n(x)\}$, $n=1, 2, \cdots$, is asymptotically subadditive (hence E is supposed complete) and if $\limsup_{n\to\infty}\|u_n(x)\|<\infty$ for every $x\in E$, then the sequence of norms $|u_n|$ is bounded. (Received November 13, 1951.)

174. H. H. Germond: *Properties of trinomial coefficients.*

Certain trinomial coefficients are analogous to Bessel functions and satisfy corresponding recurrence relationships, difference equations, expansions, and summations. (Received November 13, 1951.)

175. Samuel Goldberg: *A singular diffusion equation.*

The Fokker-Planck partial differential equation $u_t=(au)_{xx}-(bu)_x$, $0<x<1$, $t>0$, for a probability density function $u=u(t, x)$ is studied, where $a=x(1-x)$, $b=\beta-(\alpha+\beta)x$, α and β are positive constants. This equation arises in the study of the diffusion of a gene through a population undergoing random Mendelian mating in the presence of reversible mutation (S. Wright, *Statistical genetics in relation to evolution*, Paris, 1939). Using a separation of variables technique, explicit solutions as infinite series of Jacobi polynomials are obtained corresponding to the usual conditions that the boundary be an absorbing or reflecting barrier. If α and β are both greater than one, the solution obtained is unique. Let $L(t)$ and $R(t)$ be the accumulated mass, at time t, at the boundaries $x=0$ and $x=1$, respectively. Boundary conditions, developed by W. Feller, relating these functions to the density $u(t, x)$ are applied and corresponding solutions obtained. In the case where α or β is less than one, it is shown that the stationary solutions may have a positive accumulation of mass at the boundaries. (Received November 13, 1951.)

176. J. W. Green: *On the level surfaces of potentials of masses with fixed center of gravity.*

Admissible distributions are those distributions of positive mass which lie in the closed unit sphere about the origin in three dimensions, have total mass 1, and have their centers of gravity at the origin. At a point P at distance a greater than 1 from the origin, the potential $u(P)$ of an admissible distribution satisfies the inequality $1/(1+a^2)^{1/2}\leqq u(P)\leqq a/(a^2-1)$. If a level surface, $u=u_0$, has minimum and maximum distances r and R, respectively, from the origin, where $r>1$, then $R\leqq(r^2+1)^{1/2}+(r^2+5)^{1/2}$. These bounds are sharp and are attained by the distribution consisting of diametrically opposed equal point masses on the surface of the sphere. The proofs depend only on the decreasing and convex nature of the function $1/r$, and so analogous bounds exist for the potential based on any decreasing convex function; for example, the logarithmic potential. (Received November 7, 1951.)

177t. C. A. Hayes: *Differentiation with respect to ϕ-pseudo-strong blankets and related problems.*

In his paper, *A theory of covering and differentiation*, Trans. Amer. Math. Soc. vol. 55 (1944) pp. 205–235, A. P. Morse considered the class of ϕ-strong blankets, whose covering families are disjointed, and showed, among other things, that these blankets may be used for the purpose of differentiating any given member of his class $\mathcal{V}$ with respect to ϕ. In the present paper we prove that these differentiability properties still hold for a weaker class of blankets, called ϕ-pseudo-strong, whose covering families are allowed to overlap somewhat, the measure of the overlapping being expressed by a certain integral. That Morse's ϕ-strong blankets form a proper subclass of these blankets is shown by a concrete example. A further example is constructed to show that the condition on the overlapping cannot be relaxed much further without a serious loss in differentiability properties. (Received December 4, 1951.)

178. R. V. Kadison: *A generalized Schwarz inequality and algebraic invariants for operator algebras.*

The pure state space of a C^*-algebra together with its set of representing functions on this space is shown to be a complete set of algebraic invariants for the Jordan algebra of self-adjoint elements in the C^*-algebra (Jordan C^*-algebra). The principal tool is a generalized Schwarz inequality which states: if ϕ is a linear order-preserving map of one C^*-algebra into another then $\phi(A^2) \geqq \phi(A)^2$ for A self-adjoint. It follows that a linear isomorphism between two C^*-algebras which preserves the identity and, together with its inverse, preserves order, is a C^*-(Jordan) isomorphism. It is then shown that an isometry between two Jordan C^*-algebras deviates from a C^*-isomorphism at most by multiplication with a self-adjoint unitary in the center of the image algebra. The function representation of a C^*-algebra on its pure state space yields a function algebra under pointwise multiplication if and only if the representation is an algebraic isomorphism with all continuous functions on the space of pure states. It is proved that a linear map of one C^*-algebra into another which preserves the identity operator and preserves absolute values (of self-adjoint operators) is a C^*- (Jordan) homomorphism. Extensions to the non-unit situation are treated. (Received September 25, 1951.)

179t. M. S. Klamkin: *On the real and imaginary parts of a positive real function.*

Bode has shown that if $Z(p)$ is a positive real function having no poles on the imaginary axis, and if $R(w)$ and $X(w)$ are the attenuation and phase characteristics respectively (i.e., $Z(iw)=R(w)+iX(w)$), then $R(w)-R(\infty)$ or $R(w)-R(0)$ can be expressed as a function of the phase characteristic $X(w)$ provided that $R(\infty)$ or $R(0)$ exist. These results are extended such that $R(w)-R(w_0)$ is expressed as a function of $X(w)$ but subject to the condition that $|Z(iw)/w^2| \to 0$ as $|w| \to \infty$. This latter result includes both of Bode's results. $R(w)-R(\infty)$ is obtained by setting $w_0=\infty$ provided that $|X(w_0)/w_0| \to 0$ as $|w_0| \to \infty$. $R(w)-R(0)$ is obtained by setting $w_0=0$ provided that $|w_0X(w_0)| \to 0$ as $|w_0| \to 0$. The expression for $R(w)-R(w_0)$ is established by evaluating a certain line integral along a suitable contour. (Received November 13, 1951.)

180. P. D. Lax: *Operator theoretic treatment of hyperbolic equations.* Preliminary report.

According to a theorem of Hille-Yosida the operator equation $dU/dt = AU$ has a solution tending to I as $t \to 0$ if and only if $\|(\lambda - A)^{-1}\| \leqq 1/\lambda + O(1/\lambda^2)$ for large real positive λ. This is applied to the mixed initial-boundary value problem for second order hyperbolic equations $u_{tt} = a_{ij}(x)u_{x_ix_j} + b_i(x)u_{x_i} + c(x)u + d(x)u_t = Mu + du_t$; u, u_t prescribed at $t=0$ in D, $u=0$ for all t on the boundary of D. The equation is reduced to first order by introducing $u_t = v$; the metric $\|[u; v]\|^2 = \int a_{ij}u_{x_i}u_{x_j} + \int v^2$ is chosen and the Hille-Yosida condition is verified. The solution thus constructed is twice differentiable provided the coefficients of M and the initial data have enough derivatives. Mixed initial-boundary problems for hyperbolic systems are also handled; the operators A that figure here have no finite spectrum. (Received August 27, 1951.)

181t. Walter Leighton: *On self-adjoint differential equations of second order.*

This paper contains a number of theorems dealing with the behavior of solutions of the differential equation (1): $[r(x)y']' + p(x)y = 0$. Typical results are a generalization of the Sturm separation theorem and the following: Let $r(x) = 1$ and suppose

$g^2(x) = x^2p(x) - 1/4$ is a continuous positive monotone function of x on the interval $a \leqq x < \infty$. Then a necessary condition that solutions of (1) be oscillatory on this interval is that $\int_a^\infty x^{-1}g(x)dx = \infty$. (Received December 5, 1951.)

182. Benjamin Lepson: *Note on non-monotone Dirichlet series.*

Let $\{\lambda_n\}$ be a sequence of non-negative real numbers. Consider the series (1) $\sum_{n=0}^\infty a_ne^{-\lambda_n s}$, where s is a complex variable and $\{a_n\}$ a sequence of complex numbers. The set E of points in the plane at which (1) converges is investigated, and the following theorem is proved: *The necessary and sufficient condition on the sequence $\{\lambda_n\}$ in order that E be a half-plane (or the whole plane or the empty set) plus a portion of its boundary for every sequence $\{a_n\}$ is that $\sum_{n=0}^\infty \left|e^{-\lambda_{n+1}\alpha} - e^{-\lambda_n\alpha}\right| < \infty$ for all positive α.* Let D be the interior of E. The determination of D is sufficient for many purposes. It is conjectured that D *is always a half-plane* (in the above sense). (Received November 13, 1951.)

183. Lee Lorch: *Asymptotic expressions for some integrals which include certain Lebesgue and Fejér constants.*

All integrals in this abstract are definite integrals over the interval $(0, \pi/2)$. Asymptotic expressions are found for the following integrals: $L(x, r) = (2/\pi)\int\left|\sin\ (2x+1)t\right|^r/\sin\ t dt$, $L_E(x, r) = (2/\pi)\int \cos^x t\left|\sin\ (x+1)t\right|^r/\sin\ t dt$, and $L_B(x,r) = (2/\pi)\int \exp\ (-2x\sin^2 t)\left|\sin\ (x\sin 2t+t)\right|^r/\sin\ t dt, r \geqq 1$. For $r=1$, these are the respective (extended) Lebesgue constants arising in the theory of Fourier series from summation by ordinary convergence, Euler's $(E, 1)$-means, and Borel's exponential method. For $r=2$, they are the respective (extended) "Fejér constants," which play the same role in the divergence theory of Fourier series as do the Lebesgue constants. It is shown that $L(x, r) = (2/\pi)m_r \log x + \alpha + O(1/x)$, $L_E(x, r) = (1/\pi)m_r \log x + \delta + O(1/x^{1/2}) = L_B(x, r)$, as x becomes infinite, where $m_r = (2/\pi)\int \sin^r tdt = (2^r/\pi) \cdot \{\Gamma(r/2+1/2)\}^2/\Gamma(r+1)$, and α, δ are certain (rather complicated) constants. (Received November 13, 1951.)

184*t*. G. G. Lorentz: *Hahn-Banach theorem for semi-groups and multiply subadditive functions.*

Let S be a semi-group with commutative and associative addition and with a set of real numbers a as operators. For any $x_0 \in S$ and any real α_0, $q(x_0) \leqq \alpha_0 \leqq p(x_0)$, there is a linear functional on S with $f(x_0) = \alpha_0$, $q(x) \leqq f(x) \leqq p(x)$. Here p is subadditive, q superadditive, $q(x+y) \leqq p(x) + q(y) \leqq p(x+y)$, and p, q are homogeneous. Necessary and sufficient conditions for set-functions $\Phi(e)$, $\Psi(e)$ are obtained under which $\Phi(e)$ is the least upper and $\Psi(e)$ is the greatest lower bound of $\phi(e)$ for a class of additive set-functions ϕ. Special cases are results of the author on multiply subadditive set-functions. Another application concerns the existence of positive functionals on S invariant under a semi-group of linear operators which leave invariant a cofinal sub-semigroup $S_0 \subset S$ (this contains a theorem of Krein and Rutman). (Received November 7, 1951.)

185. T. S. Motzkin and I. J. Schoenberg: *On lineal entire functions of n complex variables.*

A polynomial $P(z_1, \cdots, z_n)$ with real or complex coefficients is called *lineal* if it splits into a product of linear functions of the variables in the complex field of coefficients. A lineal polynomial $P(z_1, \cdots, z_n)$ is called *really lineal*, or *positively* *lineal*, provided the coefficients of its linear factors are all real, or all real and non-negative, respectively. A sequence of entire functions $\{f_k(z_1, \cdots, z_n)\}$ is said to converge regularly to a function $f(z_1, \cdots, z_n)$, provided the convergence holds uniformly in every bounded portion of space. We denote by $L^{(n)}$, $L_r^{(n)}$, $L_p^{(n)}$, the three classes of entire functions $f(z_1, \cdots, z_n)$ which are regular limits of a sequence of polynomials $\{P_k(z_1, \cdots, z_n)\}$ whose elements are all lineal, really lineal, or positively lineal, respectively. For $n=1$ the class $L^{(1)}$ comprises all entire functions $f(z_1)$, while the classes $L_r^{(1)}$, $L_p^{(1)}$ have been determined by Laguerre and Pólya [Pólya, Rendiconti di Palermo vol. 36 (1913) pp. 1–17]. In this paper a complete characterization of the lineal classes $L^{(n)}$, $L_r^{(n)}$, $L_p^{(n)}$ is obtained for every $n \geqq 1$, in terms of their Weierstrass product representations. (Received November 13, 1951.)

186. T. S. Motzkin and I. J. Schoenberg: *On quadral entire functions of n complex variables.*

Following an oral suggestion of E. G. Straus, the problems treated in the preceding abstract are generalized as follows. A polynomial $P(z_1, \cdots, z_n)$ is called *quadral* if it splits into a product of quadratic (or linear) functions in the complex field of coefficients. Similarly a quadral polynomial $P(z_1, \cdots, z_n)$ is called *really quadral*, or *positively quadral*, provided the coefficients of its quadratic factors are all real, or all real and non-negative, respectively. Let $Q^{(n)}$, $Q_r^{(n)}$, $Q_p^{(n)}$ denote the classes of entire functions $f(z_1, \cdots, z_n)$ which are regular limits of polynomials which themselves are quadral, really quadral, or positively quadral, respectively. These quadral classes are here determined for all $n=1, 2, \cdots$. Similar results are obtained for *cubal* and generally for *N-al* functions. (Received November 13, 1951.)

187t. E. J. Moulton: *Characterization of the solutions of certain differential equations.*

For an ordinary differential equation of the first order it is sometimes possible to determine the general features of the solution curves without an integration or numerical calculations. This paper discusses available methods, with illustrations in the solution of one type of differential equation: $y' = A \exp Q(x, y)$, where $Q(x, y)$ is quadratic in (x, y). (Received November 15, 1951.)

188. Jacqueline L. Penez: *Approximation by boundary values of analytic functions.* Preliminary report.

Let D be a finite domain of the z-plane bounded by n-simple analytic curves C. Let $L_p(C)$ be the class of all complex-valued functions f defined on C which have a finite Lebesgue integral, $(\|f\|_p)^p = \int_C |f(z)|^p |dz|$ where $p > 1$. The problem considered is to minimize $(\int_C |f(z) - g(z)|^p |dz|)^{1/p}$ where f is a given function in $L_p(C)$, and g ranges through the boundary functions on C, of functions analytic in D, which are in $L_p(C)$. F. Riesz, Kakeya, Doob, Golusin, P. R. Garabedian, Schiffer, Spitzbart, MacIntyre, and Rogosinski (unpublished) are among those who have considered problems of this type. The author shows that there exists a function, g_0, uniquely determined, neglecting sets of measure zero, which minimizes the above integral. The difference, $f - g_0$, is called a minimal function. A characterization formula for a minimal function is obtained. Results of J. L. Doob for the case where f is a rational function and D is a simply-connected domain have been generalized in part. Also, this extremal problem is shown to be related to the problem of maximizing $|L(f)|$ where $L(f) = \int_C f(z)\omega(z)dz$,

f is in $L_p(C)$ and is the boundary function of an analytic function in D and $\|f\|_p \leq 1$, ω is in $L_q(C)$, and $p+q=pq$. (Received November 8, 1951.)

189. G. O. Peters: *Binomial expansion of negative factorials.*

The conditions for the binomial expansion, $(a+b)^n=\sum_{i=0}^{\infty} {}_nC_i a^{n-i}b^i$, n real, to hold are well known. Defining the factorials of x by the relations $x^{(n)}=x(x-1)\cdots(x-n+1)$ and $x^{(-n)}=1/(x+1)(x+2)\cdots(x+n)=1/(x+n)^{(n)}$, we can readily show, by induction or by use of Newton's interpolation formula, that the binomial expansion $(a+b)^{(n)}=\sum_{i=0}^{n} {}_nC_i a^{(n-i)}b^{(i)}$ holds for $n=0, 1, 2, \cdots$. The author proves that the binomial expansion $(a+b)^{(-n)}=\sum_{i=0}^{\infty} {}_{-n}C_i a^{(-n-i)}b^{(i)}$, $n=1, 2, 3, \cdots$, holds provided the real part of $a+b$ is greater than $n-2$, i.e. $\Re[a+b]>n-2$. (Received November 14, 1951.)

190. M. H. Protter: *Uniqueness of the problem of Tricomi.*

Let $K(y)$ be a monotone increasing function with one continuous derivative; suppose $K(0)=0$. Consider the equation $(^*)$ $K(y)U_{xx}+U_{yy}=0$. Let D_1 be the domain bounded by a rectifiable arc α lying in the upper half of the xy-plane with end points on the x-axis at $(a, 0)$ and $(b, 0)$, and the segment of the x-axis from a to b. Let β and γ be the characteristic curves of $(^*)$ emanating from $(a, 0)$ and $(b, 0)$ respectively which meet at the point $((a+b)/2, c)$. Let D_2 be the domain bounded by β, γ and the segment of the x-axis from a to b. Suppose values are prescribed along the arcs α and β. Then it is shown that there is at most one solution of $(^*)$ in the domain $D=D_1+D_2$ assuming the prescribed values on α and β. (Received September 25, 1951.)

191t. E. D. Rainville: *Generating functions for Bessel and related polynomials.*

Krall and Frink (Trans. Amer. Math. Soc. vol. 65 (1949) pp. 100–115) studied generalized Bessel polynomials in some detail. Their work was without benefit of generating functions, except in the special case of the simple Bessel polynomials. Burchnall (Canadian Journal of Mathematics vol. 3 (1951) pp. 62–68) obtained a generating function for generalized Bessel polynomials. This paper obtains two generalized hypergeometric generating functions which contain as special cases two generating functions, one of them Burchnall's, for the generalized Bessel polynomials. The methods used are distinct from Burchnall's method. (Received November 13, 1951.)

192. P. C. Rosenbloom: *A sufficient condition for uniform convexity.*

A sufficient condition for uniform convexity is obtained in terms of the second differential of the norm of a Banach space. This yields a simple proof of the uniform convexity of the L_p spaces for $1<p\leq 2$. A simple proof of Clarkson's inequalities for $2\leq p<+\infty$ is also given. A different proof of the uniform convexity of these spaces, which overlaps in generality with the present ones, but which gives the best possible estimates for the quantities involved, was found by Beurling (unpublished work) in 1947. (Received November 16, 1951.)

193. H. L. Royden: *The interpolation problem for schlicht functions.*

Let $z_0, z_1, \cdots, z_n$ be $n+1$ points in the interior of the circle $|z|<1$. The interpolation problem for schlicht functions is the problem of determining the possible sets of values $w_0, w_1, \cdots, w_n$ which can be taken at $z_0, z_1, \cdots, z_n$ by a function $f(z)$

which is regular and schlicht in the circle $|z|<1$ and normalized by $f(z_0)=0$, $f(z_1)=1$. By minimizing the maximum modulus of the functions taking a given a set of values, it is shown that whenever such a set of values is taken by a schlicht function, this set of values is also taken by a schlicht function satisfying a hyperelliptic differential equation. In the particular case $n=2$, the region of variability of w_2 is a region whose image in the elliptic modular plane is a circle. The parameters of the circle can be expressed in terms of certain complete hyperelliptic integrals involving the points z_0, z_1, and z_2. These results hold true if the problem is refined by considering the different homotopy classes to which the functions $f(z)$ may belong. (In fact for $n=2$ the different fundamental regions in the elliptic modular plane correspond to the different homotopy classes of the function $f(z)$.) (Received November 5, 1951.)

194. Walter Rudin: *L^2-approximation by partial sums of orthogonal developments.*

Let $\|f\|=\{\int_0^1 f^2(x)dx\}^{1/2}$. For any set $\{\phi_n\}$, orthonormal on $[0, 1]$, put $s_n(f; x) = \sum_1^n \phi_k(x)\int_0^1 f(t)\phi_k(t)dt$. Let $V(f)$ be the total variation of f on $[0, 1]$, and put $N(f) = \text{l.u.b.}\ |(f(x)-f(t))/(x-t)|$ $(0\leqq x\leqq 1,\ 0\leqq t\leqq 1)$. Define $\mu_n=\text{l.u.b.}\ \|f-s_n\|/V(f)$, $\lambda_n=\text{l.u.b.}\ \|f-s_n\|/N(f)$, the l.u.b. being taken over all nonconstant $f\in L^2$. The main result of the paper is Theorem I: There exist absolute positive constants A and B such that $\mu_n>An^{-1/2}$ and $\lambda_n>Bn^{-1}$ for every complete orthonormal set $\{\phi_n\}$. The proof of Theorem I yields Theorem II: For every orthonormal set $\{\phi_n\}$, the sequences $\{V(\phi_n)\}$, $\{N(\phi_n)\}$ are unbounded. If the functions are arranged so that $\{V(\phi_n)\}$ is nondecreasing, then $V(\phi_n)>Cn^{1/2}$. If $\{N(\phi_n)\}$ is nondecreasing, then $N(\phi_n)>Dn$. Here C and D are absolute positive constants. In all the above cases, consideration of the trigonometric set shows that the orders of magnitude obtained for the lower bounds cannot be improved. (Received November 13, 1951.)

195. Charles Saltzer: *Ring conjugate functions.*

If $f(z)$ is a function which is single-valued and analytic in the region $\lambda<|z|<1$, and $f(z)=u_1+iv_1$ on $|z|=\lambda$ and $f(z)=u_2+iv_2$ on $|z|=1$ where u_1, v_1, u_2, and v_2 are real functions, then v_1 and v_2 are said to be the inner and outer Villat λ-ring conjugates of u_1 and u_2. Since there is no loss of generality, the case where u_1 is a constant is considered and representations of the λ-ring conjugates are discussed. It is shown that if the conjugate of u_2 exists in the usual sense, then the λ-ring conjugate also exists, and more generally, if the partial sums of the Fourier series of the conjugate (in the usual sense) of u_2 are bounded, then the Fourier series of the λ-ring conjugate of u_2 converges and represents u_2. In addition, inequalities comparing the integral means of v_1 and v_2 with the integral mean of u_2 are derived together with approximation theorems. Applications of these results to certain conformal mapping problems are given. (Received November 13, 1951.)

196. James Sanders: *Uniqueness theorems for some classes of fourth-order partial differential equations.*

The system of equations (1) $\mathcal{L}(u)=((\sigma(x)/\tau(y))u_x)_x+((\sigma(x)/\tau(y))u_y)_y=-(k/\tau(y))\phi_x$, (2) $\overline{\mathcal{L}}(v)=((\tau/\sigma)v_x)_x+((\tau/\sigma)v_y)_y=-(k/\sigma)\phi_y$, (3) $\phi=\sigma u_x+\tau v_y$, (4) $\mathcal{L}'(\phi)=((1/\sigma\tau)\phi_x)_x+((1/\sigma\tau)\phi_y)_y=0$, where $\sigma\tau>0$ and k is a constant, is considered. This system is a generalization of the equations for the displacements in the theory of elasticity. If $k\neq -1$, (4) is a consequence of (1), (2), (3). If u and v are given on the boundary C of a domain D, then the solution of the system is shown to be unique provided $k>-1$. The method used consists of applying Green's theorem to

$\int_D u\mathcal{L}(u)dxdy$ and $\int_D v\overline{\mathcal{L}}(v)d(x)dy$. The method can be extended to equations in n variables. If $k=-1$, uniqueness is shown by proving that $\int_D(1/\tau\sigma)\phi^2dxdy=0$. In the case $k=-1$, the above system is equivalent to the fourth-order equation $\mathcal{L}'[\sigma\tau\mathcal{L}'(\psi)]=0$, and so it has been proved that the solution of the boundary-value problem $\mathcal{L}'[\sigma\tau\mathcal{L}'(\psi)]=g(x, y)$ in D, $\psi=f(s)$, $\partial\psi/\partial n=g(s)$ on C, is unique. Finally a direct proof of the uniqueness of the boundary-value problem $L[f(x, y)L(\psi)]=g(x, y)$ in D, $\psi=f(s)$, $\partial\psi/\partial n=g(s)$ on C, is given. Here, $L(u)=(\sigma_1(x, y)u_x)_x+(\sigma_2(x, y)u_y)_y$, $\sigma_1\sigma_2>0$ and $f(x, y)>0$ in D. The method used here consists of applying Green's theorem to $\int_D\psi L[f(x, y)L(\psi)]dxdy$. All domains considered in the above theorems are bounded. (Received November 13, 1951.)

197. Arthur Sard: *Remainders as integrals of partial derivatives.*

The kernel theorem of an earlier paper (Acta Math. vol. 84 (1951) pp. 319–346) is applied to a number of particular functionals. The variety of integral formulas possible in each case is indicated. A lemma is established which simplifies the calculation of kernels. Among the functionals studied is $Rf=\pi f(0, 0)-\iint_{x^2+y^2\leq 1} f(x, y)dxdy$. Kernels k, k^* exist such that $Rf=\int_{-1}^{1}f_{2,0}(x, 0)k^*(x)dx+\iint_{x^2+y^2\leq 1}f_{1,1}(x, y)k(x, y)dxdy+\int_{-1}^{1}f_{0,2}(0, y)k^*(y)dy$ whenever f is a function with partial derivatives $f_{2,0}(x, 0)$, $f_{1,1}(x, y)$, $f_{0,2}(0, y)$ that are continuous in x, (x, y), y, respectively. The kernels k, k^* are given explicitly. (Received November 9, 1951.)

198. Leo Sario: *On the construction of mappings onto slits domains.*

The mapping of a planar Riemann surface R onto a horizontal slits domain, the existence of which follows by Dirichlet's principle, can be performed in a purely constructive way by the use of the linear operator method. Take the initial function $s=\text{Re}\,(1/z)$ in a parameter disc S_0, $s=0$ in $S_1=R-\bar{S}_0$. The condition $\int d\bar{s}=0$ is then fulfilled. There is a normal linear operator L_0 in $S=S_0+S_1$, associating with any harmonic function v on $\Gamma=R-S$ a harmonic function $u=L_0v$ on S such that $u=v$ on Γ, $D_S[u]-\int_{\Gamma_0+\Gamma_1} v\,d\bar{u}=\min$. This function can be directly constructed, using an exhaustion of S_1. The corresponding principal function p_0, characterized by $p_0-s=L_0(p_0-s)$ in S, is the real part of the desired mapping function. The vanishing of the total area of the slits is implied by $D_{S_1}[u]=\int_\Gamma v\,d\bar{u}$. For simply-connected surfaces with an arbitrary boundary, this provides a new proof of the Riemann mapping theorem. (Received November 13, 1951.)

199*t*. Seymour Sherman: *On a conjecture of Kakutani concerning doubly-stochastic matrices.*

A doubly-stochastic (d.s.) matrix is a real $n\times n$ matrix $P=(p_{ij})$ such that $p_{ij}\geq 0$, $1\leq i\leq n$, $i\leq j\leq n$; $\sum_j p_{ij}=1$, $1\leq i\leq n$, and $\sum_i p_{ij}=1$, $1\leq j\leq n$. Introduce a partial order among d.s. matrices (henceforth designated by P's with superscripts) by defining $P^1\prec P^2$ to mean there exists a d.s. matrix P^3 such that $P^1=P^3P^2$. Introduce a partial order among real vectors $a=(a_1, \cdots, a_n)$ of real n-dimensional space E by defining $a\prec b$ to mean for each real convex ϕ, $\sum_i\phi(a_i)\leq\sum_i\phi(b_i)$. It is known that for each real n-vector a, $P^1\prec P^2\rightarrow P^1a\prec P^2a$. Kakutani has raised the following conjecture: If, for each real n-vector a, $P^1a\prec P^2a$, then $P^1\prec P^2$. This conjecture is settled in the affirmative. (Received November 5, 1951.)

200*t*. Seymour Sherman: *On a paper by B. O. Koopman concerning a probabilistic generalization of matric Banach algebras.*

In a recent note Koopman [*A probabilistic generalization of matric Banach algebras*, Proceedings of the American Mathematical Society vol. 2 (1951) pp. 404–413], whose notation and definitions are followed here, showed that a certain set of functions $\phi(x, E)$ with appropriate operations and norm constitute a Banach algebra. Additional information as well as motivation for his theorem and a simplification of his proof is provided by observing with Yosida and Kakutani [*Operator-theoretical treatment of Markoff's process and mean ergodic theorem*, Ann. of Math. vol. 42 (1941) pp. 188–228]: THEOREM. Let (M^*) be the Banach space of bounded complex-valued measurable functions on X with $\|f\| = \sup_{x\in X}\{|f(x)|\}$. Then each $\phi(x, E)$ represents a bounded linear operator $\overline{T}_\phi$: $(M^*) \to (M^*)$ given by $\overline{T}_\phi f = g$ where $g(y) = \int_X \phi(y, dx) f(x)$, i.e., for fixed y integrate f with respect to the measure $\phi(y, E)$. In this representation the bound and algebraic operations involving operators go into Koopman's norm and operations. From the theorem above and the well known fact that complete algebras of bounded linear operators on a Banach space when normed by bound form Banach algebras, Koopman's result follows. (Received November 5, 1951.)

201. Seymour Sherman: *On a theorem of Hardy, Littlewood, Pólya, and Blackwell.*

Let $U = \{u\}$ be a real vector space. Let $a(b)$ be a measure on U with a finite spectrum, $S_a = \{u_i \mid 1 \leq i \leq n_a\}$ $(S_b = \{u^j \mid 1 \leq j \leq n_b\})$. Define $b \prec_1 a$ to mean for each real convex $\phi \mid U$, $\sum_i a(u_i)\phi(u_i) \geq \sum_j b(u^j)\phi(u^j)$. Define $b \prec_2 a$ to mean $b(u^j)u^j = \sum_i p_{ij} a(u_i) u_i$, where $p_{ij} \geq 0$, $\sum_i p_{ij} a(u_i) = b(u^j)$, and $\sum_j p_{ij} = 1$. From the definition of convex function and the properties of p_{ij} it follows immediately that $b \prec_2 a \rightarrow b \prec_1 a$. This note is devoted to the proof of: THEOREM. $b \prec_1 a \rightarrow b \prec_2 a$. (Received November 5, 1951.)

202. M. L. Slater: *On a duality property*. Preliminary report.

Let $\alpha(x)$ and $\beta(x)$ be fixed functions $\in C[0, 1]$. Let F be the set of non-negative $f \in C'[0, 1]$ which satisfy $-f'(x) \leq \alpha(x)$ for all $x \in [0, 1]$. Let G be the set of non-negative $g \in C'[0, 1]$ which satisfy $g'_1(x) \geq \beta(x)$ for all $x \in [0, 1]$ and also $g(0) = g(1) = 0$. The pair (α, β) will be said to have property I if F is nonvoid and $\operatorname{Sup}_{f\in F} \int_0^1 f\beta dx < \infty$; (α, β) will be said to have property II if G is nonvoid and $\operatorname{Inf}_{g\in G} \int_0^1 g\alpha dx > -\infty$. Theorem: (α, β) *has property* I *if and only if it has property* II. *Furthermore, if* (α, β) *has either property, then* $\operatorname{Sup}_{f\in F} \int_0^1 f\beta dx = \operatorname{Inf}_{g\in G} \int_0^1 g\alpha dx$. This theorem is an analogue of a discrete problem solved by Gale, Kuhn, and Tucker (*Linear programming and the theory of games*, Cowles Commission Monograph no. 13). Cf. also Hardy, Littlewood, and Pólya, *Inequalities*, Theorem 399. Remark: In the definition of property I, "F is nonvoid" may be deleted since F is never empty. To conform with further problems now under study, the phrase has been retained. (Received November 13, 1951.)

203t. R. L. Sternberg: *Non-oscillation theorems for systems of differential equations.*

Consider the system of differential equations (*) $[R(x)y' + Q(x)y + \phi^*(x)\mu]' - [Q^*(x)y' + P(x)y + \theta^*(x)\mu] = 0$, $\phi(x)y' + \theta(x)y = 0$ with R, Q, P given $n \times n$ matrices, ϕ, θ given $m \times n$ matrices, $0 \leq m < n$, y, μ vectors, R, P symmetric, R, Q, ϕ of class C' and P, θ of class C, and with R satisfying the strengthened Clebsch condition and, finally, with a suitable $(n+m)$-rowed square matrix in R and ϕ nonsingular on $[a, \infty)$.

Call (*) non-oscillatory if x_1, x_2 large and $y(x_1)=0=y(x_2)$ imply $y=0$ on $[x_1, x_2]$. The reduced form of (*) is $[G(x)u'+\psi^*(x)v]'+F(x)u=0$, $\psi(x)u'=0$ where u is related to y by a nonsingular transformation. If $\pi^*G\pi \geqq \pi^*\pi$ when $\psi\pi=0$ and if there exists a symmetric constant matrix H such that, for large x, $|\eta^*[\int_x^\infty tF(t)dt-H]\eta| < \eta^*\eta/2$, then (*) is non-oscillatory. If $m=0$, $\eta^*G\eta \leqq \eta^*\eta$, and $\eta^*F\eta \geqq 0$, then (*) is non-oscillatory if and only if $\int_a^\infty x^rF(x)dx$ exists for $0 \leqq r < 1$ and there exists an $n\times n$ symmetric matrix W of class C' satisfying $W(x)=\int_x^\infty W(t)G^{-1}(t)W(t)dt+\int_x^\infty F(t)dt$ for large x. These and other results which are analogues of theorems of E. Hille and A. Wintner for a single differential equation are proved in the paper using a theorem from the calculus of variations of W. T. Reid. (Received November 7, 1951.)

204. Otto Szász: *On the product of two summability methods.*

We denote by $T_1 \cdot T_2$ the iteration product of two summability methods T_1 and T_2, that is, the transform T_1 of the transform T_2 of a sequence (or a function). It is clear that summability T_2 always implies summability $T_1 \cdot T_2$, when T_1 is regular; however the relationship of the methods T_1 and $T_1 \cdot T_2$ is quite complicated. We have shown in a previous paper that T_1 implies $T_1 \cdot T_2$ in the following cases for T_1 and T_2 respectively: (1) Abel and Cesàro, (2) Borel and Cesàro, (3) Borel and Euler. We now show that the same is true for the following more general methods: (a) Abel and Hausdorff, (b) Borel and Hausdorff, (c) Abel and the circle method. Obviously (a) and (b) include the previous results (1), (2), and (3), as Cesàro and Euler are special Hausdorff means. (Received November 9, 1951.)

205. J. L. Walsh and L. Rosenfeld: *On the boundary behavior of a conformal map.* Preliminary report.

Carathéodory's theory of conformal mapping of variable regions is applied to the study of a conformal map in the vicinity of a zero angle of the boundary. A function $\phi(u)$, not necessarily single-valued, is said to possess *property* B at $u=1$ if uniformly in every interval I: $|U| \leqq U_0$, we have $\lim_{u\to 1^-}\phi[U\phi(u)+u]/\phi(u)=1$. This condition is weaker than the requirement for an L-tangent and is sufficient for many applications. If a region R_w contains the interval $0 \leqq w < 1$ and if its boundary near $w=1$ consists of two parts respectively represented by functions which possess property B at $u=1$, then most of the well known (Ostrowski, Warschawski) results concerning the map of R_w onto a half-plane can be established. The analogue of property B is sufficient to obtain most of the known results on the mapping of an infinite strip. (Received December 21, 1951.)

206. J. L. Walsh and D. M. Young: *An upper bound for the moduli of continuity of harmonic functions.*

Let G denote a closed simply-connected region with interior R whose boundary S is a closed Jordan curve with the following property: there exist constants $r>0$ and $A \geqq 0$ such that for every point P of S there exists a circular sector with vertex at P with radius r and included angle A containing no point of R. Let $u(x, y)$ be harmonic in R, continuous in G, and with modulus of continuity $\omega(\delta)$ on S. If $\delta/D \leqq (r/D)^{(\pi+B)/\pi}$, then the modulus of continuity of $u(x, y)$ in G satisfies the inequality $\omega^*(\delta) \leqq \omega[D(\delta/D)^{\pi/(\pi+B)}]+(8/\pi)M(\delta/D)^{\pi/(\pi+B)}$ where $B=\text{Max }(2\pi-A;\ \pi)$, M is the oscillation of $u(x, y)$ on S, and D is any positive constant. The proof is based on inequalities for the harmonic measure of the bounding radii of a circular sector at interior points near the vertex of the sector. The result is used in connection with the determination of upper bounds for the error in the solution of the Dirichlet problem by finite differences for rectangles and for regions consisting of overlapping rectangles (see *On the accuracy of the numerical solution of the Dirichlet problem by finite differences*, Bull. Amer. Math. Soc. Abstract 57-6-500). (Received November 13, 1951.)

207. H. F. Weinberger: *An optimum problem in the method of Weinstein.*

The method of Weinstein gives upper bounds for the eigenvalues of the projection L' into a subspace $\mathfrak{G}$ of a completely continuous positive symmetric operator L in a Hilbert space $\mathfrak{H}$. These bounds are the eigenvalues of the projections of L into a subspace $\mathfrak{H}\ominus\{p_1, \cdots, p_m\}$ of finite index m. They can be explicitly computed in terms of the eigenvalues and eigenvectors of L. The vectors p_i are restricted to lie in the space $\mathfrak{H}\ominus\mathfrak{G}$, but are otherwise arbitrary. The problem is to determine the best (lowest) upper bound which can be obtained for the kth eigenvalue λ_k' of L' by an appropriate choice of the vectors $p_1, p_2, \cdots, p_m$, the index k being fixed. It is shown that there is always a choice of the vectors p_i so that the upper bound for λ_k' is either λ_k' itself or λ_{m+1}, the $(m+1)$st eigenvalue of L. It is also shown that this is the best upper bound which can be obtained when m vectors p_i are used. (Received November 13, 1951.)

208*t*. Albert Wilansky: *On the convergence fields of row-finite and row-infinite reversible matrices.*

Erdös and Piranian (Proceedings of the American Mathematical Society (1950)) displayed a row-infinite regular matrix with no stronger normal matrix. In this paper the nonexistence of a reversible such matrix is shown. They also showed a row-finite regular matrix with no stronger row-infinite matrix. In this paper it is shown that to each row-finite reversible matrix corresponds an equipotent consistent row-infinite matrix. It is known (Banach: *Théorie des opérations linéaires*, p. 50) that the inverse transformation to a reversible matrix A is given by a sequence and a matrix. It is pointed out here that the sequence vanishes if A is row-finite. A non-reversible matrix with a two-sided inverse is shown. (Received November 7, 1951.)

APPLIED MATHEMATICS

209. R. A. Clark and Eric Reissner: *A problem of finite bending of toroidal shells.*

The nonlinear differential equations of finite axi-symmetrical bending of thin shells of revolution as given recently [E. Reissner, Proceedings of Symposia in Applied Mathematics, vol. 3, 1950, pp. 27–52] are considered for a toroidal shell with circular cross section of radius b, with constant wall thickness h and radius a of the center line of the torus. It is shown that when $b/a \ll 1$ and for edge loading in the direction of the axis of revolution these differential equations may be reduced, if terms up to the second degree are retained, to the following form: $f'' + (\mu \sin x)g = \mu[\Omega(\cos x + f \sin x) + fg \cos x]$; $g'' - (\mu \sin x)g = -2^{-1}\mu f^2 \cos x$. The constant parameter Ω is proportional to the external load and $\mu = (12(1-\nu^2))^{1/2}b^2/ah$. A representative set of boundary conditions is: $f(\pm\pi/2) = g(\pm\pi/2) = 0$. When $\mu = O(1)$ integration is possible by developing the solutions in powers of $\mu\Omega$. The range of applicability of the linearized theory is given by the condition $\mu\Omega \ll 1$. When $\mu \gg 1$ a method of asymptotic integration is used which extends to the nonlinear range earlier results of the linear theory [R. A. Clark, Journal of Mathematics and Physics vol. 29 (1950) pp. 146–178]. The range of applicability of the linearized theory is now given by the condition $\mu^{1/2}\Omega \ll 1$. (Received November 13, 1951.)

210. R. J. De Vogelaere: *On the symmetric periodic orbits in the cosmic rays problem.*

The movement in the meridian plane following the particle in the cosmic rays problem leads to a system of two differential equations of second order, depending on a parameter γ_1, which is nonintegrable and symmetric with respect to the x-axis. It is shown here how the notion of surface of section can be used to find, for a given value of γ_1, all symmetric periodic orbits, proving that this problem is equivalent to finding the intersections of some lines $\mathcal{E}_1, \mathcal{E}_2, \cdots, \mathcal{E}_n, \cdots$, of a new diagram $(x, \dot{x})$, with $\dot{x}=0$. As application, the line $\mathcal{E}_1$ was calculated for the case $\gamma_1=0.97$, using integrations of Störmer and an equivalent number of integrations that were performed to complete the line. From those calculations and from some properties of the $(x, \dot{x})$ diagram, new symmetric periodic orbits were found; one of them has only two points on the x-axis and two symmetric double points (in which it differs from the oval); ten others have on the x-axis, besides the two points where they are perpendicular to this axis, one other double point. Conjectures are made on the form of the $\mathcal{E}_1$ line for values of γ_1 different from 0.97. (Received November 8, 1951.)

211. J. B. Diaz and M. H. Martin: *Riemann's method and the problem of Cauchy.* II. *The wave equation in n dimensions.*

Riemann's method for the solution of Cauchy's problem for a linear second order partial differential equation $L(u)=0$, $u=u(x, y)$, employs the well known Lagrange's differential identity in order to obtain a line integral $I_1=\int\{B(u, v)dx-A(u, v)dy\}$ vanishing on closed paths whenever u and v are solutions of $L(u)=0$ and of its adjoint equation, respectively. Recently, one of the authors (Bull. Amer. Math. Soc. vol. 57 (1951) pp. 238–249) modified Riemann's method by using a different line integral vanishing on closed paths (whenever u and v are solutions of $L(u)=0$ and of an "associate" equation $M(v)=0$, this last equation replacing the adjoint equation in Riemann's method) and employing a differential identity other than Lagrange's. This modification permitted the extension of Riemann's method to the Cauchy problem (data given on $t=0$) for the equation $u_{xx}+u_{yy}-u_{tt}=0$, the line integral I_1 being replaced by a surface integral I_2 which vanishes on closed surfaces. The authors now extend this method to the Cauchy problem (data given on $t=0$) for the wave equation $u_{x_1x_1}+u_{x_2x_2}+\cdots+u_{x_nx_n}-u_{tt}=0$, $n\geqq 2$, using an integral I_n vanishing on closed n-dimensional surfaces. The present method employs differential identities other than Lagrange's and treats simultaneously n even and n odd. (Received October 25, 1951.)

212. A. D. Fialkow and Irving Gerst: *Transfer functions of networks without mutual reactance.* I.

The networks under consideration contain resistance, capacitance, and (self) inductance, but no mutual inductance. The following conditions are necessary and sufficient that a real rational function $A(p)$ given in simplest form by $A(p)=KN/D=K(p^n+a_1p^{n-1}+\cdots+a_n)/(p^m+b_1p^{m-1}+\cdots+b_m)$ be the transfer function of a passive three terminal network of the above kind: (1) The zeros of D are anywhere in the left half-plane or on the imaginary axis with the origin excluded. (2) At pure imaginary zeros of D, $A(p)$ has a simple pole with pure imaginary residue. (3) The zeros of N may not be positive real but are otherwise arbitrary. (4) $m \geqq n$. (5) The number K satisfies the inequalities $0<K<K_0$, where K_0 is the least of the three quantities K_d, b_m/a_n, 1 if $m=n$ and of the first two quantities if $m>n$. If $K_0 \neq K_d$, then K may equal K_0. Here K_d is the least positive value of K (if it exists) for which the equation $D-KN=0$ has a positive multiple root. A synthesis procedure is given which depends in part on methods used in an earlier paper on networks containing only two kinds of elements. (To appear in Quarterly of Applied Mathematics.) (Received November 13, 1951.)

213*t*. A. D. Fialkow and Irving Gerst: *Transfer functions of networks without mutual reactance*. II.

In the case of a passive four terminal network of the above type, the necessary and sufficient conditions that $A(p)$ be a transfer function are identical with those stated in the preceding abstract except that condition (3) is lacking (i.e., the zeros of N are arbitrary) and condition (5) must be modified to read: (5′) The number K satisfies the inequalities $-K_0<K<K_0$, where K_0 is the least of the three quantities $|K_d|$, $|b_m/a_n|$, 1 if $m=n$, and of the first two quantities if $m>n$. If $K_0 \neq |K_d|$ then K may actually equal $\pm K_0$. Here K_d is that real value of K of smallest absolute value (if it exists) for which the equation $D-KN=0$ has a positive multiple root. An algorithm for the synthesis of the network corresponding to $A(p)$ is given. (Received November 13, 1951.)

214. F. A. Ficken: *Uniqueness theorems for certain parabolic problems*.

On the domain D $(0 \leqq x \leqq 1, 0 \leqq t)$, let the problem P consist of the differential equation $u_{xx}-u_t=N(x, t, u, u_x)$, the boundary conditions $u_x-a_0u_t=n_0(u)+f_0(t)$ $(x=0)$ and $u_x+a_1u_t=n_1(u)+f_1(t)$ $(x=1)$, and the data $u(x, 0+)=g(x)$; here the constants a_i $(i=0, 1)$ are $\geqq 0$, the functions f_i are continuous, N and n_i have continuous derivatives, and N_u and N_{u_x} are bounded uniformly in t. Let the class S' consist of functions $u(x, t)$ defined on D so that u, u_t, u_x, and u_{xx} are continuous for $t>0$, $|u|$ and $|u_x|$ are uniformly bounded, and $u(x, 0+)=u(x, 0)$. If u and v solve P, set $w=v-u$, let μ be a real constant, and set $\zeta(t)=e^{-\mu t}(a_0w_0^2+a_1w_1^2+\int_0^1 w^2 d\xi)$. An elementary argument, based on showing that $\dot{\zeta} \leqq 0$ for sufficiently large μ, proves that P can have at most one solution in S' if either $a_0>0$ or $(a_0=0$ and) $n_0(u)$ does not decrease with u *and* either $a_1>0$ or $(a_1=0$ and) $n_1(u)$ does not increase with u. A second theorem gives similar sufficient conditions for uniqueness of the solution of a more special problem in a larger class of functions. The second theorem applies to a nonlinear diffusion problem of some practical interest. (Received November 5, 1951.)

215. H. E. Goheen: *On certain boundary value problems for ordinary differential equations*.

The author uses the method developed in Bull. Amer. Math. Soc. Abstract 57-3-215 for the most general linear two-point boundary value problem for the equation there discussed. A similar technique is applied to simplify H. Weyl's solution of the problem of Blasius in his paper, in Proc. Nat. Acad. Sci. U.S.A., *Concerning the differential equations of some boundary layer problems*, vol. 27 (1941) pp. 578–583. (Received November 9, 1951.)

216*t*. F. B. Hildebrand: *On the convergence of numerical solutions of the heat-flow equation.*

The problem $u_{xx}(x, t) = u_t(x, t)$, $u(0, t) = u(1, t) = 0$, $u(x, 0) = f(x)$ is replaced by the problem in which the difference equation $[u(x+h_x, t) - 2u(x, t) + u(x-h_x, t)]/h_x^2 = [u(x, t+h_t) - u(x,t)]/h_t$ is to be satisfied over the net $x_m = mh_x = m/M$ $(m = 1, 2, \cdots, M-1)$, $y_n = nh_t$ $(n = 0, 1, 2, \cdots)$, and where $u(0, nh_t) = u(1, nh_t) = 0$ $(n = 1, 2, \cdots)$ and $u(mh_x, 0) = f(mh_x)$ $(m = 1, 2, \cdots, M-1)$. If $f(x)$ is continuous in $(0, 1)$ except for a finite number of finite jumps, and is of bounded variation in $(0, 1)$, it is proved that the solution of the approximate problem converges to that of the exact problem as h_x and h_t tend to zero for any fixed $h_t/h_x^2 < 1/2$. The convergence is uniform over $(0 \leqq x \leqq 1, t \geqq t_0)$ for any $t_0 > 0$. (Received November 13, 1951.)

217. H. G. Hopkins: *The bending of an elastic plate supported on an elastic foundation.*

Hankel transforms are used to solve the general problem of an infinite elastic plate, supported on an elastic foundation, and subjected to an arbitrarily prescribed distribution of pressure, the displacements and stresses being zero at infinity. The problem is statical; thin isotropic plate theory is used, and it is assumed that at any point the foundation exerts a normal stress only, proportional to the local normal displacement, on the plate. The solution of the general problem is new; it is specialized to the problem in which there is a uniform pressure distribution over a circular area (which includes, as a limiting case, a concentrated load), and simple formulas are found for the displacements and stresses of most practical importance. (Received November 5, 1951.)

218*t*. Edward Kasner and John De Cicco: *Potential theory in space of n dimensions.*

Potential theory is extended to a Euclidean space of n dimensions by means of the following adoption of Newton's Law of Universal Gravitation. The magnitude of the force of attraction between two particles is directly proportional to the product of the masses and inversely proportional to the $(n-1)$th power of the distance between them. For $n = 3$, this reduces to Newton's well known inverse square law, and for $n = 2$, this leads to the theory of the logarithmic potential. The force of attraction F upon a particle P exerted by a material Riemannian manifold V_N of N dimensions where $1 \leqq N \leqq n$, with a continuous density function μ, is obtained. The field of force is conservative and its potential U is evaluated. As this field is solenoidal, it is found that U is a harmonic function. Applications are given to hyperspheres of N dimensions. When $N = n$, the well known theorems concerning the force exerted by a homogeneous material spherical surface or volume are extended to n dimensions. However, when $1 \leqq N < n$, the situation becomes complicated and the discussion involves hypergeometric functions. (Received December 6, 1951.)

219. E. H. Lee: *An unloading wave problem in dynamic plasticity.*

The question of the relation between elastic-plastic theory and plastic-rigid theory cannot at present be treated in general. The discussion of specific solutions therefore must be used to assess the validity of a plastic-rigid model of an actual material. Such a solution is presented and discussed. A uniformly stretching rod is considered to fracture at the section $x = 0$, at the time $t = 0$. The stress there is instantaneously reduced from the yield stress to zero. An unloading wave of stress discontinuity travels from the section with elastic wave velocity c_0 causing the cessation of plastic flow behind it. It is absorbed at $x=\alpha$, and the subsequent plastic-elastic boundary moves with constant velocities through regions of length α, with velocity magnitudes c_0/n, $n=3, 5, 7, \cdots$. The corresponding plastic-rigid analysis smooths out this polygonal plastic-unloading boundary into a parabola in the x, t plane through the vertices of the polygon. The conditions under which the plastic-rigid analysis provides a good approximation to the elastic-plastic solution are discussed in terms of strain magnitudes and energy magnitudes. (Received November 9, 1951.)

220. C. C. Lin: *Instability of the boundary layer over curved surfaces.* Preliminary report.

A study is made of the instability of the boundary layer at the wall of a circular pipe produced by the combined effect of an axial flow and the rotation of the pipe about its axis. This includes the usual theories for boundary layers over straight and curved walls. It is shown that the combined effect of axial flow and rotation produces a new kind of instability not caused by either one separately. The investigation is carried out for the inviscid case, as this is usually found to be indicative for disturbances of the cellular type found in the present analysis. The general discussion of the eigenvalue problems is based on oscillation theorems, and some simple examples are given for illustration. (Received November 13, 1951.)

221. G. G. O'Brien: *Theorems on stability and convergence in numerical solutions of partial differential equations.*

The use of numerical methods in the solution of partial differential equations has increased noticeably during and since world war II. The undertaking of such solutions has become feasible due to the availability of modern, high-speed, electronic computing machines. However, there is a great need for developing the mathematical theory related to these numerical solutions. In this paper a study is made relative to stability and convergence of the one-dimensional heat flow problem with given boundary conditions. In one section the initial temperature is taken to be constant everywhere except at the end points, where there is a discontinuity. In another section the initial conditions are more general. In both cases it is shown that if a suitable difference equation with satisfactory auxiliary conditions is used to approximate the differential equation with its auxiliary conditions, that choosing the ratio within a certain range of values was sufficient to insure convergence of the difference solution to that of the differential equation. On the other hand an unstable choice of the ratio would lead to a lack of convergence of the difference solution to that of the differential equation for that particular solution. (Received October 26, 1951.)

222t. L. E. Payne and Alexander Weinstein: *Capacity, virtual mass and generalized symmetrization.*

Let B be a body of revolution in n-dimensional space. Its capacity $C\{n\}$, virtual mass $M\{n\}$, and volume $V\{n\}$ are defined as generalizations of the classical definitions of $C\{3\}$, $M\{3\}$, and $V\{3\}$. Let xy be the meridian plane of B, x being the axis of symmetry. A line $x=$constant, $y\geqq 0$, cuts the boundary of the meridian section of B in m points $y_1(x)>y_2(x)>\cdots>y_m(x)>0$. Consider $y_*(x)$ defined by $y_*^q(x) = \sum_{k=1}^{m}(-1)^{k-1}y_k^q(x)$, $q>0$. The meridian section of a new body B, bounded by the curve $y_*(x)$, is said to be obtained by a generalized symmetrization S_q. The following results are obtained. (1) $M\{n\}+V\{n\}=\pi^{n/2}(n-1)^{-1}\Gamma^{-1}(n/2+1)C\{n+2\}$. (2) $C\{n\}$ decreases under $S_{n-1}, S_{n-2}, \cdots, S_1$. (3) $M\{n\}$ decreases under S_{n-1}, S_n, and S_{n+1}. These results generalize the following theorems established for $n=2$ and $n=3$: $C\{n\}$ and $M\{n\}$ decrease under S_{n-1}. (See G. Pólya and G. Szego, *Isoperimetric inequalities in mathematical physics*, Princeton University Press, 1951. P. R. Garabedian and D. C. Spencer, *Extremal methods in cavitational flows*, Report No. 3, Office of Naval Research, 1951.) (Received November 26, 1951.)

223. W. H. Pell: *Limit design of plates: The upper bound for the safety factor.*

Following methods initiated by W. Prager and the author (*Limit design of plates*, forthcoming in the Proceedings of the First National Congress of Applied Mechanics, Chicago, Ill, June 11–16, 1951), limit design of a thin plate, simply supported, under uniform transverse pressure, made of perfectly plastic material obeying the Prandtl-Reuss stress-strain law, is discussed. Methods for determining upper and lower bounds of the safety factor are given. The upper bound is given in terms of a kinematically admissible rate of deflection $w=w(x_1, x_2)$. If $w(x_1, x_2)$ is smooth, the internal rate of energy dissipation, E, is represented by a double integral impossible of evaluation, in general. If discontinuities in $\partial w/\partial x_1$ and $\partial w/\partial x_2$ are allowed along curves, E is modified by the adjunction of line integrals along the curves of discontinuity. If w is polyhedral, and convex on the side opposite that of loading, then $E=-\oint_{(C)}(\partial w/\partial n)ds$, where (C) is the curve bounding the plate, and n the external normal thereto. Contrary to expectation, it is not possible to calculate E for a smooth w surface as the limit of E for a polyhedral approximation to w when the approximating surface tends to the smooth surface as a limit. (Received November 13, 1951.)

224. Eric Reissner: *A problem of finite bending of circular ring plates.*

Differential equations obtained recently (E. Reissner, Proceedings of Symposia in Applied Mathematics, vol. 1, 1949, pp. 213–219) are applied to the problem of finite bending by transverse edge forces of circular ring plates. If b is the width of the plate and a the radius of the inner edge, then within the range of applicability of linearized theory this plate problem can be treated by beam theory for sufficiently small values of b/a. It is shown here that this is no longer the case in the nonlinear range. The problem is governed by two parameters ϕ_0 and μ, where ϕ_0 is the average slope of the deflected plate according to linear theory and $\mu=(12(1-\nu^2))^{1/4}b^2/ah$ where ν is Poisson's ratio and h is the thickness of the plate. The condition for beam theory to be applicable is found to be $\mu\phi_0 \ll 1$. When $\mu\phi_0=O(1)$, nonlinear plate theory must be used. When $\mu\phi_0 \gg 1$, a boundary layer phenomenon is found. The boundary layer equations are $f''=-1+fg$, $g''=-2^{-1}f^2$ with the boundary conditions $f(0)=g(0)=0$, $f'(\infty)=g'(\infty)=0$. (Received November 9, 1951.)

225t. James Sanders: *Some axially-symmetric problems in hydrodynamics and the theory of elasticity.*

A number of general methods of constructing solutions of (1) $\Delta_k\Delta_k\phi=0$ are given $(\Delta_k u=u_{xx}+u_{yy}+(k/y)u_y)$. The case $k=-1$ occurs in the theory of the slow axially-symmetric flow of a viscous fluid while the case $k=1$ occurs in the theory of the axially-symmetric deformation of an elastic body. It is first shown that if $\Delta_k\rho=0$, then $\Delta_{k+2}\Delta_{k+2}\rho=0$. Next, if $\Delta_k\Delta_k(\phi)=0$, then $\Delta_{k+2}\Delta_{k+2}((1/y)\phi_y)=0$. Finally, if $\Delta_k(A)=0$, $\Delta_k(B)=0$, then (2) $\phi=xA+B$ is a solution of (1). In fact, every solution of (1) in a simply-connected domain is of the form (2). This last fact leads to a method of "correspondence" for solving boundary-value problems connected with (1), similar to the correspondence method proposed by Bers and Gelbart (Quarterly of Applied Mathematics vol. 1 (1943) no. 2) for solving equations of the second order. If a boundary-value problem has been solved for the biharmonic equation $\Delta\Delta u = 0$, then a slightly different boundary-value problem can be solved for equation (1) by this method. (Received November 13, 1951.)

226t. Seymour Sherman: *Stability of flow in a bipropellant rocket motor.*

In a recent paper by Gunder and Friant, *Stability of flow in a rocket motor*, Journal of Applied Mechanics vol. 17 (1950) pp. 327–333, a mathematical model is proposed for examining the stability of flow in a bipropellant rocket motor. The stability computations following from this model involve constructing a Cauchy-Nyquist diagram. The ideas of Ansoff and Krumhansl, *A general stability criterion for linear oscillating systems with constant time lag*, Quarterly of Applied Mathematics vol. 6 (1948) pp. 337–341, heretofore applied to other stability phenomena involving time lag, are extended so as to simplify the Gunder-Friant calculations. In particular, one can avoid the construction of a Cauchy-Nyquist diagram. (Received November 6, 1951.)

227. Domina E. Spencer: *TEM-waves, metric coefficients, and the scalar quasi-potential.*

A rigorous proof is given that a TEM-wave is possible for any waveguide consisting of a pair of metal cylinders which conform to coordinate surfaces $u^1 = \text{const.}$ of any cylindrical coordinate system. The E-vector is always in the u^1-direction and its magnitude is simply $E_1 = (F_1(u^1)/(g_{11})^{1/2})e^{j\omega(t-(\mu\epsilon)^{1/2}z)}$. The H-vector is always in the u^2-direction and its magnitude is $H_2 = E_1(\epsilon/\mu)^{1/2}$. It is rigorously shown that, while a scalar potential cannot be used to describe the electric field in a TEM-wave, a scalar quasi-potential can always be introduced such that $E = -(1/\zeta(u^1, u^2, u^3, t))\nabla\phi^*$. This contribution gives a comprehensive view of TEM waves and the various waveguides that may be used. (Received November 5, 1951.)

GEOMETRY

228t. D. O. Ellis: *On metric representations of groups.*

Any finite or countable group is (isomorphic to) a group of motions on the metric Baire space formed on the elements of the group. (Received November 5, 1951.)

229. Arrigo Finzi: *On Liouville's theorem concerning conformal transformations.*

According to Liouville's theorem a conformal transformation of the n-dimensional space ($n \geqq 3$) transforms spheres into spheres. The usual demonstration requires the existence of the derivatives of the first three orders; on the other hand the definition of the conformal transformations requires only the existence of the first derivatives. Is it possible to demonstrate the theorem under these more general conditions? Basically, it should be necessary to demonstrate, for this purpose, that a transformation which transforms infinitesimal spheres into infinitesimal spheres and leaves infinitesimal angles invariant transforms every sphere into a sphere. The following proposition (which here is given for $n = 3$) could perhaps lead to a demonstration of the theorem. Let us consider a system of six spheres. Let us suppose each of the six spheres to be externally tangent to four of the others and all the six spheres to be externally tangent to a seventh sphere. Then the six spheres are internally tangent to an eighth sphere. Note that the proposition given above can be generalised to every $n>2$. There is no corresponding proposition for $n=2$. (Received November 13, 1951.)

230. J. W. Gaddum: *Metric methods in differential geometry.*

Let A be an arc (homeomorph of a segment) in euclidean three-dimensional space. Define $K(p_1, p_2, p_3)=[-D(p_1, p_2, p_3)]^{1/2}/(p_1p_2\cdot p_2p_3\cdot p_1p_3)$, and, if no triple of p_1, p_2, p_3, p_4 is linear, define $T(p_1, p_2, p_3, p_4)=18^{1/2}|D(p_1, p_2, p_3, p_4)|^{1/2}\ [D(p_1, p_2, p_3)D(p_1, p_2, p_4)\cdot D(p_1, p_3, p_4)D(p_2, p_3, p_4)]^{-1/4}$, where D is the Cayley determinant of the points $p_1, \cdots, p_k$. For a point p of A, let $K(p)=\lim_{p_i\to p} K(p_1, p_2, p_3)$ and $T(p)=\lim_{p_i\to p} T(p_1, p_2, p_3, p_4)$, if these limits exist. These are the metric curvature (Menger) and the metric torsion (Blumenthal). If both are different from zero, A is called *bi-regular.* Using a theorem of Pauc it is shown that A is rectifiable. A metric proof is given of the first fundamental theorem of curve theory, stated as follows: If the points of two bi-regular arcs of E_3 can be put into a one-to-one correspondence in such a way that arc length, curvature, and torsion are preserved, then the arcs are congruent. Using a result in the Missouri dissertation of J. W. Sawyer it is proved that a bi-regular arc possesses a moving trihedral at each point and that the Frenet-Serret formulas hold. (Received November 13, 1951.)

231t. P. C. Hammer: *Associated convex bodies.*

This paper gives an extension to n dimensions of some results previously obtained by the author for the plane. Let C be a closed bounded convex body. Let $C_b(r)$ be the convex body obtained from C by a similarity with ratio $r>0$ and center point b on the boundary of C. Define associated bodies $C(r)$ as the intersection of all $C_b(r)$ for $r\leqq 1$ and $C(r)$ as the union of all $C_b(r)$ for $r>1$. Then the family of nonvacuous $C(r)$ are convex sets the boundaries of which simply cover the space. Furthermore, for $r<1$, $C(r)$ is the set of points dividing chords of C in a maximum ratio at most r. For $r>1$, $C(r)$ is the minimal convex body such that points of C divide chords in a maximum ratio no greater than $r/(2r-1)$. For the ratio in which a point divides a chord through it take the ratio of the larger or equal part to the whole chord. For $r\leqq 1$ let $B=C(r)$. Then if the convex body $B(r/(2r-1))=C$ and $C(r)$ is the smallest body for which this is true, C is said to be reducible to $C(r)$. For every convex body C without a center point there exists a unique body $C(r_0)$, $r_0\leqq 1$, to which C is reducible. In a sense this body may be considered the "center" of C. (Received October 31, 1951.)

232. P. C. Hammer and Andrew Sobczyk: *Symmetrisation of convex regions.*

For a convex region C in n-dimensional linear space E_n, a convex region C is defined as the limiting region, as $r\to\infty$, of the similar regions with ratio $1/r$ to the exterior associated regions with respect to an $n-k$ dimensional linear subspace E_{n-k}, $C_r(E_{n-k})$. (See another abstract entitled, *Convex regions associated in subspaces.*) Sections of the region C in flats π_{n-k} parallel to E_{n-k} are symmetric; the region $\overline{C}$ obtained by translating each π_{n-k} to center the symmetric section of C in a complementary subspace E_k to E_{n-k} is also convex (and symmetric in sections parallel to E_{n-k}). In the case $k=0$, $\overline{C}(E_{n-k})$ is the region obtained by centering translated diameters of C on a single point. This region is called the *symmetroid* of C. In case $n=2$, the symmetroid has the same circumference as C. The process of constructing the region $\bar{C}$ with respect to E_{n-k} and E_k is a generalization of Steiner's symmetrization, with which it coincides when $k=(n-1)$. (Received November 9, 1951.)

233t. Andrew Sobczyk and P. C. Hammer: *Adjoint symmetrization of convex regions.*

If $H(u)$ is the support function for a convex region C in a linear space L (Bonnesen-Fenchel, *Theorie der konvexen Körper*, pp. 23–25), the adjoint region C^* in the conjugate space L^* is the set of u for which $H(u) \leqq 1$. It is shown that the region D^* in L^* defined by $K(u) \leqq 1$, where $K(u) = [H(u)+H(-u)]/2$, is the adjoint region of a symmetric region D, obtained from C by centering the midpoints of any subfamily of diameters of C which has the property that the extended diameters simply cover the exterior of C. In particular, if C is a constant diameter region, D^* and D are spheres. Various methods of symmetrization are introduced, for example that corresponding to $J(u) = \sup\, [H(u), H(-u)]$, and their adjoint methods are determined. (Received October 31, 1951.)

234. Andrew Sobczyk and P. C. Hammer: *Convex regions associated in subspaces.*

The union C_K of the translates of a convex set C according to a convex set K, and the set of positions $p+s$ of a point p fixed in an arbitrary set D, when all translations $D+s$ with $(D+s) \subset C$ are allowed, are convex sets. In particular, the interior and exterior associated regions of Hammer (Proceedings of the American Mathematical Society (1951)) are similar respectively to the intersection and to the union of the same set of translates. For a convex region C in n-dimensional linear space E_n, interior and exterior associated regions with respect to any $(n-k)$-dimensional linear subspace E_{n-k} are similarly defined in terms of sections and translates; it is shown that the exterior region is always convex, and by examples that for $k>0$ the interior region need not be convex. Let π_{n-k} denote a flat parallel to E_{n-k}. Then any function $y'' = t(x')$, defined on the projection $P(C)$ parallel to E_{n-k} of C onto E_k, with range in E_{n-k}, defines a set of translations of the sections $C \cdot \pi_{n-k}$; it is shown that the set T of t for which the correspondingly sliced and reassembled region is convex is a convex set in the linear space of continuous functions on $P(C)$ to E_{n-k}. (Received November 9, 1951.)

LOGIC AND FOUNDATIONS

235. S. C. Kleene: *Finite axiomatizability in the predicate calculus using additional predicate symbols.*

Let the formulas constructible in the logical symbolism of the (first order) predicate calculus with a finite list $\mathrm{P}_1, \cdots, \mathrm{P}_s$ of predicate symbols be called *P-formulas*. Let a class C of P-formulas be called *finitely axiomatizable in the predicate calculus using additional predicate symbols*, if there is a formal system S, having as its formulas those constructible in the symbolism of the predicate calculus using besides $\mathrm{P}_1, \cdots, \mathrm{P}_s$ a finite list $\mathrm{Q}_1, \cdots, \mathrm{Q}_r$ of additional predicate symbols, and as its postulates those of the predicate calculus and a finite class of non-logical axioms, such that in S a P-formula is provable if and only if it belongs to C. Sufficient (and obviously necessary) conditions for this are that (1) C be recursively enumerable and (2) C be closed under deduction in the predicate calculus. The result holds for both the classical and the intuitionistic versions of the predicate calculus; and also when instead of predicate symbols only also individual and function symbols (finitely many) are allowed in the construction of P-formulas. (Received November 9, 1951.)

236t. H. G. Rice: *Classes of recursively enumerable sets and their decision problems.*

The partial recursive functions of one variable can be generated in a systematic manner which recursively enumerates them. The set enumerated by a partial recursive function $f(x)$ is defined as the set $\{f(0), f(1), \cdots\}$ up to the least argument, if any, for which $f(x)$ is not defined. Classes whose members are recursively enumerable sets of non-negative integers are considered, and the decision problem for them is shown to be negative, in the following sense. For a class A, we let θ_A be the set of all partial recursive functions enumerating sets of A. Then no such θ_A is a recursive set. There are classes for which θ_A is recursively enumerable, and a (possibly exhaustive) method for generating them is displayed. Among the methods used in reaching the main result are diagonal constructions and transformations by recursive functions of the partial recursive functions into themselves such that every class has a unique transform. (Received November 9, 1951.)

237. H. G. Rice: *Recursively enumerable and recursive orders.* Preliminary report.

We consider sequences (without repetitions) of non-negative integers. Equivalence classes are defined by the following relation: two sequences are equivalent if the same permutation of their terms arranges each in order of size. A sequence containing all non-negative integers is distinguished as the principal sequence of its class. We call an order (an equivalence class) recursively enumerable (r.e.) if at least one of its sequences is produced by a recursive function, recursive if its principal sequence is so produced. (The algebra of infinite sets permits an analogous development.) With iteration of permutations as the operation, the group of permutations on the non-negative integers is obtained. The recursive orders form a subgroup. The inverse of a r.e. but not recursive order is not r.e. A recursive set cannot be recursively enumerated in a r.e. but not recursive order. All the recursive enumerations of a r.e. set are obtained by applying all recursive permutations to a given recursive enumeration. Among the problems mentioned is whether or not the class of r.e. but not recursive orders forms a single left coset of the subgroup of recursive orders. This, if true, implies that there is only one degree of undecidability relative to general reducibility (Post, Bull. Amer. Math. Soc. vol. 50 (1944) p. 311). (Received November 9, 1951.)

STATISTICS AND PROBABILITY

238. K. L. Chung: *Contributions to the theory of Markov chains.*

In a denumerable Markov chain with stationary transition probabilities the probability, starting from i, of hitting j before k is studied. In a positive class the mean first passage times (mfpt), starting from i, to the "union" and the "intersection" of j and k are defined. Relations between these quantities and the aforesaid probabilities are obtained. In particular Doblin's ratio ergodic theorem and central limit theorem are made more precise and recent interesting results due to T. E. Harris (unpublished) are proved. For the mfpt's extended to a finite number of states a Poincaré-like formula is obtained. Another simple result expresses the mfpt's directly in terms of the transition probabilities as follows: $\sum_{n=1}^{\infty} (P_{ij}^{(n)} - P_{kk}^{(n)}) = (m_{jk} - m_{kk})/m_{kk}$. (Received October 24, 1951.)

239. Jim Douglas: *Two equivalence methods for sequences of random variables.*

Two methods of generalizing Khintchine equivalence for sequences of random variables are given. Let $\{x_k\}$ and $\{y_k\}$ be sequences of independent random variables, and let $\{F_k(t)\}$ and $\{G_k(t)\}$ be the corresponding sequences of distribution functions. Let $\delta_k = \text{l.u.b.}\ |F_k(t) - G_k(t)|$. If $\sum_{k=1}^{\infty} \delta_k < \infty$ and $\{x_k\}$ satisfies a sufficient condition of Brunk (Duke Math. J. vol. 15 (1948) pp. 181–195) for the strong law of large numbers, $\{y_k\}$ satisfies the strong law. Applications are made for independent variables to the convergence of series, the central limit law, and the generalized law of the iterated logarithm. The second method utilizes a quasi-metric of Kakutani (Ann. of Math. vol. 49 (1948) pp. 215–224). Let m and μ be the measures generated on an infinite product space of real lines by (dependent) $\{x_k\}$ and $\{y_k\}$, respectively. Let $m^{(n)}$ and $\mu^{(n)}$ be the measures generated by $\{x_{n+1}, x_{n+2}, \cdots\}$ and $\{y_{n+1}, y_{n+2}, \cdots\}$. If $\limsup \rho(m^{(n)}, \mu^{(n)}) = 1$, m and μ are equal on the class of all measurable sets which are $(n+1, n+2, \cdots)$-cylinder sets for every n. Hence, the sequences of random variables obey or disobey together the strong law of large numbers, etc. Brunk (Proceedings of the American Mathematical Society vol. 1 (1950) pp. 409–414) previously considered the second method. The two methods are independent. (Received November 5, 1951.)

240. W. M. Hirsch: *On the maximum cumulative sum of independent random variables.* Preliminary report.

Let $X_1, X_2, \cdots$ be a sequence of independent random variables with mean 0. Put $S_n = X_1 + X_2 + \cdots + X_n$, $\bar{S}_n = \max_{1 \leq j \leq n} S_j$, and $S_n^* = \max_{1 \leq j \leq n} |S_j|$. In 1944 W. Feller called attention to the desirability of studying the random variables $\bar{S}_n$ and S_n^*. Since then counterparts of the Berry-Esseen theorem and of Feller's generalized law of the iterated logarithm have been obtained by K. L. Chung for S_n^*. Theorems like these referring to $\bar{S}_n$ are considered here. The results are as follows: If α_n is an arbitrary sequence of positive numbers, and $\Gamma_n = \max\{|X_j|^3 \mid 1 \leq j \leq n\}/\min\{X_j^2 \mid 1 \leq j \leq n\}$ satisfies a mild regularity condition, then $P\{\bar{S}_n < \alpha_n s_n\} = (2/\pi)^{1/2} \int_0^{\alpha_n} e^{-t^2/2} dt + O(s_n^{-\theta})$, where $s_n^2 = \text{var}(S_n)$ and $\theta > 0$ depends on the order of magnitude of Γ_n. A similar result referring to the "truncated" probability, $P\{\bar{S}_n < \alpha_n s_n;\ S_n > -s_n\}$, is also obtained. With the aid of these estimates it is shown that if $\psi(t) \uparrow \infty$ is arbitrary, then the probability, $P\{\bar{S}_n < s_n/\psi(s_n^2) \text{ i.o.}\}$, is equal to zero or one according as the series $\sum_{k=1}^{\infty} [\psi(2^k)]^{-1}$ is convergent or divergent. (Received November 5, 1951.)

241. Eugene Lukacs: *An essential property of the Fourier transforms of distribution functions.*

Three important theorems are valid for Fourier transforms of distribution functions, the uniqueness theorem, the convolution theorem, and the continuity theorem. These theorems make Fourier transforms an important tool in the theory of probability. The question is investigated whether any other integral transforms applicable to distribution functions have these properties. It is shown that the uniqueness theorem and the convolution theorem—without using the continuity theorem—determine the kernel. The kernel is given by $K(s, t) = e^{itA(s)}$ where $A(s)$ is a real-valued function which assumes all values of a set which is dense between $-\infty$ and $+\infty$. (Received October 1, 1951.)

242. Herschel Weil: *On a combination of correlated random variables.*

The probability density $p(r)$ of $r=(x^2+y^2)^{1/2}$ is obtained when x and y are random variables satisfying a general two-dimensional Gaussian distribution. The result is given in the form of a constant times $r \exp(-ar^2) f(r, \mu_{ij})$ where f is a series of products of Bessel functions whose arguments are functions of r and the moments μ_{ij} of the distribution of x and y and a is a function of the moments. For the special case of zero means, $\mu_{10}=\mu_{01}=0$, and for the case of a common variance $\mu_{20}=\mu_{02}$ combined with a zero correlation $\mu_{11}=0$, the series f reduces to a Bessel function of zero order. The latter special case has been worked out previously and applied to various physical problems such as the study of noise and signal in a linear detector and to the problem of random aiming errors, although the restrictions on the moments are not always valid in these applications. Except in the special cases, the integrals involved in obtaining the moments of $p(r)$ by direct integration of $r^n p(r)$ seem prohibitively difficult, but the even order moments may be obtained by direct differentiation of the generating function for the probability distribution of r^2 which is obtainable in closed form. (Received November 19, 1951.)

TOPOLOGY

243. R. D. Anderson: *Continuous collections of continuous curves in the plane.* Preliminary report.

Let G denote a continuous collection of mutually exclusive compact continuous curves in the plane. If the elements of G are all nondegenerate and G^* is compact and connected, then G with respect to its elements as points is a hereditary continuous curve such that the closure of the set of emanation points of G is totally disconnected in G. If J is such a hereditary continuous curve which is also a subset of the plane, then there exists a collection G all of whose elements are arcs such that G is homeomorphic to J. If a collection G fills up the plane, then either all of the elements of G are degenerate or at most one element of G is degenerate and in the latter case G is a ray. If a collection G all of whose elements are nondegenerate fills up a 2-cell, then G is an arc. (Received November 13, 1951.)

244*t*. B. J. Ball: *Concerning continuous and equicontinuous collections of arcs.*

It is proved that if G is a continuous and equicontinuous collection of mutually exclusive arcs in the plane and G^*, the sum of the arcs of G, is closed and compact, then there exists a reversibly continuous transformation of the plane into itself which carries every arc of G into a straight line interval. Furthermore, if G^* is connected, it is either a simple closed curve plus its interior or the sum of two mutually exclusive simple closed curves and the connected domain bounded by them. (Received December 10, 1951.)

245. A. F. Bartholomay: *On type-invariance and h-retraction.* I.

A transitive (Borsuk) class Rh of mappings is introduced; viz., continuous mappings on a separable, metric space which have right homotopic inverses. A stratification of type-invariance called $\mathfrak{R}h$-type-invariance is then obtained. In an attempt to relate this notion to retraction theory, the author defines "h-retraction": Given a mapping $\rho: A \supset B$, where $B \subset A$ (separable, metric space) and $\rho | B \cong$ identity of B, then B is called an "h-retract" of A and ρ is an "h-retraction." The following theorem (and its converse) is established: *If B is homotopically equivalent to an h-retract of A, there exists an rh-continuous mapping of A onto B.* Corollary: *Type-invariance of h-retracts $\subset Rh$-type-invariance.* The following definitions are then introduced: (1) Spaces A, B will be called "Rh-equivalent" if and only if $\mathfrak{R}h(A) = \mathfrak{R}h(B)$, where $\mathfrak{R}h(A)$ is the class of all spaces obtained as ranges of all elements of $\mathfrak{R}h$ using A as domain. (2) Sets for which $\mathfrak{R}h$-equivalence is identical with homotopic equivalence are called "$\mathfrak{R}h$-determinable." The relation of this latter notion to Borsuk's notion of "$\mathfrak{R}$-determinability" is then discussed. A study of $\mathfrak{R}h$-determinability of spaces is planned for the next part. (Received November 7, 1951.)

246. W. R. Baum: *Characterisation of a factor-group of the n-dimensional homology group.*

H. Hopf has shown that the n-dimensional homology group $\mathfrak{H}^n(K)$ of a connected complex K contains a subgroup $\mathfrak{P}^n(K)$ of homology classes represented by the n-cycles which are (simplicial) images of n-dimensional "handle-manifolds" and that the factor-group $\mathfrak{H}^n(K)/\mathfrak{P}^n(K)$ is isomorphic to the factor-group of two subgroups of of the Hurewicz homotopy group $\pi_{n-1}(K^{n-1})$ of the $(n-1)$-skeleton K^{n-1} of K. (Cf. H. Hopf, Comment. Math. Helv. vol. 17 (1945) pp. 307–326.) The group $\mathfrak{H}^n(K)/\mathfrak{P}^n(K)$ can, on the other hand, be characterized by the group $\Omega_p^{n-1}(K)$ of the reduced $(n-1)$-dimensional nullspherods which are nullhomotopic in K in the ordinary sense of homotopy: $\mathfrak{H}^n(K)/\mathfrak{P}^n(K) \cong \Omega_p^{n-1}(K)$. The group $\Omega_p^{n-1}(K)$ is a subgroup of the group $\Omega^{n-1}(K)$ of the reduced $(n-1)$-nullspherods. (Cf. Bull. Amer. Math. Soc. Abstract 57-3-200.) This fact and an earlier result about $\Omega^{n-1}(K)$ (contained in the mentioned abstract) allow furthermore an interpretation of the Hopfian group $\Delta^{n-1}(K)$ (cf. H. Hopf, loc. cit.) by such reduced $(n-1)$-nullspherods which can not be deformed on K into a point in the sense of ordinary homotopy; one shows that: $\Delta^{n-1}(K) \cong \Omega^{n-1}(K)/\Omega_p^{n-1}(K)$. (Received November 13, 1951.)

247. A. L. Blakers: *Obstruction theorems for pairs.*

Let $A = \{A^r, \delta_A\}$, $B = \{B^r, \delta_B\}$, $C = \{C^r, \delta_C\}$ be Mayer cochain complexes. (See J. L. Kelley and Everett Pitcher, Ann. of Math. vol. 48 (1947) p. 686.) Assume given cochain transformations $\alpha: A \to C$, $\beta: B \to C$. We define a new cochain complex $K = \{K^r, \delta_K\}$ as follows: $K^r = \{(a^r, b^r) | \alpha a^r = \beta b^r\}$, and $\delta_K(a^r, b^r) = (\delta_A a^r, \delta_B b^r)$. We readily verify that $\delta_K(K^r) \subset K^{r+1}$, and that $\delta_K \delta_K = 0$. Some special cases are considered and some relations shown between the cohomology groups of K and those of A, B, and C. In particularly, consider the following extension problem: $(L; L_0, M)$ is a simplicial triad; $L_0 \cap M = M_0$. Assume that we are given a mapping $f: (L_0, M_0) \to (X, A)$, where X and A are arcwise connected. Consider the problem of extending the mapping f to a map $f': (L, M) \to (X, A)$. If $A^r = \mathfrak{C}^r(M, M_0, \pi_{r-1}(A))$, $B^r = \mathfrak{C}^r(L, L_0, \pi_{r-1}(X))$, $C^r = \mathfrak{C}^r(M, M_0, \pi_{r-1}(X))$ and α, β are the cochain transformations induced by the natural homomorphism of $\pi_{r-1}(A) \to \pi_{r-1}(X)$, and the inclusion mapping $(M, M_0) \to (L, L_0)$ respectively, then it is shown that the obstructions to the extension of f lie in the groups $H^r(K)$ as defined above, and that the usual obstruction theorems hold. Similar constructions may be defined to take care of obstruction problems for more complex systems. (Received November 6, 1951.)

248*t*. Raoul Bott: *A note on Whitney's duality theorem.*

The join operation between sphere bundles B and B', originally introduced by

Whitney and modified by Wu, is extended to apply to general bundles, yielding a bundle $B \wedge B'$ over the product of the respective base spaces. The basic lemma of the paper shows that the bundle space and projection of $B \wedge B'$ can be defined in the large in terms of the mapping cylinders of the projections of B and B'. It then follows practically from the definition of the objects involved that the homotopy classes of partial cross sections in B and B' are paired to the homotopy classes of partial cross sections in $B \wedge B'$, and that the induced pairing of obstructions to extending these cross sections is the cross product pairing of cohomology classes under a natural pairing of the local coefficient groups. This result, which generalizes the first formula of Whitney's duality theorem, is then shown to imply the whole duality theorem (which applies only to sphere bundles) by two different methods. One of these consists of the observation that the Whitney theorem now is a rewriting of the Cartan expansion of the Steenrod squares of cross products, due to the recent results of Thom and Wu. (Received November 13, 1951.)

249. Raoul Bott: *Periodic involutions and the Steenrod squares*. I.

In this paper it is shown by a refinement of the P. A. Smith and M. Richardson theory of "periodic involutions" and "symmetric product" that: (1) Every periodic involution $t: K \to K$ of a finite complex K onto itself, which is of prime period p and satisfies a certain axiom of "separation," induces dimension raising homomorphisms $\phi_t^\alpha: H^i(L) \to H^{i+\alpha}(L)$ in the cohomology groups mod p of L, where L is the fixed point set of t. (2) A certain class of involutions, which includes the cyclic permutations $T_{(p)}$ of $K^{(p)} = K \times K \times \cdots \times K$ (p terms), satisfies the axiom of separation. In a subsequent paper the ϕ_{T_p} will be identified as the "reduced pth powers" of Steenrod. (Received November 13, 1951.)

250*t*. F. E. Browder: *Local homeomorphisms and covering mappings*.

Let f be a local homeomorphism of the topological space X into the connected space Y. Suppose that one of the following three conditions holds: (a) Y is locally connected and has a denumerable basis of neighborhoods at each point: (b) Y is locally connected, X is paracompact; (c) Y is locally pathwise connected and semi-locally simply connected. Then f is a covering mapping of X onto Y if and only if for each point of Y there exists a neighborhood V such that f is a closed mapping of each component of $f^{-1}(V)$ into V. In particular if f is a closed mapping, then f is a covering mapping and, if (a) holds, a finite covering mapping. (Received November 13, 1951.)

251*t*. Bailey Brown: *On imbedding in product spaces*.

All spaces considered are topological. A space R is homeomorphic to a subspace of the product of a family of spaces R_k indexed by a set H if and only if there exists a family of maps $f_k: R \to R_k$ such that (1) the collection $\{f_k^{-1}(U_k);\ U_k \text{ open in } R_k,\ k \in H\}$ is a subbase for the open sets of R, and (2) if $a \neq b$ in R, then $f_k(a) \neq f_k(b)$ for some k. This is used to characterize to within homeomorphism: trivial spaces as subspaces of products of two-element trivial spaces, T_0-spaces as subspaces of products of two-element Sierpinski spaces, zero-dimensional spaces (T_0 with a subbase of open and closed sets) as subspaces of products of two-element discrete spaces. Moreover, every space R can be topologically imbedded in the product of its quotient T_0-space and the trivial space whose elements are the elements of R. It follows that every space can be topologically imbedded in a product of two-element spaces which have the trivial or the Sierpinski topology. (Received November 5, 1951.)

252*t*. M. L. Curtis: *Deformation-free continua.*

Let M be a continuum separating S^n, and let A be a domain of $S^n - M$. M is said to be deformation free into A if there exists a deformation $h: MxI \rightarrow \overline{A}$ such that $h(x, t) \in A$ for $t > 0$. It is known that in this case there is only one other domain B of $S^n - M$, both A and B are ulc^n (rationals) and M is an $(n-1)$-gcm. This same result is obtained in this paper when M is imbedded in an LC^{n-1} n-gcm S with $p^1(S) = 0$. Various consequences of M being deformation free into A are obtained (M now in S^n). For example, $\overline{A}$ must be an ANR (although M need not be). If h is monotone open then A must be ULC^1, and if h is light open then A must be almost ULC^1 in a certain sense. Necessary and sufficient conditions that A be ULC^n are given in terms of deformation-free maps. Examples of a 3-gcm which is not 1-lc over the integers and an LC^3 3-gcm which is not a manifold are given. This paper contains the results on homotopy-regular convergence reported on previously (Bull. Amer. Math. Soc. Abstract 57-1-49). (Received November 13, 1951.)

253*t*. Mary-Elizabeth Hamstrom: *Concerning continuous collections of continuous curves.*

Let G be a continuous collection of mutually exclusive continuous curves filling up a compact metric continuum M such that G is an arc with respect to its elements. It is shown that if G has a nondegenerate element, then M is not irreducible. This is an extension of a result of E. E. Moise (*A theorem on monotone interior transformations*, Bull. Amer. Math. Soc. vol. 55 (1949) pp. 810–811). It is also shown that under the same conditions, if M is a subset of the plane it is a domain plus its boundary and if furthermore M is a continuous curve, then it is a simple closed curve plus its interior. (Received December 18, 1951.)

254*t*. S. T. Hu. *Cohomology relations in principal fiber spaces.*

If X is a compact metrizable principal fiber space over B with projection $p: X \rightarrow B$ and a compact zero-dimensional structural group G, then the homomorphism $p^*: H(B) \rightarrow H(X)$ of the Čech cohomology rings with real coefficients induced by the projection $p: X \rightarrow B$ maps $H(B)$ isomorphically onto the invariant subring $H_*(X)$ of $H(X)$. Hence, up to an isomorphism, $H(B)$ is completely determined by $H(X)$ and the operations of G on $H(X)$. It follows also that $H(B)$ is isomorphic with $H(X)$ if and only if G operates trivially on $H(X)$. If, for a given integer $n \geqq 0$, the dimension of the Čech cohomology group $H^n(X)$ with real coefficients is finite, then G operates on the real vector space $H^n(X)$ by means of a natural linear representation. For each $g \in G$, let $\chi^n(g)$ denote the character of the linear transformation determined by g in this linear representation. Then the dimension $p^n(B)$ of the Čech cohomology group $H^n(B)$ is also finite and is given by the formula $p^n(B) = \int_G \chi^n(g) dg$. Since every principal fiber bundle is a principal fiber space in the sense of Bourbaki, the foregoing formula generalizes a similar formula of Beno Eckmann on regular covering polyhedra. (Received November 2, 1951.)

255*t*. V. L. Klee: *Convex bodies and periodic homeomorphisms in Hilbert space.*

This paper contains work described in two earlier abstracts (Bull. Amer. Math. Soc. Abstract 56-4-339 and *A proof that Hilbert space is homeomorphic with its solid sphere*, Proceedings of the International Congress of Mathematicians, Cambridge, 1950), and also establishes some previously unannounced results concerning periodic homeomorphisms of Hilbert space H. Suppose $n \geqq 2$. Then: (1) *If* X *is a finite polytope in* H, H *admits a homeomorphism of period* n *with fixed-point set* X; (2) *If* Y *is a relatively open subset of* X, H *admits a homeomorphism of period* n *whose set of fixed points is homeomorphic with* Y; (3) *For each* $d>0$, H *admits a homeomorphism of period* n *whose fixed-point set is* $\{x \mid \|x\| \geqq d\}$. Also included is an extension to H of the "nearest-point" characterization of convex sets (Jessen, Matematisk Tiddskrift (1940) pp. 66–70) and an extension to more general spaces of Keller's results (Math. Ann. vol. 105 (1931) pp. 748–758) on compact convex subsets of H. (Received November 13, 1951.)

256*t*. V. L. Klee: *Invariant metrics in groups (solution of a problem of Banach)*.

By use of Sierpinski's theorem [Fund. Math. vol. 11 (1928) pp. 203–205] that a topologically complete metric space is an absolute G_δ set (relative to metric spaces), the following is proved: *If* G *is a group with two-sided invariant metric* ρ *such that the space* (G, ρ) *is topologically complete, then* G *is complete under* ρ. In conjunction with the theorem of Kakutani [Proc. Imp. Acad. Japan vol. 12 (1936) pp. 82–84] and G. Birkhoff [Compositio Math. vol. 3 (1936) pp. 427–430] on the existence of invariant metrics, this result guarantees a complete invariant metric (compatible with the topology) for every first countable abelian Hausdorff group which is topologically complete. As applied to linear metric spaces, this answers affirmatively a question of Banach [*Théorie des opérations linéaires*, p. 232]. Also included is a characterization of those groups with associated topology which admit a left-invariant metric, and of those which admit a two-sided invariant metric, thus sharpening slightly the result of Kakutani and Birkhoff. (Received November 13, 1951.)

257*t*. E. E. Moise: *Affine structures in 3-manifolds*. III. *Tubular neighborhoods of linear graphs*.

Let L be a polyhedral linear graph in Euclidean 3-space E^3 with vertices v and edges e. Let T be a sum of closed spheres U_v with centers at v and closed solid cylinders C_e with axes e. T is a tubular neighborhood of L if, v' and v'' being the end points of e, $C_e-(U_{v'} \cup U_{v''})$ is connected and intersects no U_v or $C_{e'}$, $e' \neq e$. If $K, K' \subset E^3$ are homeomorphic, then $D(K, K')$ is the greatest lower bound of the numbers ϵ for which there is a homeomorphism f, throwing K onto K', such that for each $p \in K$ the distance from p to $f(p)$ is less than ϵ. *Theorem.* If f is a homeomorphism throwing a neighborhood U of L into E^3, and $\epsilon>0$, then for every sufficiently small tubular neighborhood T of L, lying in U, there is a homeomorphism f', throwing T onto a polyhedron $K \subset E^3$, such that (1) K is a neighborhood of $f(L)$ in E^3, (2) the U_v's used in defining T are thrown by f' onto polyhedra of diameter less than ϵ, and (3) $D(K, f(T))<\epsilon$. Forthcoming in Annals of Mathematics. (Received November 13, 1951.)

258*t*. E. E. Moise: *Affine structures in 3-manifolds*. IV. *Piecewise linear approximations of homeomorphisms*.

A separable metric space is a 3-manifold with boundary if each of its points has a closed neighborhood homeomorphic to a 3-cell. If K is a (locally finite simplicial) complex, then a homeomorphism f throwing K into Euclidean 3-space E^3 is piecewise linear if there is a simplicial subdivision G of K such that f is linear over every cell of G. *Theorem.* Let K be a finite simplicial complex which (considered as a space) is a 3-manifold with boundary, let f be a homeomorphism throwing K into E^3, and let ϵ

be a positive number. Then there is a piecewise linear homeomorphism f', throwing K into E^3, such that for each point p of K, the distance from $f(p)$ to $f'(p)$ is less than ϵ. This is the solution, for $n=3$, of a problem of J. W. Alexander (*Some problems in topology*, Verhandlungen des Internationalen Mathematiker Kongresses, Zürich, 1932, 1932, vol. 1, pp. 249–257). Forthcoming in Annals of Mathematics. (Received November 13, 1951.)

259. E. E. Moise: *Affine structures in 3-manifolds.* V. *The triangulation theorem and Hauptvermutung.*

A 3-manifold is a separable metric space each of whose points has an open neighborhood homeomorphic to Euclidean 3-space E^3. A space is triangulable if it is homeomorphic to a (locally finite simplicial) complex. Two complexes K_1 and K_2 are isomorphic if there is a one-to-one correspondence between the simplices of K_1 and those of K_2, preserving incidence relations. K_1 and K_2 are combinatorially equivalent if there are simplicial subdivisions K_1' and K_2' of K_1 and K_2 respectively, such that K_1' and K_2' are isomorphic. *Theorem.* Every 3-manifold is triangulable. This is the solution (for $n=3$) of a problem of H. Poincaré. *Theorem.* If the complexes K_1 and K_2 are homeomorphic 3-manifolds, then they are combinatorially equivalent. This is the case $n=3$ of the *Hauptvermutung* of Steinitz. It implies that every combinatorial invariant of a triangulated 3-manifold is a topological invariant. Reidemeister's torsion invariant, which is sufficient to classify the 3-dimensional lens spaces combinatorially, is therefore also sufficient to classify these spaces topologically. (Reidemeister, *Homotopieringe und Linsenräume*, Abh. Math. Sem. Hamburgischen Univ. vol. 11 (1935) pp. 102–109.) Forthcoming in Annals of Mathematics. (Received November 13, 1951.)

260*t*. E. E. Moise: *On the nonseparation of the Antoine set by 2-spheres.*

L. Antoine (*Sur la possibilité d'étendre l'homéomorphisme de deux figures à leur voisinage*, C. R. Acad. Sci. Paris vol. 171 (1920) pp. 661–663) has described a 0-dimensional compact set A in 3-space, such that the complement of A is not simply connected. It was asserted by Antoine, with extremely brief indications of proof, that any 2-sphere which separates two points of A from one another in 3-space must intersect A. In the present note, a detailed argument for Antoine's statement is given, using Vietoris linking theory. This property of the set A is used to give an example of two 2-spheres in 3-space which cannot be separated by any third 2-sphere, polyhedral or not. (Received November 13, 1951.)

261. G. D. Mostow: *On the topology of homogeneous spaces.* II.

Let G be a simply connected solvable Lie group, and S a closed subgroup of G. Let N be the connected subgroup determined by the maximum nilpotent ideal of the Lie algebra of G. Then either S is in a Cartan subgroup of G or SN is closed. It follows that a homogeneous space arising from a solvable Lie group is the direct product of a euclidean space and a fibre bundle which has a toroid as base and a nilmanifold as fibre (i.e., homogeneous space of a nilpotent group; cf. Malcev, Izvestiya Akademii Nauk SSSR. Ser. Mat. (1949), Amer. Math. Soc. Translation No. 39). Necessary and sufficient conditions on the fundamental group of G/S in order that the bundle above admit a continuous cross-section are given. If G/S is compact and its fundamental group is abelian, then G/S is homeomorphic to a toroid. (Received November 13, 1951.)

262. O. M. Nikodým: *Universal locally convex linear topological spaces.*

Locally convex linear topological spaces whose vectors are some real-valued functions $M(U, \alpha)$ of two variables are constructed, into which vast classes of locally convex linear topological spaces L can be isomorphically embedded with preservation of Hyer's pseudonorm. These applications transform the convergence of directed-set-sequences in L into a generalized uniform convergence. This paper constitutes part of the work done by the author for the A.E.C. under a nonclassified project. (Received November 13, 1951.)

263. C. N. Reynolds: *The present status of the four color problem.*

Traditionally, the reductions of the four color problem have been part of synthetic geometry. Their implications, on the other hand, have been developed by arithmetic and algebraic inequalities. This paper generalizes the algebraic formulae, proves the generalizations by algebra, and shows that they lead to this proposition: A map which is irreducible with respect to the four color problem must contain a denumberably infinite set of regions. If, by definition, irreducible maps are restricted to maps with a finite set of regions, this "solves" the "four color problem." (Received November 13, 1951.)

264t. S. K. B. Stein: *Homology of the symmetric product.*

Let X be a topological space. The symmetric product of X is obtained from the cartesian product $X \times X$ by identifying the points (x, x') and (x', x). Now assume that X is triangulable. P. A. Smith (Proceedings of the Academy of Science (1933)) obtains the mod 2 homology of the symmetric product from that of X. Moses Richardson (Duke Math. J. (1935)) obtains the homology mod p^n where p is a prime other than 2. His proof for the case $p=2$ seems to contain errors. In the present paper the author computes the homology over the integers. The method is a generalization of that employed by Smith. Three exact sequences and Smith's mod 2 result are used. (Received December 6, 1951.)

265. H. C. Wang: *One-dimensional Lefschetz group of locally compact homogeneous spaces.*

Let X be a locally compact, noncompact space, and $X' = X + I$ be its one-point compactification. Following H. Cartan, by Lefschetz group $L_r(X)$ of X we mean the Čech homology group $H_r(X')$ of the compact space X'. The following is proved: Theorem. Let X be a connected, locally compact, separable metric space admitting a transitive group of isometries (in particular, X is a connected, locally compact group satisfying the first countability axiom). If X is noncompact and locally connected, the 1-dimensional Lefschetz group $L_1(X)$ of X is either 0 or isomorphic with the coefficient group. (Received November 13, 1951.)

266. M. T. Wechsler: *A characterization of certain topological spaces by means of their groups of homeomorphisms.*

A topological space F is n-homogeneous if for every pair of n-tuples $(x_1, \cdots, x_n)$ and $(y_1, \cdots, y_n)$, where the x_i and y_i are points of F and $x_i \neq x_j$ and $y_i \neq y_j$ for $i \neq j$, there exists a homeomorphism g of F onto F such that $g(x_i) = y_i$, $i = 1, \cdots, n$. F is said to be ω-homogeneous if it is n-homogeneous for each n. Let $\mathcal{J}$ be the class of all spaces F with the property that F is ω-homogeneous, nondiscrete, and that there exists at least two open sets of F which are nonempty and disjoint. Note that the class of all manifolds of dimension two or greater is a subclass of $\mathcal{J}$. Let $G(F)$ be the group of all homeomorphisms of F onto itself and give $G(F)$ the point-open topology. The result of this paper is that if F and F' are two spaces belonging to the class $\mathcal{J}$ and if there exists a mapping of $G(F)$ onto $G(F')$ which is both an isomorphism and a homeomorphism, then the spaces F and F' are homeomorphic. (Received November 16, 1951.)

267. G. W. Whitehead: *Fibre spaces and the Eilenberg homology groups.*

Let X be a 0-connected, locally equi-connected space. Then for any integer n, there exists a fibre space T_n over X such that (1) T_n is n-connected; (2) the fibre map of T_n on X induces isomorphisms onto: $\pi_i(T_n) \approx \pi_i(X)$ for $i \geqq n+1$. The space T_n is constructed as follows: let $X^* \supset X$ be a space such that $\pi_i(X^*) = 0$ for $i > n$, while the inclusion map of X into X^* induces isomorphisms onto: $\pi_i(X) \approx \pi_i(X^*)$ for $i \leqq n$. Then T_n is the space of all paths in X^* starting at a fixed point $x_0 \in X$ and ending in X. As an application it is shown that if X is a 0-connected space and $S_n(X)$ is the subcomplex of the total singular complex consisting of all singular simplexes whose n-skeletons are mapped into x_0, then $H_q(S_n(X), S_{n+1}(X)) \approx H_{q-1}(\pi_{n+1}(X), n)$ for $q \leqq 2n$. (Received October 29, 1951.)

L. W. Cohen,
Associate Secretary

APPENDIX

EXCERPTS FROM REPORT OF TREASURER

TO THE BOARD OF TRUSTEES OF THE
AMERICAN MATHEMATICAL SOCIETY:

Gentlemen:

I have the honor to submit herewith the report of the Treasurer for the fiscal year ended November 30, 1950, as is customary in two parts, (1) exhibits and schedules, and (2) a discussion and commentary.

On November 30, 1950, the market value of securities held for invested funds exceeded book value (normally cost) by $45,008, while the market value of securities held for current funds was less than book value by $444.00.

It has been our policy in recent years to keep current funds only in Government securities, as in all probability the Society will be obliged either to liquidate many of these securities or to incur bank loans secured by pledging these assets as collateral. Liquidity and safety have been our sole objectives in the investment of these funds.

On the other hand, in dealing with invested funds, which may be thought of with reasonable accuracy to be the permanent funds of the Society, we have sought a diversified portfolio of high grade securities yielding as liberal a return as is consistent with sound investment policy. Our return this year on these funds has been 5.47%, computed on average book value of investments.

The sale of the Library placed $66,000 in the hands of the Trustees. It was decided to set aside $50,000 as a fund to be known as the "Library Proceeds Fund," and securities were purchased to implement this decision of the Board. The income from these investments should cover the rent on the new headquarters building in Providence.

The balance of $16,000 was made available for current use in reconditioning and furnishing the headquarters building, and in meeting expenses incurred in moving from New York to Providence. The Board voted that the sum should be amortized over five years and repayments added to the Library Proceeds Fund until this Fund should have been built up to the sum of $66,000. Interest will be charged on the unamortized balance.

Following is a summary of the changes in the security holdings made during the year, including the securities purchased with the monies placed in the Library Proceeds Fund.

Acquired

$5,000	American Telephone and Telegraph Co. Deb $3\frac{3}{8}$s 1973
5,000	Consolidated Edison Co. of New York 1st Ref Ser G $3\frac{1}{4}$s 1981
3,000	Oregon Washington RR and Navigation Co. Ref A 3s 1960
9,000	United Gas Corp. 1st Mtg and Coll Trust $3\frac{3}{8}$ 1971
2,500	United States Treasury $2\frac{1}{4}$s 1962–1959
50	shares American Tobacco Co. Cum 6%
100	shares Ingersoll Rand Co. Common
25	shares Northern States Power Co. of Minnesota Cum $3.60
100	shares Phillips Petroleum Co. Common
100	shares Westinghouse Electric Common
5	shares Standard Oil Co. of New Jersey

Acquired as result of stock splits

100	shares Continental Oil Co.
50	shares Texas Company
2	shares Standard Oil Co. of New Jersey

Sold

25	rights American Telephone and Telegraph Co.
100	rights Consumers Power Co.
30/40	share Standard Oil of New Jersey

It will be noted that, in accordance with a recommendation made by the Treasurer in the Annual Report for 1950 and subsequently approved by the Board, the *Proceedings* and the *Transactions* have been set up as publication funds, rather than being treated as general membership activities. The deficits incurred in the publication of these journals have thus been made startlingly apparent. A reserve to cover the deficit has been set up in anticipation of an appropriation from surplus by the Board at the annual meeting to cover these deficits—approximately $23,000.

The increase in dues, both individual and institutional, was of substantial help in meeting the obligations of the Society, having produced about $12,000.

As a matter of record, it should be noted that the Society accepted Federal Social Security benefits for its staff as of January 1, 1951, in accordance with an Act of Congress offering this opportunity to scientific and educational organizations hitherto excluded from these benefits. The cost to the Society in the resulting taxes was just under $750.00.

Fortunately for the Society, a substantial balance remains in the International Congress account, for the Proceedings of the Congress have not yet been published. The fact that these funds have been held in the treasury pending the receipt of bills for printing and distributing the Proceedings has enabled us to avoid throughout the entire year bank borrowing to obtain current cash.

The Society has about one more year during which it can continue to conduct its extensive program of publication on its present scale without securing new sources of income. If additional income of $15,000 to $25,000 per year cannot be obtained, publication of our journals will then have to be curtailed. Our reserves, built up when the volume of publication was abnormally low, will have been exhausted by what is evidently not a temporarily abnormally high volume, but a new expanded normal level.

In addition, every economy in operation must be sought and embraced. The move to Providence will help. Our unified operations can be conducted at less expense than was the case when our offices were dispersed. It may be necessary for us to allow some of our older publications to go out of print, or to adopt a policy of maintaining files of back numbers for only a limited period of years.

Above all, the Society must be sure that any new activities undertaken are fully supported by clearly foreseeable income before being begun.

The Treasurer believes that the basic financial policies of the Society are soundly conceived and wisely executed, and in particular, holds that the action of the Board in authorizing unbalanced budgets during the postwar years, was necessary to the welfare of American mathematics and the furtherance of the objectives of the Society. But the time has now come when we must return, not later than fiscal 1953, to a strictly balanced budget and engage only in enterprises we can currently support.

Reference has already been made to the increased dues authorized by the membership, effective in 1951. The members, with commendable realism, began the program of finding increased income with themselves. The universities, as institutional members, have also recognized the reality of increased costs and assumed their share of the burden. Thus individual mathematicians and institutions of higher learning have assumed and are carrying at least as much of the responsibility as can rightfully be expected of them. If American mathematical research is not to be hampered and, indeed, curtailed at a time when its importance to Government and industry is greater than ever, Government and industry must accept also their rightful share of its support.

In doing this, such agencies and corporations should recognize the importance of the claims of the American Mathematical Society, at once the voice of the mathematicians themselves, the principal agency of research publication in America, and the one organization in the Country whose principal purpose—indeed, whose sole purpose—is the furtherance of mathematical research. It is believed that contributions to the work of the American Mathematical Society will be more effective in securing the desired result than would similar contributions made in any other way.

Respectfully submitted,
ALBERT E. MEDER, JR.
Treasurer

BALANCE SHEET

Assets

	November 30, 1951	November 30, 1950
CURRENT FUNDS:		
Cash	$ 39,634.51	$ 33,332.51
Account Receivable (United States Government)	3,137.82	5,842.66
Investments	67,329.31	67,328.66
	$110,101.64	$106,503.83
INVESTED FUNDS:		
Cash	$ 1,427.00	$ 629.83
Investments	244,972.71	194,242.32
	$246,399.71	$194,872.15
TOTAL ASSETS	$356,501.35	$301,375.98

Liabilities

	November 30, 1951	November 30, 1950
CURRENT FUNDS:		
Publications	$ 5,667.37	$ 26,348.27
International Congress	41,992.58	45,531.48
Policy Committee	778.30	(648.55)
Prize Funds and Other Special Funds Accumulated Income	4,346.02	3,524.71
Sinking Fund		54.04
Profit on Sales of Securities	2,049.85	2,049.85
Miscellaneous (credit)	2,033.45	1,644.02
Surplus and Reserves	46,655.93	28,000.01
Account Payable (United States Government)	28.95	
Office Transfer Fund	6,549.19	
	$110,101.64	$106,503.83
INVESTED FUNDS:		
Endowment Fund Principal	$ 71,000.00	$ 71,000.00
Library Proceeds Fund	50,000.00	
Prize Funds and Other Special Funds	33,683.22	33,683.22
Life Membership and Subscription Reserve	4,162.64	2,635.08
Mathematical Reviews	65,000.00	65,000.00
Reserve for Investment Losses	4,385.89	4,385.89
Profit on Sales of Securities	18,167.96	18,167.96
	$246,399.71	$194,872.15
TOTAL LIABILITIES	$356,501.35	$301,375.98

GENERAL RECEIPTS AND DISBURSEMENTS

GENERAL RECEIPTS:			
Dues from Ordinary Memberships		$45,677.64	
Dues from Contributing Memberships		776.75	
Dues from Institutional Memberships		14,323.46	
Publication Charges from Non-Member Institutions		213.75	
Initiation Fees		799.62	
Income from			
Henderson Estate	$ 6,850.00		
Current Funds Investments	1,509.60		
Invested Funds	5,321.96	13,681.56	
Miscellaneous			
Donations	357.90		
Meeting Fees	516.00	873.90	
TOTAL GENERAL RECEIPTS			$76,346.68
GENERAL DISBURSEMENTS:			
Secretaries and Executive Director		$27,396.78	
Treasurer		7,859.00	
Librarian		1,056.81	
Policy Committee		785.03	
Furniture and Fixtures, N. Y.		2,198.56	
Rent		1,021.20	
Headquarters Utilities		113.94	
Miscellaneous		1,796.75	
TOTAL GENERAL DISBURSEMENTS			$42,228.07
EXCESS OF GENERAL RECEIPTS OVER GENERAL DISBURSEMENTS			$34,118.61

ANALYSIS OF SURPLUS

	Additions	Deductions	
SURPLUS AT DECEMBER 1, 1950			$28,000.01
Excess of General Receipts over General Disbursements (see above)	$34,118.61		
Bulletin	3,262.66	$13,431.63	
Appropriations:			
Accumulated Income—Reilly	2,300.00		
Mathematical Reviews		500.00	
American Journal		2,091.16	
Canadian Journal		1,000.00	
Pacific Journal		1,300.00	
Annals		1,175.00	
Miscellaneous	374.47	1,902.03	
	$40,055.74	$21,399.82	
Net Additions			$18,655.92
Surplus before Reserves			$46,655.93
Reserves for Proceedings and Transactions Deficit			23,170.67
SURPLUS, November 30, 1951			$23,485.26

SUMMARY OF CURRENT FUNDS

	Balance 12/1/50	Additions	Deductions	Balance 11/30/51
CURRENT FUNDS				
Publications				
Mathematical Reviews	$ 6,912.09	$ 53,911.30	$ 56,280.27	$ 4,543.12
Colloquium	11,530.24	13,962.33	12,231.28	13,261.29
Mathematical Surveys	(3,457.77)	5,474.54	7,669.35	(5,652.58)
Proceedings		3,515.34	19,129.50	(15,614.16)
Transactions		9,963.35	17,519.86	(7,556.51)
Symposia on Applied Mathematics	(1,901.77)	1,191.02	168.12	(878.87)
Birkhoff's Collected Papers	(3,145.63)	2,015.30	117.19	(1,247.52)
Memoirs	4,290.44	1,653.00	3,614.99	2,328.45
Bulletin Reprinting	7,631.43	1,523.09		9,154.52
Transactions Reprinting	5,159.28	2,435.15		7,594.43
Russian Vocabulary	(49.72)	284.33	128.64	105.97
Translations	(620.32)	810.62	561.07	(370.77)
TOTAL PUBLICATIONS	$26,348.27	$ 96,739.37	$117,420.27	$ 5,667.37
International Congress	$45,531.48	$ 421.17	$ 3,960.07	$41,992.58
Office Transfer Fund		16,000.00	9,450.81	6,549.19
Policy Committee	(648.55)	1,823.55	396.70	778.30
Prize Funds and Other Special Funds Accumulated Income	3,524.71	3,121.31	2,300.00	4,346.02
Sinking Fund	54.04		54.04	
Profit on Sales of Securities	2,049.85			2,049.85
TOTAL	$76,859.80	$118,105.40	$133,581.89	$61,383.31

SUMMARY OF INVESTED FUNDS

	PRINCIPAL				INCOME			
	Balance 12/1/50	Additions	Deductions	Balance 11/30/51	Balance 12/1/50	Additions	Deductions	Balance 11/30/51
INVESTED FUNDS								
Endowment	$ 71,000.00			$ 71,000.00		$ 3,887.55	$ 3,887.55*	
Library Proceeds		$50,000.00		50,000.00		1,277.02		$1,277.02
Prize Funds								
Bôcher	$ 1,188.00			$ 1,188.00	$ 748.89	$ 65.04		$ 813.93
Brown	1,000.00			1,000.00	56.07	54.74		110.81
Cole	2,093.13			2,093.13	1,055.33	114.62		1,169.95
Henderson	1,000.00			1,000.00	56.07	54.74		110.81
Hutchinson	1,000.00			1,000.00	56.07	54.74		110.81
Merrill	650.00			650.00	52.19	35.57		87.76
Moore	2,100.62			2,100.62	117.78	115.05		232.83
Reilly	23,651.47			23,651.47	1,326.24	1,295.05	2,300.00†	321.29
Whittemore	1,000.00			1,000.00	56.07	54.74		110.81
TOTAL	$ 33,683.22			$ 33,683.22	$3,524.71	$ 1,844.29	$ 2,300.00	$3,069.00
Mathematical Reviews	$ 65,000.00			$ 65,000.00		$ 3,559.04	$ 3,559.04‡	
Life Membership and Subscription Reserve	2,635.08	1,902.03	374.47	4,162.64		201.81	201.81*	
Reserve for Investment Losses	4,385.89			4,385.89		238.09	238.09*	
Profit and/or Loss on Sales of Securities	18,167.96			18,167.96		994.51	994.51*	
TOTAL	$194,872.15	$51,902.03	$374.47	$246,399.71	$3,524.71	$12,002.31	$11,181.00	$4,346.02

* Transferred to General Receipts.

† Transferred to Surplus.

ÉLIE CARTAN AND HIS MATHEMATICAL WORK

SHIING-SHEN CHERN AND CLAUDE CHEVALLEY

After a long illness Élie Cartan died on May 6, 1951, in Paris. His death came at a time when his reputation and the influence of his ideas were in full ascent. Undoubtedly one of the greatest mathematicians of this century, his career was characterized by a rare harmony of genius and modesty.

Élie Cartan was born on April 9, 1869 in Dolomieu (Isère), a village in the south of France. His father was a blacksmith. Cartan's elementary education was made possible by one of the state stipends for gifted children. In 1888 he entered the "École Normale Supérieure," where he learned higher mathematics from such masters as Tannery, Picard, Darboux, and Hermite. His research work started with his famous thesis on continuous groups, a subject suggested to him by his fellow student Tresse, recently returned from studying with Sophus Lie in Leipzig. Cartan's first teaching position was at Montpellier, where he was "maitre de conférences"; he then went successively to Lyon, to Nancy, and finally in 1909 to Paris. He was made a professor at the Sorbonne in 1912. The report on his work which was the basis for this promotion was written by Poincaré;[1] this was one of the circumstances in his career of which he seemed to have been genuinely proud. He remained at the Sorbonne until his retirement in 1940.

Cartan was an excellent teacher; his lectures were gratifying intellectual experiences, which left the student with a generally mistaken idea that he had grasped all there was on the subject. It is therefore the more surprising that for a long time his ideas did not exert the influence they so richly deserved to have on young mathematicians. This was perhaps partly due to Cartan's extreme modesty. Unlike Poincaré, he did not try to avoid having students work under his direction. However, he had too much of a sense of humor to organize around himself the kind of enthusiastic fanaticism which helps to form a mathematical school. On the other hand, the bulk of the mathematical research which was accomplished at the beginning of this century in France centered around the theory of analytic functions; this subject, made glamorous by the achievement represented by Picard's theorem, offered many not too difficult problems for a young mathematician to tackle. In the minds of inexperienced beginners in mathematics, Cartan's teaching, mostly on geometry, was sometimes very wrongly mistaken for a remnant of the earlier Darboux tradition of rather hollow geometric elegance. When, largely under the influence of A. Weil, a breeze of fresh air from the outside came to blow on French mathematics, it was a great temptation to concentrate entirely on the then ultra-modern fields of topology or modern algebra, and the ideas of Cartan once more, though for other reasons, partially failed to attract the amount of attention which was their due. This regrettable situation was partly corrected when Cartan's work was taken (at the suggestion of A. Weil) in 1936 to be the central theme of the seminar of mathematics organized by Julia. In 1939, at the celebration of Cartan's scientific jubilee, J. Dieudonné could rightly say to him: " . . . vous êtes un "jeune," et vous comprenez les jeunes"—it was then beginning to be true that the young understood Cartan.

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[1] This report was in part published in Acta Math. vol. 38 (1921) pp. 137–145. It should be of considerable historic interest to have now a complete version of this report.

In foreign countries, particularly in Germany, his recognition as a great mathematician came earlier. It was perhaps H. Weyl's fundamental papers on group representations published around 1925 that established Cartan's reputation among mathematicians not in his own field. Meanwhile, the development of abstract algebra naturally helped to attract attention to his work on Lie algebra. However, the reception of his contributions to differential geometry was varied. This was partly due to his approach which, though leading more to the heart of the problem, was unconventional, and partly due to inadequate exposition. Thus Weyl, in reviewing one of Cartan's books [41],[2] wrote in 1938:[3] "Cartan is undoubtedly the greatest living master in differential geometry. . . . I must admit that I found the book, like most of Cartan's papers, hard reading. . . ." This sentiment was shared by many geometers.

Cartan was elected to the French Academy in 1931. In his later years he received several other honors. Thus he was a foreign member of the National Academy of Sciences, U.S.A., and a foreign Fellow of the Royal Society. In 1936 he was awarded an honorary degree by Harvard University.

Closely interwoven with Cartan's life as a scientist and teacher has been his family life, which was filled with an atmosphere of happiness and serenity. He had four children, three sons, Henri, Jean, and Louis, and a daughter, Hélène. Jean Cartan oriented himself towards music, and already appeared to be one of the most gifted composers of his generation when he was cruelly taken by death. Louis Cartan was a physicist; arrested by the Germans at the beginning of the Résistance, he was murdered by them after a long period of detention. There is no need to say here that Henri Cartan followed in the footsteps of his father to become a mathematician.

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[2] Numbers in brackets refer to the bibliography at the end of the paper.

[3] Bull. Amer. Math. Soc. vol. 44 (1938) p. 601.

Cartan's mathematical work can be roughly classified under three main headings: group theory, systems of differential equations, and geometry. These themes are, however, constantly interwoven with each other in his work. Almost everything Cartan did is more or less connected with the theory of Lie groups.

S. Lie introduced the groups which were named after him as groups of transformations, i.e., as systems of analytic transformations on n variables such that the product of any two transformations of the system still belongs to the system and each transformation of the system has an inverse in the system. The idea of considering the abstract group which underlies a given group of transformations came only later; it is more or less implicit in Killing's work and appears quite explicitly already in the first paper by Cartan. Whereas, for Lie, the problem of classification consisted in finding all possible transformation groups on a given number of variables—a far too difficult problem in the present stage of mathematics as soon as the number of variables is not very small—for Killing and Cartan, the problem was to find all possible abstract structures of continuous groups; and their combined efforts solved the problem completely for simple groups. Once the structures of all simple groups were known, it became possible to look for all possible realizations of any one of these structures by transformations of a specified nature, and, in particular, for their realizations as groups of linear transformations. This is the problem of the determination of the representations of a given group; it was solved completely by Cartan for simple groups. The solution led in particular to the discovery, as early as 1913, of the spinors, which were to be re-discovered later in a special case by the physicists.

Cartan also investigated the infinite Lie groups, i.e., the groups of transformations whose operations depend not on a finite number of continuous parameters, but on arbitrary functions. In that case, one does not have the notion of the abstract underlying group. Cartan and Vessiot found, at about the same time and independently of each other, a substitute for this notion of the abstract group which consists in defining when two infinite Lie groups are to be considered as isomorphic. Cartan then proceeded to classify all possible types of non-isomorphic infinite Lie groups.

Cartan paid also much attention to the study of topological properties of groups considered in the large. He showed how many of these topological problems may be reduced to purely algebraic questions; by so doing, he discovered the very remarkable fact that many properties of the group in the large may be read from the infinitesimal structure of the group, i.e., are already determined when some arbitrarily small piece of the group is given. His work along these lines resembles that of the paleontologist reconstructing the shape of a prehistoric animal from the peculiarities of some small bone.

The idea of studying the abstract structure of mathematical objects which hides itself beneath the analytical clothing under which they appear at first was also the mainspring of Cartan's theory of differential systems. He insisted on having a theory of differential equations which is invariant under arbitrary changes of variables. Only in this way can the theory uncover the specific properties of the objects one studies by means of the differential equations they satisfy, in contradistinction to what depends only on the particular representation of these objects by numbers or sets of numbers. In order to achieve such an invariant theory, Cartan made a systematic use of the notion of the exterior differential of a differential form, a notion which he helped to create and which has just the required property of being invariant with respect to any change of variables.

Raised in the French geometrical tradition, Cartan had a constant interest in differential geometry. He had the unusual combination of a vast knowledge of Lie groups, a theory of differential systems whose invariant character was particularly suited for geometrical investigations, and, most important of all, a remarkable geometrical intuition. As a result, he was able to see the geometrical content of very complicated calculations, and even to substitute geometrical arguments for some of the computations. The latter practice has often been baffling to his readers. But it is an art whose presence is usually identical with the vigor of a geometrical thinker.

In the 1920's the general theory of relativity gave a new impulse to differential geometry. This gave rise to a feverish search of spaces with a suitable local structure. The most notable example of such a local structure is a Riemann metric. It can be generalized in various ways, by modifying the form of the integral which defines the arc length in Riemannian geometry (Finsler geometry), by studying only those properties pertaining to the geodesics or paths (geometry of paths of Eisenhart, Veblen, and T. Y. Thomas), by studying the properties of a family of Riemann metrics whose fundamental forms differ from each other by a common factor (conformal geometry), etc. While in all these directions the definition of a parallel displacement is considered to be the major concern, the approach of Cartan to these problems is most original and satisfactory. Again the notion of group plays the central rôle. Roughly speaking, a generalized space (espace généralisé) in the sense of Cartan is a space of tangent spaces such that two infinitely near tangent spaces are related by an infinitesimal transformation of a given Lie group. Such a structure is known as a connection. The tangent spaces may not be the spaces of tangent vectors. This generality, which is absolutely necessary, gave rise to misinterpretation among differential geometers. As we shall show below, it is now possible to express these concepts in a more satisfactory way, by making use of the modern notion of fiber bundles.

We can perhaps conclude from the above brief description that Cartan's mathematical work, unlike that of Poincaré or Hadamard, centers around a few major concepts. This is partly due to the richness of the field, in which his pioneering work has opened avenues where much further development is undoubtedly possible. While many of Cartan's ideas have received clarification in recent years, the difficulties of conceiving the proper concepts at the early stage of development can hardly be overestimated. Thus in writing on the psychology of mathematical thinking, Hadamard had to admit "the insuperable difficulty in mastering more than a rather elementary and superficial knowledge of Lie groups."[4] Thanks to the development of modern mathematics, such difficulties are now eased.

Besides several books Cartan published about 200 mathematical papers. It is earnestly to be hoped that the publication of his collected works may be initiated in the near future. Not only do they fully deserve to find their place on the bookshelves of our libraries at the side of those of other great mathematicians of the past, but they will be, for a long time to come, a most indispensable tool for all those who will attempt to proceed further in the same directions.

We now proceed to give a more detailed review of some of the most important of Cartan's mathematical contributions.

I. GROUP THEORY

Cartan's papers on group theory fall into two categories, distinguished from each other both by the nature of the questions treated and by the time at which they were written. The papers of the first cycle are purely algebraic in character; they are more concerned with what are now called Lie algebras than with group theory proper. In his thesis [3], Cartan gives the complete classification of all simple Lie algebras over the field of complex numbers. They fall into four general classes (which are the Lie algebras of the unimodular groups, of the orthogonal groups in even or odd numbers of variables, and of the symplectic groups) and a system of five "exceptional" algebras, of dimensions 14, 52, 78, 133, and 248. Killing had already discovered the fact that, outside the four general classes, there can exist only these five exceptional Lie algebras; but his proofs were incorrect at several important points, and, as to the exceptional algebras, it is not clear from his paper whether he ever proved that they actually existed. Moreover, in his work, the algebra of dimension 52 appears under two different forms, whose equivalence he did not recognize. Cartan gave rigorous proofs that the classification into four general classes and five exceptional algebras is complete, and constructed explicitly the exceptional algebras.

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[4] J. Hadamard, *The psychology of invention in the mathematical field*, Princeton, 1945, p. 115.

Let $\mathfrak{g}$ be any Lie algebra; to every element X of $\mathfrak{g}$ there is associated a linear transformation, the adjoint ad X of X, operating on the space $\mathfrak{g}$, which transforms any element Y of $\mathfrak{g}$ into $[X, Y]$. Because of the relation $[X, X]=0$, this linear transformation always admits 0 as a characteristic root; those elements X of $\mathfrak{g}$ for which 0 is a characteristic root of least possible multiplicity of ad X are called regular elements. Let H be a regular element; then those elements of $\mathfrak{g}$ which are mapped into 0 by powers of ad H are seen to form a certain subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, and this subalgebra is always nilpotent (which means that, in the adjoint representation of such an algebra, every element has 0 as its *only* characteristic roots). A subalgebra such as $\mathfrak{h}$ has been called a Cartan subalgebra of $\mathfrak{g}$. It is a kind of inner core of the algebra $\mathfrak{g}$, and many properties of the big algebra $\mathfrak{g}$ are reflected in properties of this subalgebra $\mathfrak{h}$. In the case where $\mathfrak{g}$ is semi-simple, $\mathfrak{h}$ is always abelian (which means, for a Lie algebra, that $[X, Y]$ is always 0 for any X and Y in the algebra). Moreover, $\mathfrak{g}$ has a base which is composed of elements which are eigenvectors simultaneously for all adjoint operations of elements of $\mathfrak{h}$. The factors by which these elements are multiplied when bracketed with elements of $\mathfrak{h}$ are called the *roots* of the Lie algebra; it is the study of the properties of these roots which leads to the classification of simple Lie algebras. In establishing these properties, Cartan made a systematic use of the "fundamental quadratic form" of $\mathfrak{g}$, whose value at an element X is the trace of the square of ad X (if $\mathfrak{g}$ is semi-simple, or more generally if it coincides with its derived algebra, then the trace of ad X itself is always zero). One of the most important results of Cartan's thesis is that a necessary and sufficient condition for $\mathfrak{g}$ to be semi-simple is that its fundamental quadratic form be nondegenerate (i.e. that its rank be equal to the dimension of $\mathfrak{g}$). Incidentally, Cartan also applied similar methods to the study of systems of hypercomplex numbers (cf. [4]) and obtained in this manner the main structure theorems for associative algebras over the fields of real and of complex numbers; however, these results were superseded by the work of Wedderburn, which applies to algebras over arbitrary basic fields. By studying those algebras which have only one integrable (or, as we say now, solvable) ideal, Cartan also laid the foundations in his thesis for his subsequent study of linear representations of simple Lie algebras; in particular, he determined, for each class of simple groups, the linear representation of smallest possible degree.

The general theory of linear representations is the object of the paper [5]. As above let $\mathfrak{g}$ be any semi-simple Lie algebra over the field of complex numbers (any algebraically closed field of characteristic 0 would do just as well); a linear representation of $\mathfrak{g}$ is a law which assigns to every X in $\mathfrak{g}$ a linear transformation $\rho(X)$ on some finite-dimensional space; $\rho(X)$ depends linearly on X, and is such that $\rho([X, Y])=\rho(X)\rho(Y)-\rho(Y)\rho(X)$ for any X and Y in $\mathfrak{g}$. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. Then it turns out that the matrices which represent the elements of $\mathfrak{h}$ may all be reduced simultaneously to the diagonal form; the diagonal coefficients which occur in these matrices, considered as linear functions of the element which is represented, are called the *weights* of the representation. The roots of $\mathfrak{g}$ are the weights of a particular linear representation, viz. the adjoint representation. Cartan proved that all relations between weights of one or several representations are consequences of certain linear relations with rational coefficients between these weights, a fact which can now be explained in two different manners: it reflects the properties of characters of compact abelian groups, and also the properties of algebraic groups of linear transformations. Cartan then introduces an order relation in the system of all weights and roots, and proves that any irreducible representation is uniquely determined by its highest weight for this order relation. The problem of finding all irreducible linear representations of $\mathfrak{g}$ is thereby reduced to that of finding all possible highest weights of representations. The sum of the highest weights of two irreducible representations is again the highest weight of an irreducible representation, which is contained in the tensor product (or rather, sum, if we speak of representations of Lie algebras and not of groups) of the two given representations. If r is the rank of the Lie algebra $\mathfrak{g}$ (i.e. the dimension of any Cartan subalgebra of $\mathfrak{g}$), Cartan established that all possible highest weights of irreducible representations may be written as linear combinations with non-negative integral coefficients of r particular linear functions which depend only on the structure of $\mathfrak{g}$ and the order relation in the system of roots. Considering one by one the various types of simple Lie algebras, he established that every one of these r basic functions is the highest weight of some irreducible representation; this led to a complete classification of all irreducible linear representations of simple Lie algebras. This theory of linear representations was later completed in an important point by H. Weyl, who established by transcendental methods that every representation of a semi-simple Lie algebra is completely reducible, and who expressed the degree of an irreducible representation in terms of its highest weight. It has also recently become possible to give a direct proof of the existence of irreducible representations corresponding to the possible highest weights predicted by Cartan's theory, a proof which applies not only to simple but also to semi-simple algebras (Lie algebras behave differently from associative algebras in this respect that algebras which are not simple may have faithful irreducible representations; this happens for instance for the Lie algebra of the orthogonal group in 4 variables). In the process of classifying all possible linear representations, Cartan discovered the spin representations of the orthogonal Lie algebras, which later played such an important rôle in physics. In a book published later (*Leçons sur la théorie des spineurs*, Hermann, Paris, 1938), Cartan developed the theory of spinors from a geometric point of view.

In [6] Cartan classifies all simple Lie algebras over the field of real numbers instead of that of complex numbers. The method is to study the "complexification" of the Lie algebra under consideration; this complexification is either simple or the sum of two simple Lie algebras, and there is defined in it an operation of passing to the imaginary conjugate which admits the elements of the given simple Lie algebra as fixed elements. Starting with a suitable complex algebra, Cartan determines all possible operations of conjugation in it and arrives in this manner at a complete classification of simple real Lie algebras (the method was later simplified by himself and by others, making use of the compact real forms and replacing the determination of all possible conjugations by that of all classes of involutive automorphisms of the compact form).

It turned out that, but for one exception, the structure of a simple real Lie algebra is characterized by that of its complexification and by its *character*, i.e., by the index of inertia of its fundamental quadratic form. Cartan noticed that, for every complex form, there is a unique real form whose fundamental quadratic form is negative definite; this real form is the Lie algebra of a compact Lie group. This fact was to play a very important rôle in the subsequent theory of Lie groups, because it establishes a one-to-one correspondence between semi-simple connected complex Lie groups and compact semi-simple Lie groups. While the former are more readily amenable to an algebraic study, because of the algebraically closed character of the basic field, the latter lend themselves more easily to study by transcendental methods because the volume of the whole group (in the sense of Haar measure) is finite.

The last paper of the first cycle is [7] in which Cartan determined all real linear representations of simple real Lie algebras.

The work of Cartan's second group-theoretic period is concerned with the groups themselves, and not with their Lie algebras, and in general with the global aspect of the group. This period opens with a paper [8] which contains a study of group manifolds from a local differential geometric point of view. A group G may operate on itself in three different ways: by the left translations, by the right translations, and by transformation (we call here transformation a mapping $t \rightarrow sts^{-1}$, where s is a fixed element of the group). In relation to this, Cartan shows that there are three affine connections which are intrinsically defined on G. Two of these connections (those which correspond to the left and right translations) are without curvature but in general have torsion; the third one has no torsion but has in general a curvature. The geodesic lines are the same for all three connections: they are the cosets with respect to the one-parameter subgroups. Cartan determines also the totally geodesic varieties on G (varieties such that any geodesic which is tangent to it is entirely contained in it); they are of two different kinds. The varieties of the first kind are the subgroups of G and their cosets. The varieties of the second kind are determined by what have since been called the Lie triple systems contained in the Lie algebra $\mathfrak{g}$ of G, i.e., the linear subspaces of $\mathfrak{g}$ which, together with three elements X, Y, and Z, contain the element $[[X, Y], Z]$.

After the paper we have just mentioned, Cartan's interest orients itself very definitely towards the topological study of Lie groups in the large.

This period begins around 1925, at the time when H. Weyl had just published his fundamental papers on the theory of compact Lie groups. It is difficult to appreciate to what extent Cartan was influenced by Weyl's methods and results; at any rate his book *Leçons sur la Géométrie des espaces de Riemann* shows clearly that, even before Weyl's paper, Cartan was already getting more and more interested in topological questions. Whereas Weyl's line of attack was, if we may say so, brutally global, depending essentially on the method of integration on the whole group, the work of Cartan puts the emphasis on the connection between the local and the global. This essential difference has a great bearing on the nature of the results it is possible to expect from these two methods. Weyl's methods are not bound to the differentiable structure of the group under consideration; as soon as the possibility of integrating on any locally compact group was established Weyl's results could be extended to all compact topological groups. However, the assumption of compactness is essential (it insures the convergence of the integrals on the group), and Weyl's methods give nothing on noncompact groups, whereas those of Cartan, applied in the domain of Lie groups, have led to a very complete knowledge of the topology of these groups, whether compact or not.

In his paper [9] Cartan studies the topology of compact semi-simple Lie groups and of their complexifications. Let G be a compact semi-simple Lie group, $\mathfrak{g}$ the Lie algebra of G, and $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$. Cartan establishes that every element of G belongs to a one-parameter subgroup (at least), and that every infinitesimal transformation may be transformed by an operation of the adjoint group into an element of $\mathfrak{h}$. Every element H of $\mathfrak{h}$ gives rise to a one-parameter subgroup with a definite parameter on it; let us denote by $\exp H$ the point of parameter 1 on this subgroup. Then every element of G is conjugate to some element of the form $\exp H$; the next question is to find out under which condition two elements $\exp H$ and $\exp H'$ of this form are conjugate to each other. Cartan shows that a necessary and sufficient condition for this to happen is that H' can be transformed into H by an operation of a certain discontinuous group S operating on the space $\mathfrak{h}$. This group admits a fundamental domain which is a polyhedron P with a finite number of vertices. The points of P, with suitable identification of faces, will represent in a one-to-one manner the classes of conjugate elements of G. The inner points of P correspond to the regular elements of the group, i.e., to those elements which may be represented in the form $\exp H$, where H is a regular element of $\mathfrak{h}$. It follows that any closed path in G which does not meet the set of singular (i.e., not regular) elements will be represented by a continuous path in P, not necessarily closed. Now, it turns out that the singular elements of G form a set of dimension at least 3 less than the dimension of the whole group; as a consequence they may be entirely disregarded in the determination of the fundamental group. This allows one to proceed to the determination of this fundamental group π by the mere consideration of the polyhedron P itself; the order of π is the number of vertices H of P such that $\exp H$ is the unit element of G. From this follows Weyl's theorem to the effect that the fundamental group of a compact semi-simple Lie group is finite. This implies that, if the fundamental quadratic form of a Lie group G is definite negative, then not only does there exists at least one compact group which is locally isomorphic with G, but G itself is compact. Moreover Cartan's methods allow one to determine, for every semi-simple infinitesimal structure with a negative definite fundamental quadratic form, the number of times that the simply-connected group with this structure covers the adjoint group, and to study the various types of closed one-parameter groups (or geodesics) in the space of a compact group. The last part of the paper, devoted to the study of simple complex groups, is a prelude to the future theory of noncompact simple groups; it is proved in particular that, for any simple Lie algebra over the complex numbers, there always exists a simply-connected complex *linear* group having this Lie algebra.

After the determination of the fundamental group was accomplished, the next step was to determine the higher-dimensional Betti numbers of compact Lie groups. The method of accomplishing this was indicated in Cartan's paper [11]. Cartan considers a homogeneous space E whose group of transformations G is compact; the space may then be considered to be the space of cosets of G modulo some closed subgroup g. Let us say that two exact differential forms are equivalent when their difference is the differential of a form of degree $p-1$, and let b_p be the maximal number of forms of degree p no linear combination of which is equivalent to 0. It was conjectured, but not yet proved at the time, that b_p is equal to the pth Betti number of the space E (this was established soon afterwards by de Rham). Let ω be any exact form of degree p. Any operation s of the group G transforms into a new exact form $s\omega$, which Cartan proves to be equivalent to ω. He then constructs the average of the form $s\omega$, s running over all elements of the group G; this new form is still exact, is equivalent to ω, and is furthermore invariant under the group G. Moreover, Cartan proves by a similar argument that, if ω is invariant and equivalent to 0, then ω is the differential of a form which is itself invariant. These theorems reduce the determination of the number b_p to that of the integral invariants of the space E. Cartan then shows that the latter problem may be reduced to a purely algebraic problem depending only on the Lie algebra of the group G and the subalgebra corresponding to the subgroup g (this, at any rate, in the case where the group g is connected). This algebraic problem has been solved since for the case where E is either the group G itself or a symmetric Riemann space with G as its group of isometric transformations.

We now come to the results obtained by Cartan in the study of Lie groups which are not compact. The proof he gave of the converse of Lie's third theorem (i.e., every Lie algebra over the field of real numbers is the Lie algebra of some group) implied that every simply-connected Lie group is topologically the product of a Euclidean space by the space of a simply connected semi-simple group. The problem was therefore reduced to the special case in which the group G under consideration is semi-simple. Let then $\mathfrak{g}$ be its Lie algebra, and let $\mathfrak{g}'$ be the complexification of $\mathfrak{g}$. Since $\mathfrak{g}'$ is semi-simple, it has a compact real form $\mathfrak{g}_c$; i.e., it may be considered as the complexification of a Lie algebra $\mathfrak{g}_c$ which is the algebra of a compact semi-simple group. Cartan proves (in his paper [10]) that there exists an involutory automorphism a of $\mathfrak{g}_c$ such that $\mathfrak{g}$ is spanned by those elements of $\mathfrak{g}_c$ which are invariant under a and by the products by i of those elements of $\mathfrak{g}_c$ which are changed into their opposites by a. The subalgebra $\mathfrak{g}\cap\mathfrak{g}_c$ is the Lie algebra of a maximal compact subgroup g of G. Cartan then considers the space R whose points are the conjugates of g in G; the essential fact is that the existence of the automorphism a implies that R is a symmetric Riemannian space, having the adjoint group Γ of G as its group of isometries (more precisely, Γ is the component of the identity in the group of isometries of R). This opened the way to the application of the theory of symmetric Riemannian spaces which Cartan had already developed for its own merits (cf. the part of this article which is concerned with the geometric aspects of Cartan's work). Assume that G is Γ itself. Then every operation of G may be decomposed uniquely into a rotation around the origin (the origin in R being the point which represents the group g itself) and a transvection. The transvections are the operations of G which result from the integration of those infinitesimal transformations which are changed into their negatives by the operation a. They form a subvariety of G, homeomorphic to R, and it turns out that this variety has the topological structure of a Euclidean space. This establishes that the adjoint group of any connected semi-simple Lie group is topologically the product of the space of a compact group by a Euclidean space. Moreover, making use of the fact that any compact group of isometric transformations of the space R admits a fixed point, Cartan proves that every compact subgroup of G is conjugate to a subgroup of g. These results relative to the adjoint groups may be extended without any difficulty to the simply-connected groups with the same infinitesimal structure; they are also true for all intermediary groups, as follows from the recent work of Iwasawa.

II. SYSTEMS OF DIFFERENTIAL EQUATIONS

The principal paper of Cartan on the theory of differential systems is [15]. The reader will find a very clear exposition of the theory of Pfaffian systems in Cartan's book: *Les systèmes différentiels extérieurs et leurs applications géométriques*, Paris, Hermann, 1945.

A Pfaffian system is a system consisting of a certain number of equations of the form $A_1dx_1+ \cdots +A_ndx_n=0$, where $A_1, \cdots, A_n$ are functions of $x_1, \cdots, x_n$, and possibly of certain equations of the form $F(x_1, \cdots, x_n)=0$. A parametric r-dimensional manifold, given by $x_i=f_i(t_1, \cdots, t_r)$ $(1\leqq i\leqq n)$ is a solution of the system if the equations of the system become identically satisfied when one replaces the variables x and their differentials by their expressions in terms of the variables t and their differentials.

It is first necessary to indicate how any system of differential equations may be reduced to a Pfaffian system. If the system contains equations of order higher than one, we may first reduce it to the order one by introducing new unknown functions which represent certain derivatives of the original ones. This being done, we obtain a system of equations of the form $F_i(x_1, \cdots, x_m; z_1, \cdots, z_p; \cdots, \partial z_r/\partial x_s, \cdots)=0$ where the z's are the unknown and the x's the independent variables. If we set $\partial z_r/\partial x_s=t_{rs}$, the original system of partial differential equations may be replaced by the Pfaffian system composed of the equations $F_i(x; z; t)=0$, $dz_r-\sum_s t_{rs}dx_s=0$. The solutions of the original system correspond to those solutions of the Pfaffian system which are manifolds of dimension r on which the variables $x_1, \cdots, x_n$ are independent.

The essential originality of Cartan consists in having introduced, besides the Pfaffian forms, the exterior differential forms of higher degree. The algebra of exterior forms had been developed by Grassmann for geometric purposes; before it could be used in the theory of differential systems, it was necessary to introduce the operation of exterior differentiation. An exterior differential form is an expression of the form $\sum A_{i_1\cdots i_p}\, dx_{i_1}\cdots dx_{i_p}$, where the coefficients $A_{i_1\cdots i_p}$ are functions of the variables x; such expressions may be multiplied with each other with the convention that $dx_i dx_j = -dx_j dx_i$ (in particular, $(dx_i)^2 = 0$). The exterior differential of the form written above is $\sum dA_{i_1\cdots i_p}\, dx_{i_1}\cdots dx_{i_p}$ where it is understood that the differentials $dA_{i_1\cdots i_p}$ are expressed as linear combinations of $dx_1, \cdots, dx_n$. The fundamental property of the operation of differentiation is that it is invariant with respect to any change of variables. Now, let us consider any Pfaffian system $\omega_1 = 0, \cdots, \omega_k = 0$, $F_1 = 0, \cdots, F_m = 0$, where $F_1, \cdots, F_m$ are functions and $\omega_1, \cdots, \omega_k$ Pfaffian forms; then we see immediately that any solution of the system will also be a solution of the system obtained by adjoining to the original one the equations $dF_1 = 0, \cdots, dF_m = 0$, $d\omega_1 = 0, \cdots$, $d\omega_k = 0$. More generally, let I be the smallest set of differential forms containing $F_1, \cdots, F_m, \omega_1, \cdots, \omega_k$ and such that, whenever ω and ω' are in I, then $\omega + \omega'$ is in I, the product of ω by an arbitrary differential form is in I, and $d\omega$ is in I; I is called the differential ideal generated by $F_1, \cdots, F_m, \omega_1, \cdots, \omega_k$. Then any solution of the original system will be a solution of the system obtained by equating to 0 all forms in I. The operation of adjoining to a system the exterior differentials of its forms is an invariant counterpart of the method of obtaining conditions of compatibility by writing that certain higher derivatives which may be computed in two different ways by means of the equations of the system have the same value.

By a contact element E_p of dimension p in the Cartesian n-space R^n is meant a pair (M, P) formed by a point M of R^n and a p-dimensional linear subspace P of R^n going through M. It is sometimes necessary to generalize this notion to the case of an arbitrary n-dimensional manifold V instead of R^n; M is then any point of V, and P any p-dimensional vector subspace of the n-dimensional tangent vector space to V at M. The totality of all p-dimensional contact elements of a given manifold V is itself a manifold V_1, the so-called *first prolonged manifold* of V. If $x_1, \cdots, x_n$ are local coordinates at the origin of the contact element (M, P), P may be represented parametrically by equations of the form $dx_i = L_i(v_1, \cdots, v_p)$, where the L_i's are linear forms in p parameters $v_1, \cdots, v_p$. Let ω be a differential form; if we substitute the coordinates of the origin of a contact element E_p for the variables $x_1, \cdots, x_n$ in the coefficients of ω, and the linear forms L_i for the differentials dx_i which occur in ω, we obtain an exterior form in the variables v. If this form is 0, then we say that ω is 0 at E_p. If we are given a differential ideal I of differential forms, and if every form of I is 0 at E_p, then we say that E_p is *an integral element* of I.

Let W be a p-dimensional submanifold of V. If M is a point of W, we may represent W locally around M by equations of the form $x_i=f_i(u_1, \cdots, u_p)$, the u_i's being parameters. The contact element (M, P) formed by M and by the p-dimensional tangent space P to W at N is called the tangential element of W at M; the space P may be represented by the equations $dx_i=df_i$, the du_i's taking the place of the parameters v_i considered above. A solution (or integral manifold) of the system obtained by equating to 0 the forms of a differential ideal I is a manifold W whose tangential contact elements are integral elements of I. The problem of looking for such solutions may be decomposed into two parts: the determination of all integral elements, which is an algebraic problem, and the determination of the ways of grouping these integral elements together in such a way that they may be the tangential elements of some manifold.

The ideal I may contain forms of degree 0, i.e., functions of the variables. Assume that these functions, equated to 0, represent an irreducible analytic manifold V_0. Assume that any point of V_0 is the origin of ∞^{r_1} integral elements of dimension 1. There may be points of V which are the origins of more than ∞^{r_1} integral elements of dimension 1, but they form lower-dimensional submanifolds of V. A point of V which does not lie on any one of these submanifolds is called an *ordinary* point. The smallest manifold (in the space of integral elements of dimension 1) which contains the integral elements whose origins are ordinary points is called the manifold of *general integral elements* of dimension 1; the origin of a general integral element is not necessarily ordinary. Assume now that any general integral element of dimension 1 is contained in ∞^{r_2} integral elements of dimension 2; then those integral elements of dimension 1 which are not contained in more than ∞^{r_2} integral elements of dimension 2 are called ordinary; proceeding as above, we define the notion of a general integral element of dimension 2. We may continue in the same manner, and define inductively the integers $r_1, r_2, \cdots, r_n$. For a certain dimension n, the r_{n+1} will be 0, which means that not every general integral element of dimension n will be contained in an integral element of dimension $n+1$. The number n is called the *genus* of the system, and the system is said to be *in involution* for every dimension not greater than n. A p-dimensional general integral element E_p is called *regular* if there exists a chain $E_0 \subset E_1 \subset \cdots$

$\subset E_{p-1} \subset E_p$ where, for each $i<p$, E_i is an ordinary *general* integral element of dimension i. A p-dimensional *general* solution of I is a p-dimensional manifold whose tangential contact elements are general integral elements of I, at least one of these integral elements being regular. In this manner, Cartan succeeded for the first time in giving a precise definition of the notion of the general solution of any differential system. The existence theorem for general solutions states that any regular p-dimensional integral element E_p is a tangential contact element of some manifold which is a solution of I. More precisely, if E_{p-1} is a regular element of dimension $p-1$ contained in E_p and is a tangential element of a $(p-1)$-dimensional integral manifold V^{p-1} of I, then V^{p-1} is contained in at least one integral manifold V^p tangent to E_p. This general theorem allowed Cartan to determine exactly the degree of indetermination of the general solution (i.e. on how many arbitrary constants, arbitrary functions of 1, 2, $\cdots$ arguments, it depends). Its application is however limited to the consideration of *analytic* differential systems and to the determination of *analytic* solutions.

The next step was to try to determine the *singular solutions* of the system, i.e. the solutions which are not in the general solution (for instance, for a differential equation in the plane, the envelope of the general solutions). Here the idea of Cartan was to construct from the given differential system new systems which are obtained from it by *a method of prolongation*, in such a manner that any singular solution of the original system should become a general solution of one of these new systems. The method consists, generally speaking, in adjoining new unknowns which are the coordinates of integral elements which are not general, and constructing *a priori* the finite and differential equations these new unknowns must satisfy. However, an exact description of the method would be a little too long to be given here. In every concrete case in which it was applied, Cartan's method led to the complete determination of all singular solutions. But a general proof that it always does so is still missing; this is a theme of research which would richly deserve to attract the attention of ambitious young mathematicians.

One of the main applications of Cartan's theory of systems of differential equations is his theory of infinite groups of transformations (cf. [16; 17; 18; 19]). We touch here a branch of mathematics which is very rich in results but which very badly needs clarification of its foundations. For the infinite Lie groups, in spite of their name, are probably no groups at all, in the precise sense the word has received in modern algebra; what they really are is not clear yet. Lie defined them as follows: he considers a set of analytic transformations $x'_i = F_i(x_1, \cdots, x_n)$ on n variables which is closed with respect to the ordinary operations of forming the product of two transformations and taking the inverse of a transformation of the set, and which has furthermore the property of being composed of all transformations of the form indicated above for which the functions F_1 satisfy a certain system of partial differential equations. The hitch is of course that nothing is said about the domains in which the transformations are to be defined and invertible, and that this domain may apparently vary from one transformation to another. Cartan establishes that any Lie group, whether finite or infinite, may be defined (after possible adjunction of new variables, which transform in a suitable way when the original variables are transformed by an operation of the group) to be the group of all transformations which leave invariant a certain number of functions and Pfaffian forms. A simple but not typical example is the group of transformations of the form $x' = F(x)$, $y' = G(y)$ on two variables x and y (where F and G are arbitrary analytic functions). It may be considered to be the group which leaves invariant the two Pfaffian forms udx and vdy, where u and v are new variables, which are transformed as follows: we have $u' = u(dF/dx)^{-1}$, $v' = v(dG/dx)^{-1}$. Having written a group in the form we have just indicated, Cartan was able to extend to infinite groups the structure theory which Lie had developed for finite groups. Assume that we have a group G which is defined by the conditions that some of the variables, say $x_{r+1}, \cdots, x_n$, are invariant and that some Pfaffian forms $\omega_1, \cdots, \omega_k$ are invariant, the ω_i's containing only the differentials of the variables $x_1, \cdots, x_n$ but their coefficients involving possibly certain other variables u. Then we may write

$$d\omega_i = \sum c_{ijk}\omega_j\omega_k + \sum a_{ijk}\omega_j\eta_k$$

where the η_k's are certain linear combinations of the differentials of the auxiliary variables u. Cartan shows that it is always possible to assume that the coefficients c_{ijk}, a_{ijk} depend only on the invariants $x_{r+1}, \cdots, x_n$ (if the group may be put into a form where it is transitive, which always happens for finite-dimensional groups, then the c_{ijk}'s and a_{ijk}'s are constants). These coefficients define the *structure* of the group. Just as in the case of finite-dimensional groups, they cannot be taken arbitrarily; Cartan gave the conditions they must satisfy in order to define a group, thus generalizing to infinite groups Lie's third fundamental theorem.

The operation of adjoining new variables to those which are transformed by a group is called the prolongation of the group. Cartan says that two groups are isomorphic to each other if they admit prolongations which are similar, i.e., which can be deduced from each other by a change of independent variables. He showed how it is possible to recognize whether two infinite groups whose structures are known are isomorphic or not. He applied this method to the problem of the classification of simple infinite groups, and found that they fall into 8 general types.

Cartan's theory of infinite groups had its origin in the study he made of equivalence problems. The general problem can be formulated as follows: Let G be a linear group acting in a space of n dimensions. Let $\theta_1, \cdots, \theta_n$ and $\bar{\theta}_1, \cdots, \bar{\theta}_n$ be two sets of linearly independent Pfaffian forms in the variables $x_1, \cdots, x_n$ and $\bar{x}_1, \cdots, \bar{x}_n$, respectively. Determine whether there exists an admissible transformation of coordinates

$$\bar{x}_i = \bar{x}_i(x_1, \cdots, x_n), \qquad i = 1, \cdots, n,$$

such that

$$\bar{\theta}_i = \sum_{j=1}^{n} a_{ij}(x)\theta_j, \qquad i = 1, \cdots, n,$$

where the linear transformation belongs to G. To treat this problem let $u_1, \cdots, u_m$ be the parameters of G and let us introduce the Pfaffian forms

$$\omega_i = \sum_{j=1}^{n} a_{ij}(u)\theta_j,$$

$$\bar{\omega}_i = \sum_{j=1}^{n} a_{ij}(\bar{u})\bar{\theta}_j, \qquad i = 1, \cdots, n,$$

in which we regard the u's and the $\bar{u}$'s as auxiliary variables. The sets of forms θ_i and $\bar{\theta}_i$ are equivalent in the above sense if and only if $\bar{x}^i$, $\bar{u}^r$ can be determined as functions of x^j, u^s ($i, j=1, \cdots, n$; $r, s=1, \cdots, m$) so that

$$\bar{\omega}_i = \omega_i \qquad i = 1, \cdots, n.$$

Such a system may be discussed by the general methods for dealing with Pfaffian systems. The first step is, of course, to adjoin to the system the equations $d\bar{\omega}_i = d\omega_i$; if we express the forms $d\omega_i$ by means of the forms ω_i themselves and of the differentials of the auxiliary variables, the coefficients of these expressions, when they contain the variables x only, will yield invariants $I_k(x)$, and we may enlarge the original system by adjoining the equations $I_k(\bar{x}) = I_k(x)$, together with the equations which are obtained from them by differentiation.

Cartan shows that the continuation of this procedure eventually leads to a complete system of invariants which can be obtained by operations of differentiation only. However, when the systems one is led to consider are not in involution for the dimension n, the completeness of the system of invariants depends on the theorem that all singular solutions of a differential system may be obtained by the method of prolongation, a theorem which is not completely proved as yet (cf. above).

Among other applications which Cartan made of his theory of differential equations we mention the following: (1) Various applications to differential-geometric problems; (2) Principle of integral invariants in analytical dynamics; (3) Theory of general relativity.

As a matter of fact, the study of differential equations arising from problems of differential geometry had always interested him, and his papers on this subject run through most of his scientific career. The numerous examples given in [23] show quite decisively the advantages of using differential forms. One of the notable results is his proof of a conjecture of Schläfli to the effect that every Riemann metric of n dimensions can be imbedded locally into a Euclidean space of dimension $n(n+1)/2$ [22]. This theorem played no small part in Levi-Civita's original definition of his parallelism and has attracted the attention of differential geometers.

Also his work on integral invariants in analytical dynamics can be considered as an application of the theory of differential equations [21]. Mathematically the problem is to determine the trajectories, which are to be solutions of a differential system of the type

$$\frac{dx_i}{dt} = X_i(x_1, \cdots, x_n, t), \qquad i = 1, \cdots, n.$$

The standard way is by means of Hamilton's principle, which defines the trajectories as the extremals of a certain variational problem. Unfortunately the integrand of the latter does not have a simple physical interpretation. An alternative way was suggested by Poincaré. He called a multiple integral

$$\int \underset{p}{\cdots} \int \sum_{i_1, \cdots, i_p} a_{i_1 \cdots i_p}(x_1, \cdots, x_n, t) dx_{i_1} \cdots dx_{i_p}$$

invariant when its value over a domain covered by the trajectories is invariant under the motion. In fact, it is called absolute if the domain is arbitrary and relative if the invariance is true only for closed domains. If p_i, q_i, $i=1, \cdots, n$, are the canonical variables of a dynamical system with n degrees of freedom, Poincaré's principle asserts that the trajectories can be characterized as the curves admitting the relative integral invariant

$$\sum_{i=1}^{n} p_i dq_i.$$

Cartan's principle is a modification of Poincaré's. He derived his ideas from his theory of differential systems. The differential system of the trajectories has $2n-2$ functionally independent first integrals. Cartan observed that the property of an exterior differential form to be invariant and thus to depend only on the trajectories is that it is a form in these first integrals. Expressed in terms of the original variables, it may involve the independent variable t. Omitting from this the terms involving dt, we obtain the integrand of an invariant integral in the sense of Poincaré. Thus the latter is the truncated form of an invariant differential form (i.e., a form in the first integrals of the trajectories). Conversely, it can be proved that, given the integrand of an invariant integral of Poincaré, terms involving dt can be added to it so as to obtain an invariant differential form. Cartan's principle characterizes the trajectories as admitting an invariant differential form. Moreover, the latter has a simple physical interpretation. The work therefore furnishes an interesting complement to formal dynamics.

In connection with the general theory of relativity and the unified field theory Cartan studied on several occasions the question of the possible forms of the equations of gravitation and of the unified gravitational and electro-magnetic field. He made a very detailed analysis and determined all possible forms of such differential systems. He was also the first one to introduce the notion of Riemann spaces without curvature and with torsion, which later served as the basis of Einstein's unified field theory. Apparently these studies are not of the same importance as his studies on pure mathematics.

III. GEOMETRY

Although the theory of Lie groups has an intimate relationship with differential geometry, Cartan did not begin his substantial work on differential geometry until a relatively late stage. His first series of papers on differential geometry was concerned with the problem of deformation [27; 28]. It is clear that he had then all the essential ideas of the method of moving frames, one of his favorite subjects in later years, which has not been fully exploited even now.

The method was not new. It is a generalization, to an arbitrary homogeneous space, of the method of moving trihedrals, so successfully used by Darboux, Ribaucour, and others [39; 41]. Even in the most general case some of its essential ideas had been given by Emile Cotton. It is also closely related to Cesaro's "intrinsic method" in differential geometry, as later developed by Kowalewski. To Cartan the attraction was not the method, but the geometrical results to which it so effectively leads. It is interesting to notice in his book [41] how he took pleasure in studying numerous examples and did not care to discuss the generalities, except in very sketchy outlines.

We attempt to give a description of this method in modern terminology. The problem is the local theory of a p-dimensional submanifold M in a homogeneous space E of dimension n, acted on by a Lie group G of dimension r. Let O be a point of E and H the subgroup of G leaving O fixed. Then the set of all transformations of G carrying O to a point P of E is a left coset gH of G relative to H, and E can be identified with the space of left cosets G/H. Under this identification the action of G on E is represented by left multiplication. This process depends on the choice of O. If we replace O by O' and if g_O is a transformation of G carrying O' to O, the subgroup of G leaving O' fixed will be $g_O^{-1}Hg_O$ and the set of transformations of G carrying O' to P will be gHg_O. In other words, the latter is defined up to multiplication by a fixed element to the right.

The method of moving frames is a method for the determination of differential invariants of M under G, and in fact for determining enough of them to enable us to decide whether two given submanifolds differ from each other only by a transformation of G. Its main idea is that of passing from the homogeneous space E to the group space G. In fact, denote by $\psi: G \rightarrow G/H$ the natural projection which assigns to an element $g \in G$ the coset gH. From M we get the submanifold $F_O = \psi^{-1}(M) \subset G$, determined up to multiplication by a fixed element on the right (depending on the choice of the point O). F_O is in general a manifold of dimension higher than p and will be called the manifold of frames of order 0 of M. Now the Lie algebra $\mathfrak{h}$ of H is a subalgebra of the Lie algebra $\mathfrak{g}$ of G. There is therefore in the dual space $\mathfrak{g}^*$ of $\mathfrak{g}$, whose elements are the so-called Maurer-Cartan forms, a linear subspace $\mathfrak{n}^*(G, H)$ of dimension n consisting of all elements of $\mathfrak{g}^*$ orthogonal to $\mathfrak{h}$. The dual mapping of the identity mapping $i_0; F_0 \rightarrow G$ maps the elements of $\mathfrak{n}^*(G, H)$ into Pfaffian forms on F_0, called by Cartan the principal components of order 0. Among them there are exactly p linearly independent ones, the others being their linear combinations. The coefficients of such linear combinations play an important rôle in the method of moving frames. From them it is sometimes possible to derive differential invariants of M by elimination.

In order to get more information about M, we have to pass to the elements of contact of higher order. The general principle is to extend the above considerations to them. Let M be defined locally by the equations

$$x_i = f_i(u_1, \cdots, u_p), \qquad i = 1, \cdots, n,$$

where the functions f_i possess sufficiently many continuous partial derivatives or even are analytic. The values, at a certain point, of the functions x_i and their partial derivatives up to the order s inclusive, subject to the usual laws of transformation when either the x's or the u's undergo an admissible coordinate transformation, constitute an element of contact C_s of order s. Thus an element of contact of order 0 is the point itself. Moreover, an element of contact of order $s>0$ determines uniquely an element of contact of order $s-1$, obtained by ignoring the derivatives of order s.

The totality of the elements of contact of order s, for all submanifolds of dimension p in E, is a space E_s on which G acts. The elements of contact of order s of M constitute a submanifold M_s of E_s. The case for general s differs from the case $s=0$ in two essential aspects: (1) The group G does not necessarily act in a transitive manner on E_s, so that the latter decomposes into domains of transitivity; (2) The subgroup H_s of G leaving fixed a given C_s may not be connected, as for instance in the case when E is the Euclidean space with the group of rigid motions, and $p=1$, $s=1$. In this example M is a curve and C_1 can be identified with the tangent direction; the motions leaving a line fixed have two connected components.

This phenomenon shows that the same C_s may carry several "oriented" elements of contact of order s, obtained by replacing H_s by the component of the unit element in H_s. The latter process involves an arbitrary choice, because only the class of conjugate subgroups of H_s is defined by C_s. On the other hand, the first fact, together with generality assumptions, allows us to coordinatize the generic oriented elements of contact of order s by a finite set of numbers. These are the differential invariants of order not greater than s.

Denote by T a domain of transitivity of E_s under G, by $H_s(T)$ the subgroup of G leaving one of its points fixed, and by $H_s'(T)$ the connected component of the unit element of $H_s(T)$. Then the space of oriented elements of contact of order s can be identified with the union $\cup_T G/H_s'(T)$, and those of M can be considered as a submanifold M_s of $\bigcup_T G/H'_s(T)$. If $\psi_{s,T}: G \to G/H'_s(T)$ is the natural projection, the submanifold

$$F_s = \bigcup_{a_s \in M_s} \overset{-1}{\psi}_{s,T}(C_s)$$

is called the manifold of frames of order s of M. Generalizing the situation on frames of order 0, let $\mathfrak{h}_s$ denote the Lie algebra of $H_s(T)$ and $\mathfrak{n}^*(G, H_s(T))$ the linear subspace of the dual space $\mathfrak{g}^*$ of $\mathfrak{g}$ which consists of all elements of $\mathfrak{g}^*$ orthogonal to $\mathfrak{h}_s$. The dual mapping of the identity mapping $i_s: F_s \to G$ maps the elements of $\mathfrak{n}^*(G, H_s(T))$ into Pfaffian forms on F_s, called the principal components of order not greater than s. The main feature of the method of moving frames is the result that the study of the manifold F_s in G gives some of the most important of the local geometric properties of M in E. The determination of the manifolds of frames of different orders is achieved by induction on s.

Cartan had a more geometrical picture of the frames. To him they are configurations in E such that there exists exactly one transformation of G carrying one such configuration into another. In Euclidean space with the group of rigid motions we can take as frames sets of points (P, U_1, U_2, U_3) with the properties: (1) the points U_1, U_2, U_3 are at a distance 1 from P; (2) any two of the lines PU_1, PU_2, PU_3 are perpendicular; (3) the vectors PU_1, PU_2, PU_3 form a right-handed system. For a surface in Euclidean space the frames of order 0 are those with P on the surface. The frames of order one satisfy the further condition that PU_3 is normal to the surface. Here an orientation has to be made according as U_3 is along one sense of the normal or the other. If P is not an umbilic, the frame of order two is uniquely determined at each point by the condition that PU_1, PU_2 are in the principal directions. The two principal curvatures are invariants of order two.

So far we have restricted our discussion to the generic elements of contact. Among the most interesting properties of differential geometry are perhaps those concerning the nongeneric ones. Thus the four-vertex theorem for closed plane curves and the theorem on the existence of umbilics on closed surfaces of genus $\neq 1$ are statements on the existence of certain types of elements of contact on a closed submanifold. No general result along this direction is known, and it is hoped that the method of moving frames will give some clues.

From this viewpoint the problem of deformation naturally presents itself. Two submanifolds M^p and M^{*p} in E are said to be applicable or deformable of order s relative to G if there is a transformation of G which carries the elements of contact of order s of M^p into those of

M^{*p}, that is, if there is a one-one mapping between the submanifolds under which the invariants and the principal components of order not greater than s are equal. When G is the group of motions in Euclidean space and $p=2$, $s=1$, this notion of applicability reduces to the classical one studied by Gauss, Minding, Darboux, and others. When G is the group of projective collineations in a real projective space and $p=2$, $s=2$, the problem was known as the projective deformation of surfaces and was studied at great length by Fubini and Čech.

When the order s is sufficiently large, the method gives a solution of the fundamental problem of local differential geometry in a homogeneous space, namely, that of deciding whether two submanifolds differ from each other by a transformation of the group G.

Actually it may be quite complicated to carry out the method (i.e., to determine the invariants and frames of different orders) in concrete cases, particularly when p is large. Moreover, the generality assumptions may soon become unrealistic. Cartan developed various ways of simplifying the computations and adapting the method to special cases. As frequently happens in mathematics, the generality of the viewpoint helps also to treat the special cases in a more effective way.

Some of the applications he made are: the conformal deformation of hypersurfaces [28], the projective deformation of surfaces [27], and the theory of submanifolds of constant curvature in a Euclidean or non-Euclidean space [25; 26]. The second was a problem which had received considerable attention from the Italian geometers. Cartan proved that, except for a class of surfaces which depends on six arbitrary functions of one variable, a surface is not deformable in a nontrivial way. Moreover, if a surface is projectively deformable (i.e., in a nontrivial way), the surfaces to which it is deformable depend at most on three arbitrary constants. This does not settle the problem. It perhaps makes the study of projectively deformable surfaces even more interesting. To cite an example, the question whether there exist surfaces projectively deformable to ∞^3 projectively inequivalent surfaces is not yet solved.

The study of the submanifolds of constant curvature in a Euclidean or non-Euclidean space generalizes the classical treatment of developable surfaces [25; 26]. Cartan made an exhaustive study and determined the degree of generality of such submanifolds. They do not exist under all circumstances. For instance, if the curvature of the submanifold of dimension p is less than the curvature of the space, the latter must have dimension not less than $2p-1$. His emphasis was on the necessity of investigating the existence of submanifolds with prescribed properties, and their degree of generality, before their study can be undertaken with sense. For this purpose his theory of differential systems in involution was applied to the best advantage.

Most remarkable among his works along this line are the results on the isoparametric families of hypersurfaces in a spherical space [42; 43]. It started with a problem of Levi-Civita: to study the scalar functions in a Riemannian space which are functionally dependent on both their first and second Beltrami differential parameters. The hypersurfaces obtained by equating such a scalar function to a constant are said to form an isoparametric family. When the Riemannian space is of zero or constant negative curvature, the determination of its isoparametric families of hypersurfaces does not cause much difficulty. This is due to the fact that every such hypersurface has at most two distinct principal curvatures. When the Riemannian space is of constant positive curvature, the situation is very complicated, but also most interesting. Cartan proved that in this case there do exist isoparametric families of hypersurfaces having three distinct principal curvatures but that this can happen only when the dimension of the space is 4, 7, 13, or 25. The last family admits the exceptional simple Lie group in 52 parameters; this was the first time that this group was geometrically realized. Similar results hold for isoparametric families of hypersurfaces in spherical space with four distinct principal curvatures. These exist only in spaces of 5 and 9 dimensions. This is one of the few instances of geometrical problems where the dimension of the ambient space plays an essential rôle.

Einstein's theory of general relativity gave a new impetus to differential geometry. In their efforts to find an appropriate model of the universe geometers have broadened their horizon from the study of submanifolds in classical spaces (Euclidean, noneuclidean, projective, conformal, etc.) to that of more general spaces intrinsically defined. The result is an extension of the work of Gauss and Riemann on Riemannian geometry to spaces with a connection, which may be an affine connection, a Weyl connection, a projective connection, of a conformal connection. In these generalizations, sometimes called non-Riemannian geometry, an important tool is the absolute differential calculus of Ricci and Levi-Civita. The results achieved are of considerable geometric interest. For instance, in the theory of projective connections, developed independently by Cartan, Veblen, Eisenhart, and Thomas, it is shown that when the space has a system of paths defined by a system of differential equations of the second order, a generalized projective geometry can be defined in the space which reduces to ordinary projective geometry when the differential system is that of the straight lines. Numerous other examples can be cited. The problem at this stage is twofold: (1) to give a definition of "geometry" which will include most of the existing spaces of interest; (2) to develop analytic methods for the treatment of the new geometries, it being increasingly clear that the absolute differential calculus is inadequate.

For this purpose Cartan developed what seems to be the most comprehensive and satisfactory program and demonstrated its advantages in a decisive way [31; 39]. This contribution clearly illustrates his geometric insight and we consider it to be the most important among his works on differential geometry. It can be best explained by means of the modern notion of a fiber bundle. Let $p: B \rightarrow X$ be a fiber bundle with fiber Y and structural group G. We assume X to be a differentiable manifold and G a Lie group. Let $U, V, W, \cdots$ be a covering of X by coordinate neighborhoods. A point of B belonging to $p^{-1}(U \cap V)$ has, with respect to U and V, respectively, the coordinates (x, y) and $(x, g_{UV}(x)y)$, $x \in U \cap V, y \in Y$, where we denote the action of G on Y by multiplication to the left. The function $g_{UV}(x)$ is defined for $x \in U \cap V$, with values in G. Its dual mapping maps the Maurer-Cartan forms of G into forms in $U \cap V$, to be denoted by ω^i_{UV}, $i=1, \cdots, r$. Let $(a^i_j(g))$, $g \in G$, denote the adjoint representation of G in the space of the Maurer-Cartan forms. A connection in the bundle is defined in each coordinate neighborhood U by a set of Pfaffian forms θ^i_U, $i=1, \cdots, r$, such that in $U \cap V$,

$$\theta^i_U = \omega^i_{UV} + \sum_{j=1}^{r} a^i_j(g_{UV})\theta^j_V, \qquad i = 1, \cdots, r.$$

It is easy to verify that this condition is coherent in the intersection of any three coordinate neighborhoods U, V, W.

The curvature tensor is given by the exterior quadratic differential forms

$$\Theta^i_U = d\theta^i_U - \frac{1}{2} \sum_{j,k=1}^{r} c^i_{jk}\theta^j_U \wedge \theta^k_U, \qquad i = 1, \cdots, r$$

where the c^i_{jk} are the constants of structure of G. Under a change of the coordinate neighborhood they are transformed according to the adjoint representation.

Actually Cartan proceeded in a different way. Guided by the classical notion of parallelism he laid more emphasis on the possibility of "developing" the fiber along a parametrized curve. In the present formulation this possibility arises from the fact that the differential system

$$\sum_{j=1}^{r} a_j^i(g)\theta^j + \omega^i(g) = 0, \quad i = 1, \cdots, r; g \in G,$$

is independent of the choice of the coordinate neighborhood. Corresponding to a parametrized curve in X there is a uniquely determined integral curve $g(t)$ of the differential system through the unit element of G. The curve $g(t)$ gives rise to a one-parameter family of transformations in Y. Cartan called this process the development along a parametrized curve, and took it as the definition of a connection.

Without the notion and terminology of fiber bundles it was difficult to explain these concepts in a satisfactory way. The situation was further complicated by the fact that Cartan called tangent space what is now known as fiber while the base space X, being a differentiable manifold, has a tangent space from its differentiable structure. But he saw clearly that the geometrical situation demands the introduction of fiber bundles with rather general fibers. Attempts by several other mathematicians to tie up the fiber with the differentiable structure of the base space were suggested by their experience from affinely connected spaces and led to complicated computations which have nothing to do with the geometrical problem.

When one takes the notion of a connection as the guiding principle in differential geometry, the fundamental problem is to define the fiber bundle and the connection in every geometrical problem. This is not at all a routine matter, and Cartan carried it out in various cases. Some of the more important ones are: (1) the projective connection of the geometry of paths [29]; (2) the conformal connection in the conformal theory of Riemann spaces; (3) the metrical connection in Finsler spaces; (4) the metrical connection in spaces based on the notion of area of hypersurfaces, now known as Cartan spaces [38]; (5) the geometry of the integral $\int F(x, y, y', y'')dx$, which is a generalization of plane Finsler geometry.

Results from these particular examples tend to substantiate the belief that the notion of a connection is a guiding principle in differential geometry. For instance, in Finsler geometry, the spaces of interest are those defining a "regular" problem in the calculus of variations. This fact manifests itself clearly when one tries to define a connection in the space.

In order that the connection in a space may be useful it must have a further property: it should give all the geometrical properties of the space. This can be made precise by the requirement that two spaces are to be equivalent under admissible transformations of the coordinates if and only if the connections are equivalent. This naturally leads to an equivalence problem whose analytical aspects have been discussed in Part II. It suffices to remark here that, while in simpler geometrical problems (such as Riemannian geometry) the introduction of a connection in the space automatically solves the equivalence problem, it is advisable in more general cases to go the other way around by first solving the equivalence problem and then interpreting the solution geometrically. Cartan's treatment of the equivalence problem furnishes a method which is particularly suitable for such geometrical problems.

Cartan, first in his Mémorial volume and later in his *Leçons*, applied his general ideas on connections to the case of Riemannian geometry [**30**; **35**]. Although he himself never refrained from computations, he did not hide his distaste for the computational work on differential geometry which was then very fashionable and much of which was of little geometrical interest. He stated his aim in the preface of [**35**] as that of bringing out the simple geometrical facts which have often been hidden under a debauch of indices. The result is a very original account of Riemannian geometry, still the standard book in the field.

His most important work on Riemannian geometry is undoubtedly the theory of symmetric Riemann spaces [**33**; **34**; **37**]. It is well known that the local properties of a Riemann metric are given by Riemann-Christoffel curvature tensor and its successive covariant derivatives. Besides the locally Euclidean spaces the simplest Riemann spaces are therefore the ones for which the covariant derivative of the Riemann-Christoffel tensor is zero. These spaces, which include the Riemann spaces of constant curvature, are called symmetric by Cartan, for an obvious reason which will be brought out below. He published papers on the subject during the period from 1927 to 1935. Perhaps because of their extensiveness the subject did not receive the attention it deserves. Its significance for the determination of the real forms of complex simple Lie groups has been discussed above. We proceed to give a very brief survey of the geometrical aspects of the theory, together with its relations to classical geometries, the theory of analytical functions of several complex variables, number theory, and topology.

Cartan soon discovered that the definition can be put in a more geometrical form. A symmetric Riemann space can be defined either as one for which the Levi-Civita parallelism preserves the sectional Riemannian curvature or as one for which the symmetry about a point is an isometry. From the second equivalent definition it follows immediately that the space admits a transitive group of isometries and that the connected component of the subgroup of isometries leaving fixed a point of the space is compact. This result brings the symmetric Riemann spaces into relation with homogeneous spaces.

The enumeration of all the symmetric Riemann spaces is not a simple problem. Cartan first observed that if a symmetric Riemann metric can be decomposed (locally) into a sum of two lower-dimensional Riemann metrics, each of the latter is symmetric. The problem is thus reduced to the irreducible case in which such a decomposition is not possible. Cartan then applied two different methods to the problem.

The first method consists in the determination of the subgroups of the orthogonal groups which can be the groups of holonomy of an irreducible symmetric Riemann space. Such a subgroup leaves invariant a form

$$\sum R_{ij,kl}x^iy^jx^ky^l,$$

where R_{ijkl} is the Riemann-Christoffel curvature tensor. It is thus not a most general subgroup of the orthogonal group, and this limitation makes it possible to carry out the program to the end. Unfortunately the method leads to very complicated computations.

It is the second method that opens up entirely unexpected views. Denote by G the connected component of the group of all isometries, and by H the connected component of the subgroup of G leaving a point O fixed. Then H is compact. If σ denotes the symmetry about O, the mapping which sends $g \in G$ into $\sigma g \sigma \in G$ is an involutory automorphism of G. Under this automorphism all elements of H remain fixed. Conversely, when a connected Lie group G has an involutory automorphism such that the connected component of the set of fixed elements is compact, the homogeneous space G/H can be given a symmetric Riemann metric. Now choose a base in the Lie algebra of G such that the endomorphism induced by the involutory automorphism changes the signs of some of the base vectors and leaves the remaining ones fixed. This normalization allows one to draw far-reaching conclusions on the infinitesimal structure of G. In fact, it follows that if the space, which we can now denote by G/H, is irreducible and is not locally Euclidean (that is, its Riemannian-Christoffel tensor is not 0), the group G is simple or is the direct product of two isomorphic compact simple groups.

In the latter case the elements of G can be written as (a, b), $a, b \in H$, H being a simple group. Then the involutory automorphism has to be $(a, b) \rightarrow (b, a)$, and the space can be identified with the space of H. In other words, this case reduces to the geometry of the space of a compact simple Lie group.

More interesting is the case in which G is simple. If G is a complex simple Lie group and H its compact real form, the space G/H is homeomorphic to a Euclidean space and is the only symmetric Riemann space with G as its group of isometries. Cartan called it the fundamental Riemann space of G. When G is a noncompact simple real Lie group, we consider its corresponding complex group G and the fundamental Riemann space E of G. The involutory automorphism in G, which maps every element into its complex conjugate, induces a symmetry in E leaving invariant a totally geodesic manifold of E. The latter is homeomorphic to a Euclidean space and is the only symmetric Riemann space with the group G. The situation is more complicated when G is a compact simple real Lie group, as then the symmetric Riemann space with G as the group of isometries is not unique. Thus there is, from the point of view of the infinitesimal structure, one and only one symmetric Riemann space belonging to a given noncompact simple group G: it is that of the homogeneous space G/H, where H is a maximal compact subgroup of G (which turns out to be uniquely determined up to an inner automorphism of G). For instance, the symmetric Riemann space belonging to the unimodular real linear group $GL_n(R)$ is the space of positive definite quadratic forms in n variables; to the unimodular complex linear group $GL_n(C)$ there belongs similarly the space of positive definite Hermitian forms in n variables. This undoubtedly accounts for the rôle played by those forms both in classical and in modern number theory.

The study of symmetric Riemann spaces also throws considerable light on the relations between Riemannian geometry and the classical geometries, and helps to unify and explain some of the phenomena in classical geometries. Cartan carried out this idea for the case of complex projective geometry in his book [36]. It is known, for instance, that there is a correspondence between the geometry on the complex projective line and non-Euclidean hyperbolic geometry in space. In the present terminology the hyperbolic space is the fundamental Riemann space of the group of projective collineations on the complex projective line.

Actually Cartan's interest in symmetric Riemann spaces was aroused by a related but different problem. It is the study of Riemann spaces which admit an absolute parallelism, whose autoparallel curves are geodesics. The guiding example is the Clifford parallelism in non-Euclidean elliptic space. The first result in this direction was achieved jointly by Cartan and J. A. Schouten. They found that the irreducible spaces with absolute parallelism are exactly the spaces of compact simple Lie groups with one exception, which is the 7-dimensional elliptic space. The existence of the latter is related to properties of Cayley numbers.

Another application of the theory of symmetric Riemann spaces is to functions of several complex variables. Henri Cartan studied the group of all pseudo-conformal transformations which leave invariant a bounded domain in a space of several complex variables and proved that it is a Lie group. Using this result, Cartan studied the domains which are homogeneous, that is, which admit a transitive group of pseudo-conformal transformations. He did not succeed in determining all such domains, because there are perhaps too many. However, he did determine all those which are also symmetric, that is, which have the further property that to every point O of the domain there exists an involutory pseudo-conformal transformation of the domain onto itself which admits O as an isolated invariant point. This is due to the fact that the group is then semi-simple. The irreducible bounded symmetric homogeneous domains form four large classes and two exceptional cases, of dimensions 16 and 27 respectively. These domains have recently been found to play an important rôle in Siegel's work on automorphic functions and analytic number theory, where the discontinuous subgroups of these groups are studied. No bounded homogeneous domain is known which is not symmetric.

The notion of a symmetric space can be extended to the case which Cartan called non-Riemannian. It is a homogeneous space G/H such that there is an involutory automorphism σ of G with H as the connected component of the subgroup of invariant elements of σ, where H is not necessarily compact. His main contribution to general symmetric homogeneous spaces is concerned with their Betti numbers, of which an account was given above in connection with the Betti numbers of compact Lie groups [11]. The results given there are valid for any compact symmetric homogeneous space, so that the determination of the Betti numbers of such a space can be reduced to a purely algebraic problem. Non-Riemannian symmetric spaces have otherwise hardly been studied.

Besides the importance of integral invariants in studying the topological properties of a space as a consequence of de Rham's theorems, they play a rôle in another field of geometry, now known as integral geometry. There again Cartan's exterior differential forms can be applied to the best advantage. In 1898 he devoted a paper to multiple integrals in the spaces of lines and planes of Euclidean space, which are invariant under the group of motions. The paper marks an important step in integral geometry, a subject founded by the English mathematician Crofton and later developed by Blaschke and his school. It is quite curious that, although Cartan laid much emphasis on the idea of defining a group as the set of transformations leaving invariant a set of linear differential forms and took this to be the starting point of his theory of infinite groups, he did not come back to invariant differential forms of higher degree, except in the paper discussed above. His exterior differential forms have now become an indispensable tool in integral geometry.

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GILBERT AMES BLISS
1876–1951

The contributions made by Gilbert Ames Bliss to mathematics and to educational and scientific activities in the United States were many and varied. The following is a quite inadequate summary of his life and work, with special reference to his mathematical activities.

Gilbert Bliss was born in Chicago on May 9, 1876, and died in Ingalls Memorial Hospital in Harvey, Illinois, on May 8, 1951. In 1893, one year after the University of Chicago first opened its doors to students, he enrolled there as a student, and was awarded the degrees of B.S. in 1897, M.S. in 1898, and Ph.D. in 1900. He studied at the University of Göttingen during the year 1902–1903. His first teaching was done as a substitute for a member of the staff of Kalamazoo College for several weeks during the year 1898–1899. He was an instructor in mathematics at the University of Minnesota 1900–1902, an associate at Chicago 1903–1904, assistant professor at the University of Missouri 1904–1905, preceptor (=assistant professor) at Princeton 1905–1908, and at Chicago was associate professor 1908–1913, professor 1913–1933, Martin A. Ryerson Distinguished Service Professor 1933–1941, and chairman of the department 1927–1941. He retired from active service in 1941. In various summer or autumn terms from 1906 to 1911 he gave courses at Wisconsin, Chicago, Princeton, and Harvard.

During his student days Bliss first fell under the influence of F. R. Moulton, then a young assistant who was teaching at Chicago, and his first published paper was entitled *The motion of a heavenly body in a resisting medium*. The fellowship in astronomy for which he applied was not granted, and he eventually decided to devote himself to pure mathematics. His thesis for the M.S. was entitled *The geodesic lines on the anchor ring*, and his thesis for the Ph.D. bore the same title. The first developed explicit formulas for the geodesics in terms of elliptic integrals and discussed some of their properties. In the Ph.D. dissertation (**D**3)[1] it was shown that the points on the inner equator of the anchor ring are of the first kind (in the classification due to Mangoldt), i.e., each geodesic passing through such a point contains no conjugate point, while all other points on the anchor ring are of the second kind, i.e., not of the first kind. Previous investigations by Jacobi and by Mangoldt had shown that on a surface of negative curvature all points are of the first kind, while on a closed surface of positive curvature all points are of the second kind.

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[1] The letters in bold face refer to the sections of the bibliography at the end of this article, and the numbers refer to the individual items.

During his time as a graduate student Bliss made a copy of Bolza's record of Weierstrass' 1879 course on the calculus of variations. This record together with Bolza's influence undoubtedly helped fix in Bliss' mind the interest in the subject which dominated his research. During his stay at Minnesota he studied Kneser's book on the subject, which was the first printed exposition of Weierstrass' ideas, and published a paper (**D4**) on the second variation and sufficient conditions for a minimum when one end point is variable. A second paper (**D5**), taking up the case when both end points are variable, was written during his stay in Göttingen. In both these papers, as in his thesis, geometric considerations play a prominent role. There are frequent evidences of his continuing interest in geometry. In his papers **D11, 14, 27**, he discussed "Finsler" geometry of two dimensions, where the arc length is given by an integral of the form

$$\int f(x, y, r)(x'^2 + y'^2)^{1/2} dt, \quad \tan r = \frac{y'}{x'} \cdot$$

Finsler's thesis treating the n-dimensional case did not appear until 1918. The tensor analysis of such spaces was developed still later by Élie Cartan. Bliss kept in touch with these ideas, and developments of them appeared in the dissertations of Taylor (**G18**), Johnson (**D28**), Householder (**G47**), and Stokes (**G51**), which he suggested and supervised. During the years 1908–1910, immediately following the death of Maschke, he lectured on geometry at Chicago, but when Bolza returned to Germany in 1910, Bliss seemed glad to make analysis again the focus of his teaching.

A number of papers applying the methods of Weierstrass to a variety of problems in calculus of variations appeared from Bliss' pen during the years before the first world war. Some of these were written in collaboration with Max Mason (**D12, 17, 19**) and one was a joint paper with A. L. Underhill (**D25**).

An idea which has become basic for much subsequent work in the calculus of variations appeared in the paper **D30**. This is the consideration of the minimum properties of the second variation. It makes possible simple proofs of the necessity of the Jacobi condition, and of the analogous condition for the more general problem of Bolza, and is a guiding principle in the construction of sufficiency proofs. It led to the paper **D44** on the transformation of Clebsch, which expresses the second variation in the form

$$I_2(\eta) = \int_{x_1}^{x_2} F_{y_i'y_k'}(\eta_i' - \pi_i)(\eta_k' - \pi_k)dx.$$

This formula is valid when there exists a conjugate system of accessory extremals whose determinant does not vanish on the interval $[x_1, x_2]$ and the vector function π is suitably defined in terms of the admissible variation η (vanishing at x_1 and x_2) and of the given conjugate system. It is derived very simply with the help of the theory of fields of extremals and the formula of Weierstrass. A simple direct proof is also given. The methods of this paper are basic for the various sufficiency proofs subsequently given for the problems of Lagrange, Mayer, and Bolza.

During the years 1916–1946 Bliss devoted much time to improving and extending the theories of the problems of Lagrange, Mayer, and Bolza. In this he had the cooperation of several of his students, notably Hestenes. The results appeared in definitive form in 1946 in the second part of his "Lectures" (A6), where the problem of Bolza is treated in detail. If one compares this with earlier expositions of these general problems—for example, of the problem of Lagrange in Bolza's *Vorlesungen*, or in Bliss' paper D49—one is immediately impressed with the greater scope of the theory, due to weakening of hypotheses both for necessary conditions and for sufficient conditions, and with the simplifications obtained in the proofs. The first part of the "Lectures" contains an unusually clear presentation of the theory of the calculus of variations for cases when there are no side conditions.

Although Bliss published only one paper on multiple integrals in the calculus of variations (D59) he began to discuss the subject in courses and seminars in the 1920's, and continued at frequent intervals up to the summer of 1942. Coral, Courant, McShane, Radó, and Smiley were among those from outside the University of Chicago who participated in some of these seminars. Bliss expounded the subject of multiple integrals in mimeographed lecture notes (B2, 3), where various improvements in the theory may be found. Some of his ideas were developed in the doctoral dissertations of Simmons (G19), Coral (G35), Raab (G39), Cosby (G41), Nordhaus (G50), and Landers (G52). One of the outstanding problems in this domain is to find conditions ensuring the existence of a field suitable for use in a sufficiency proof.

During the 1920's Bliss also lectured on boundary value problems associated with the calculus of variations, and on applications to quantum mechanics and relativity. Some of his work on boundary value problems appeared in the papers **D45, 46, 50, 58**. For the sake of the greatest symmetry and generality it is convenient to study differential systems of the form

$$(1)\qquad \frac{dy}{dt} = [A(x) + \lambda B(x)]y, \qquad My(a) + Ny(b) = 0.$$

(Here capital letters are used to denote square matrices of order n, y and z denote matrices of one column, and the transpose of A, for example, is denoted by A'.) The adjoint system to (1) is defined to be

$$\frac{dz}{dt} = -z'[A + \lambda B], \qquad z'(a)P + z'(b)Q = 0,$$

where $MP - NQ = 0$. The system (1) is said to be self-adjoint if it is equivalent to its adjoint under a transformation $z = T(x)y$ where the matrix T is nonsingular on the interval $[a, b]$. In the paper **D58**, a self-adjoint system (1) was defined to be *definite* in case: (a) the matrix $T'B$ is symmetric and positive semidefinite, and (b) the system (1) has no nontrivial solutions for which $By = 0$. The boundary value problems arising from problems of Bolza satisfying certain mild restrictions are of this type. The definition of "definitely self-adjoint" given in the earlier paper **D46** excluded many such problems. However, in both papers properties of the system are derived which are like those which hold for the Fredholm equation with real symmetric kernel. Student theses related to the subject of boundary value problems include those of Miss Stark (**G21**), Bamforth (**G22**), Cope (**G23**), Hickson (**G26**), Miss Jackson (**G27**), Hu (**G38**), and Miss Wiggin (**G43**).

The inverse problem of the calculus of variations drew some attention from Bliss, although it was not in the main line of his interest. (See the paper **D15**.) Among his students Davis (**G24**), LaPaz (**G29**), and Moscovitch (**G46**) contributed to the discussion of the problem.

Calculus of variations theory requires the use of existence theorems for implicit functions and differential equations which yield more information than those commonly given in the textbooks and treatises on analysis. Hence Bliss was led to write a number of papers on the subject (**D6, 9, 20, 33**), and his Colloquium lectures delivered before the American Mathematical Society in 1909 (**A1**) were devoted to these and related topics. The analysis of singular points for transformations of the plane was one of the topics treated in the Colloquium. This work grew out of special cases arising in the calculus of variations. The dissertation of Lovitt (**G6**) treated some cases for transformations of three-space. The transformations of the plane considered in the Colloquium are real analytic ones, and real branches of plane analytic curves play a prominent role. This led to the two papers on the factoring of power series (D18, 21), and later to the study of algebraic curves (D41, 42, A4). At the date of these studies the geometric proofs for the theorems on the reduction of the singularities of algebraic curves were quite unsatisfactory. In recent years modern algebra has provided the means for a very abstract and general treatment of the theory of algebraic curves. Bliss gave a clear treatment from the viewpoint of analysis of the case when the base field is the complex number field. This phase of his work indicates that he was not always interested in the maximum abstractness, although he always sought for simplicity, clarity, and comprehensiveness in his mathematical exposition.

Upon the urging of Veblen in 1918 Bliss went to Aberdeen as a scientific expert in the Range Firing Section. His work on the differential corrections of trajectories (D35, 36, 37) made possible enormous savings in the time required to compute the range corrections which are necessary in allowing for the rotation of the earth, effects of wind, and variations in the density of the air, powder charge, etc. His methods continued to be used during World War II. While the advent of high speed computing machines has led to some changes in methods, the basic theoretical ideas remain the same. These basic ideas were expounded in two papers in Trans. Amer. Math. Soc. (D39, 40), and later in a book (A5). Bliss' interest in functional analysis was not confined to this period. In earlier years the dissertations of Fischer (D5), Lamson (G10), Le Stourgeon (G11), and Barnett (G13) were related to this field.

Bliss was known the world around as one of the leading authorities on the calculus of variations, although he did not contribute to the new directions of study opened up by Tonelli and by Morse. He was primarily interested in mathematical research, but he heeded also the call of other duties. His broad interest in mathematics was evidenced, for example, by his regular attendance at the meetings of the Mathematical Club of the University of Chicago, where he contributed comments and questions on a wide variety of topics. He felt the duty of exposition, and wrote the first of the series of Carus Monographs, for which he also served as a member of the editorial committee. In mathematical publication he took a middle ground between those who pour forth undigested ideas and those who insist on excessive polishing. In 1925 he was the recipient of the first award of the Chauvenet Prize by the Mathematical Association of America for his paper *Algebraic functions and their divisors* (D43).

Bliss was an associate editor of the Annals of Mathematics from 1906 to 1908, and of the Transactions of the American Mathematical Society from 1909 to 1916. He was chairman of the Editorial Committee of the University of Chicago Science Series from 1929 until his retirement. From 1924 to 1936 he served on the Fellowship Board of the National Research Council. He was a trustee of the Teachers Insurance and Annuity Association for several years. A wide variety of organizations called on him to make informal addresses, and as a result he gave quite a number of such talks on mathematics and related topics.

He was elected to the National Academy of Sciences in 1916, to the American Philosophical Society in 1926, and was made a Fellow of the American Academy of Arts and Sciences in 1935. During his term (1921 and 1922) as president of the American Mathematical Society, he conducted a campaign for new members, in cooperation with E. R. Hedrick, Chairman of the Membership Committee. As a result the membership of the Society increased from 770 to 1127 during this biennium. In 1930 he was Vice President and Chairman of Section A of the American Association for the Advancement of Science. He was also a member of the Mathematical Association of America, the Illinois Academy of Science, the London Mathematical Society, the Deutsche Mathematische Verein, and the Circolo Matematico di Palermo. In 1935 the University of Wisconsin awarded him the honorary degree of Doctor of Science.

Bliss played a central role in the planning of Eckhart Hall, which houses the Department of Mathematics at Chicago, and he closely supervised its erection. It is very largely due to his care and insight that this building fills so well the needs of the Department, and it will stand as a beautiful and enduring memorial to his efforts. He accumulated a rather substantial personal library of mathematical books and periodicals, which he presented to the Department of Mathematics in 1950 for the use of the staff. This "Gilbert Ames Bliss Library" is now housed in the Faculty Conference Room in Eckhart Hall. It includes an unusually complete collection of works on the calculus of variations.

Bliss believed wholeheartedly in the importance of teaching as an accompaniment of mathematical research. All those who experienced the stimulation of his courses retain a high admiration for him as very nearly the ideal scientific teacher. His judgment was highly regarded, and his counsel was sought from many quarters, by other universities as well as his own, and by other organizations besides the mathematical ones. He was conservative in tendency, and was not one to rush into new domains of mathematics or into new projects in war time. In his view fundamental mathematical research and teaching were activities too important to be interrupted except for very strong reasons. Attempts to restrict student enrollments by high admission requirements seemed to him unwise, since some poorly trained students develop real intellectual power, and some brilliant ones gradually fade. With respect to the value of the Ph.D. training, he stated:

"The real purpose of graduate work in mathematics, or in any other subject, is to train the student to recognize what men call the truth, and to give him what is usually his first experience in searching out the truth in some special field and recording his impressions. Such a training is invaluable for teaching, or business, or whatever activity may claim the student's future interest."

Gilbert Bliss was married to Helen Hurd in 1912. Their children are Elizabeth (Mrs. Russell Wiles), born in 1914, and Ames, born in 1918. His wife was stricken and died in the influenza epidemic of December, 1918. In 1920 he married Olive Hunter, who survives him. The Blisses had a summer home in Flossmoor, Illinois, for many years, and beginning in 1931 they made their year round residence in Flossmoor, where Gilbert was at one time a member of the Village Board of Trustees and head of the Police Commission. They were exceptionally friendly in manner and in spirit, with a high sense of humor, and enjoyed entertaining students, faculty colleagues, and friends in their home. Golf at one of the nearby country clubs was a favorite recreation until declining health forced its discontinuance. Bliss was interested in competitive sports throughout his life, and in his earlier years participated actively in bicycle racing, tennis, and racquets.

His influence on mathematics and mathematicians was widespread and deep, and his contributions will be long remembered.

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G. DOCTORAL DISSERTATIONS SUPERVISED BY G. A. BLISS

1910

1. William Hunt Bates (begun with Maschke), *An application of symbolic methods to the treatment of mean curvatures in hyper-space.*
2. Egbert J. Miles (begun with Bolza), *The absolute minimum of a definite integral in a special field.*
3. Marion Ballantyne White, *The dependence of the focal point on curvature in space problems of the calculus of variations.*

1911

4. Lloyd Lyne Dines, *The highest common factor of a system of polynomials, with an application to implicit functions.*

1912

5. Charles Albert Fischer, *Some contributions to the theory of functions of lines.*

1914

6. William Vernon Lovitt, *A type of singular points for a transformation of three variables.*

1916

7. Gillie Aldah Larew, *Necessary conditions for the problem of Mayer in the calculus of variations.*
8. Archibald Shepard Merrill, *An isoperimetric problem with variable end-points.*
9. David Melville Smith, *Jacobi's condition for the problem of Lagrange in the calculus of variations.*

1917

10. Kenneth Worcester Lamson, *A general implicit function theorem, with an application to problems of relative minimum.*
11. Flora Elizabeth Le Stourgeon, *Minima of functions of lines.*
12. William Pinkerton Ott, *The general problem of the type of the brachistochrone with variable end-points.*

1918

13. Israel Albert Barnett, *Differential equations with a continuous infinitude of variables.*

1922

14. Elbert Howard Clarke, *On the minimum of the sum of a definite integral and a function of a point.*
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1924

16. Lawrence Murray Graves, *The derivative as independent function in the calculus of variations.*
17. Jewell Constance Hughes, *A problem in the calculus of variations in which one end-point is variable on a one-parameter family of curves.*
18. James Henry Taylor, *A generalization of Levi-Civita's parallelism and the Frenet formulas.*

1925

19. Harvey Alexander Simmons, *The first and second variations of a double integral for the case of variable limits.*

1926

20. Irvine Rudsdale Pounder, *A method of successive approximations for a partial differential equation of hyperbolic type.*
21. Marion Elizabeth Stark, *A self-adjoint boundary value problem associated with a problem of the calculus of variations.*

1927

22. Frederic Richard Bamforth, *A classification of boundary value problems for a system of ordinary differential equations of the second order.*
23. Thomas Freeman Cope, *An analogue of Jacobi's condition for the problem of Mayer with variable end points.*
24. David Roy Davis, *The inverse problem of the calculus of variations in higher space.*

25. Ernest Lloyd Mackie (with L. M. Graves), *The Jacobi condition for a problem of Mayer with variable end points.*

1928

26. Arthur Owen Hickson, *An application of the calculus of variations to boundary-value problems.*

1928

27. Rosa Lea Jackson, *The boundary-value problem of the second variation for parametric problems in the calculus of variations.*
28. Marie Mathilda Johnson, *Tensors of the calculus of variations.*
29. Lincoln La Paz, *An inverse problem of the calculus of variations.*

1930

30. Moffatt Grier Boyce (with L. M. Graves), *An envelope theorem and necessary conditions for a problem of Mayer with variable end-points.*
31. William Larkin Duren, Jr., *The development of sufficient conditions in the calculus of variations.*
32. Aline Huke, *An historical and critical study of the fundamental lemma in the calculus of variations.*
33. Edward James McShane (with L. M. Graves), *Semi-continuity in the calculus of variations, and absolute minima for isoperimetric problems.*
34. Frank Lynwood Wren, *A new theory of parametric problems in the calculus of variations.*

1931

35. Max Coral, *The Euler-Lagrange multiplier rule for double integrals.*
36. Ralph Grafton Sanger, *Functions of lines and the calculus of variations.*

1932

37. Magnus Rudolph Hestenes, *Sufficient conditions for the general problem of Mayer with variable end-points.*
38. Kuen-Sen Hu (with L. M. Graves), *The problem of Bolza and its accessory boundary value problem.*
39. Albert William Raab (with L. M. Graves), *Jacobi's condition for multiple integral problems of the calculus of variations.*

1933

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41. Byron Cosby II, *Fields for multiple integrals in the calculus of variations.*
42. Van Bauman Teach (died before taking examination), *The Hamilton-Jacobi theory for the problem of Lagrange in parametric form.*

1936

43. Evelyn Prescott Wiggin (with W. T. Reid), *A boundary value problem of the calculus of variations.*
44. Ross Harvey Bardell, *The inequalities of Morse for a parametric problem of the calculus of variations.*

1937

45. Carl Herbert Denbow (with L. M. Graves and W. T. Reid), *A generalized form of the problem of Bolza.*
46. Nathan Abraham Moscovitch, *Studies of the inverse problem of the calculus of variations.*
47. Alston Scott Householder, *The dependence of a focal point upon curvature in the calculus of variations.*

48. Frederick Albert Valentine (with L. M. Graves), *The problem of Lagrange with differential inequalities as added side conditions.*

1938

49. Carroll Parker Brady (with M. R. Hestenes), *The minimum of a function of integrals in the calculus of variations.*

1939

50. Edward Alfred Nordhaus (with M. R. Hestenes), *The problem of Bolza for double integrals in the calculus of variations.*
51. Ellen Clayton Stokes, *Applications of the covariant derivative of Cartan in the calculus of variations.*
52. Aubrey Wilfred Landers, Jr. (with M. R. Hestenes), *Invariant multiple integrals in the calculus of variations.*
53. Mary Kenny Landers (with M. R. Hestenes), *The Hamilton-Jacobi theory for problems of Bolza and Mayer.*

L. M. GRAVES

BOOK REVIEWS

Topologie I. *Espaces métrisables, espaces complets.* By C. Kuratowski. (Monografie Matematyczne, vol. 20.) 2d. ed. Warsaw-Wrocław, 1948. 12+452 pp. $7.50.

Topologie II. *Espaces compacts, espaces connexes, plan euclidien.* By C. Kuratowski. (Monografie Matematyczne, vol. 21.) Warsaw-Wrocław, 1950. 8+444 pp. $6.00.

Noteworthy in the development of topology is the large number of fundamental results which, however originally discovered or proven, eventually find their true setting among those theorems most simply and easily established by set theoretic methods. This is due in large measure, of course, to extensive and powerful new tools which have been discovered from time to time. For example, the development and exploitation of the homotopy relation between mappings has very greatly extended the range of applicability of the set theoretic method and has rendered many results in topology, analysis and other fields of mathematics accessible to this approach. The same is true of studies in the development and uses of restricted types of mappings and in many other phases of topology.

A better illustration of this point will hardly be found than in the two volumes of Kuratowski under review. Here the hand of the master is apparent and guides the reader on every page, presenting him with a perfectly organized and beautifully simple pathway to a large body of the deepest results in topology and in other fields which are essentially topological in character. One finds here the topological invariance of the Euclidean domain, the invariance of the property of separating Euclidean n-space, as well as the theorems of Rouché and Runge from analysis—to give a few examples—skillfully obtained by the author with what seems to be small effort and with tools readily accessible to the beginning graduate or advanced undergraduate student. No use is made of homology theory and only restricted use of groups is made, largely in connection with homotopy.

The new edition of volume I offers a considerable body of new material and some other changes from the original edition all of which adapt it perfectly for use along with volume II. The two volumes together constitute a simple, effective, and definitive treatment of topological results at present obtainable by set theoretic methods. As the manuscript for volume II was nearly complete at the time of the outbreak of the war in 1939, developments made during the last decade are covered to only a limited extent.

The reader is referred to a review of the original edition of volume I by the present reviewer (Bull. Amer. Math. Soc. vol. 40 (1934) p. 787) for comments still largely applicable to the new edition, to a review of the new edition of volume I by J. H. Roberts (Mathematical Reviews vol. 10 (1949) p. 389) for a detailed comparison of the old and new editions and to a review of volume II by E. G. Begle (Mathematical Reviews vol. 12 (1951) p. 517) for detailed indication of the content of volume II. The two volumes comprise a historic contribution to mathematical and topological literature and will need to be a part of the library of every individual interested in or making use of the results of topology.

G. T. WHYBURN

The elements of mathematical logic. By P. C. Rosenbloom. New York, Dover, 1950. 4+214 pp. $2.95.

This book is intended for mature mathematicians with no previous knowledge of mathematical logic. Chapter I deals with Boolean algebras and includes the Stone representation theorem. Chapter II is entitled *The logic of propositions*. Truth tables are explained and there are three alternative formulations of the propositional calculus and a number of tautologies are proved. There is also a finitary formulation which incorporates part of the syntax in the object language which is therefore unusually rich. Next the relation between Boolean algebras and propositional calculus is explained and the final section of this chapter contains a very interesting discussion of many-valued and modal logics and of intuitionism. It includes some material on Post algebras and formulations of intuitionistic propositional calculus and Lewis' basic logic. Chapter III, on *The logic of propositional functions*, begins with an informal discussion of intuitive class theory and the Russell paradox. It shows that some restriction on the method of class formation is necessary and different methods of doing this are briefly mentioned. After this there is in §2 a formulation of the monadic first order functional calculus and an extension to polyadic functional calculus. The Peano axioms are then adjoined to get a system adequate for arithmetic. §3 begins with an exposition of the pure first order functional calculus, followed by a brief account of the theory of types, the system of Quine's new foundations and finally of Zermelo's system. Bernays' system is also mentioned, but von Neumann's only in the bibliographical notes (p. 202): "Other formulations of Zermelo's system have been given by von Neumann and Ackermann." It is debatable whether it is fair to consider the system of von Neumann merely a reformulation of Zermelo's. §4 contains an account of the work of Curry on the analysis of the notions of substitution and variable and the development of combinatory logics. It is proved that the logic presented is functionally complete. Church's λ-conversion is also introduced. §5 sketches a development of mathematics in the system of Quine's new foundations. This is followed in §6 by a very brief discussion of the Epimenides and Richard paradoxes. The 7th and last section of this chapter deals with the axiom of choice and gives several equivalent formulations. Chapter IV is called *The general syntax of language*. It depends largely on the work of Post. The first section discusses what is meant by a language and a simple language. §2 introduces the concepts of productions, axioms, theorems, canonical languages, extensions of languages, canonical classes of statements, and also contains an examination of the character of definitions. In §3 we find a definition of normal languages which are especially simple in that they have only one axiom and the productions are of a very symmetric form. A theorem of Post states (p. 170): "Every canonical language has a conservative normal extension." As almost all known systems of mathematical logic (those with no rules of inference with an infinite number of premises) can be formulated in terms of canonical languages, this theorem says that all problems of each of these systems can be reduced to problems of a language with a very simple structure. In order to conserve space, the proof of this theorem is however omitted since it is somewhat lengthy. The chapter concludes with a proof of Gödel's incompleteness theorem. The appendix contains canonical formulations of pure first order functional calculus, of Quine's system, and of Zermelo's system, and also a proof of Church's theorem. There are exercises throughout the book following each section and also a very illuminating set of bibliographical notes.

Professor Rosenbloom's book is a very interesting addition to the texts of logic now available, particularly since it contains material not found in others, for instance discussions of the work of Curry and Post. This important and often neglected work is stressed a great deal and very well presented. The presentation of the traditional subject matter of an elementary course in logic, that is, propositional calculus, functional calculus, and the various systems of set theory, is often unusual. The reviewer would however criticize the author on several minor points.

First, we quote from page 28: " '$\sim p$' shall denote the proposition that it is false that p." This is not a good interpretation, for "$\sim p$" is a sentence of the object language, while "it is false that p" is a sentence of the syntax language. (Compare for instance Quine, *Mathematical logic*, p. 27.) In particular this use causes confusion when iterated and coupled with the interpretation of the word "true" in this book, for then "$\sim(\sim p)$" must be read: "it is not a theorem that it is not a theorem that p."

The second disagreement concerns the use of the word "categorical." This is defined on p. 44: " . . . In many deductive theories we wish the axioms to be categorical, that is, that the system should be adequate to decide the truth or falsehood of any proposition which can be formulated in the system. In the frame of A 1″–A 7″ (a formulation of the propositional calculus-rev.) we can give this demand the strong form that for every $p \in C$ (C is the class of wff's-rev.) either $\vdash p$ or $\vdash \sim p$" as this is contrary to the general usage of this word (cf. Fraenkel, *Einleitung i.d. Mengenlehre*, 3d ed., p. 349), the reviewer would in this case suggest the use of the word "complete." A similar objection applies to the use of the word "true." On page 94 we find the definition "A sentence q is said to be true if there is a proof of q." The reviewer would prefer the word "provable" in this connection. If "true" is used with Rosenbloom's meaning, every undecidable sentence is false. This contradicts the following statement on p. 179: "Thus any canonical language which is consistent and adequate for arithmetic will contain undecidable sentences expressing elementary arithmetical propositions. There will even be such sentences which we can prove to be true by an argument in the syntax language." The reviewer noted a few misprints, also a few misreferences (e.g. T9″ referred to on p. 44 could not be found, however T13 (p. 35) could be used here, also there is no Lemma 6 (cf. pp. 44–45), only Theorem 6 (on p. 22), also on p. 54, the ref. to A5″ seems incorrect).

I. L. NOVAK

Espaços vetoriais topologicos 1. By L. Nachbin. (Notas de Matematica, no. 4.) Rio de Janeiro, Boffoni, 1948. 2+100 pp. 70 Cruzeiros.

This is intended as the first volume of a self-contained treatise on topological vector spaces. Of the 9 chapters it contains, chapters 1 to 4 are devoted to algebraic and topological preliminaries (topological spaces, fields, topological fields, vector spaces); in addition chapter 7 discusses mainly absolute values on fields and their generalizations, so that only 4 chapters remain for topological vector spaces proper.

The author presents the theory from a very general point of view, making as few assumptions as possible on the field of scalars, which for most results is allowed to be an arbitrary topological field (very little labour would have been needed to extend almost all definitions and theorems to the case in which the scalars are in a noncommutative topological sfield). A topological vector space is defined in chapter 5 as a vector space E over a topological field K, with a topology on E such that $x+y$ and λx are continuous functions of (x, y) and (λ, x) respectively; such a topology is characterized by properties of a system of neighborhoods of 0, and the notions of continuous linear transformation and of quotient topological vector space are carefully studied. Chapter 6 is devoted to the definition and elementary properties of bounded sets in a topological vector space, such a set B being characterized by the fact that λB "tends to 0" with λ in an obvious sense.

Chapter 7 is the longest and most elaborate of the book. Together with the classical notion of absolute value, the author introduces quasi-absolute values $v(x)$, which satisfy the usual axioms with the exception of the triangle inequality, which is weakened to $v(x+y) \leq m(v(x)+v(y))$ for a fixed $m>1$. After a discussion of archimedean and non-archimedean absolute values, he proves that for every quasi-absolute value v, v^h is an absolute value for sufficiently small $h>0$ (theorem of Artin). Next he introduces the topology defined on a field by an absolute value, and proves Shafarevich's criterion characterizing such topologies. Finally he introduces a new notion, that of *strictly minimal* topological field: this is a field K with a Hausdorff topology $\mathfrak{T}$ such that no Hausdorff topology strictly coarser than $\mathfrak{T}$ can make K into a topological vector space (the scalars retaining their original topology $\mathfrak{T}$). It turns out that that condition (which is satisfied by any field with an absolute value) is necessary and sufficient for the validity of the following theorem: in order that a hyperplane defined by an equation $f(x)=0$ in a topological vector space E over K be closed, it is necessary and sufficient that f be a continuous linear form.

Chapter 8 treats the *strong* topologies defined by Mackey: a topology on a vector space E is strong if there is no strictly finer topology giving the same bounded sets. This is equivalent to saying that any linear mapping sending bounded sets of E into bounded sets of a topological vector space F is always continuous. Metrizable vector spaces over metrizable fields are always strong.

Finally the last chapter is devoted to *weak* topologies, but practically does not go beyond their definition; in particular, the theory of weak duality, which alone can give a meaning to the notion of weak topology, is not touched at all. There are other topics which one misses in chapters 5 and 6, where they would have been in their proper setting, such as the discussion of finite-dimensional topological vector spaces, or of locally compact vector spaces. On the whole, in the reviewer's opinion, the book suffers from a lack of balance, due to the overemphasis laid on chapter 7, at the expense of more relevant matters. However, the author has done a very valuable service to mathematicians in bringing together in book form a large number of results which up to now were scattered in periodicals, and not always very explicitly. His style moreover deserves high praise for its remarkable clarity and thoroughness, so that the book genuinely vindicates its claim of being self-contained, although of course the motivation for the whole theory can only be understood with a considerable background of functional analysis.

J. DIEUDONNÉ

Rekursive Funktionen. By R. Péter. Budapest, Akademischer Verlag, 1951. 206 pp.

Although recursions have been used since Archimedes, and have played a part in foundational investigations by Dedekind (1888), Peano (1891), and Skolem (1923), the theory of recursive functions consists largely of two recent developments, which we call here the "special theory" and the "general theory."

The stimulus to the special theory came from Hilbert's lecture *Über das Unendliche* (published 1926) in which he proposed to attack the continuum problem of set theory by showing that there is no inconsistency in supposing that the number-theoretic functions are all definable by use of forms of recursion associated with the transfinite ordinals of Cantor's second number class. (This program has not yet been carried out, though Gödel in 1938 used an analogous idea to show the consistency of the continuum hypothesis within axiomatic set theory.) For Hilbert's proposal it was necessary to show that higher forms of recursion do give new functions; and the first demonstration of the existence of a function definable by a double recursion but not by use only of simple or "primitive" recursion was given by Ackermann in 1928 in a paper entitled *Zum Hilbertschen Aufbau der reellen Zahlen.* Beginning in 1932, Rósza Péter has published a series of papers, examining the relationship of various special forms of recursion, and showing the definability of new functions by successively higher types of recursion, which establish her as the leading contributor to the special theory of recursive functions.

The general theory of recursive functions dates from the formulation nearly simultaneously (in publications appearing in 1934–1937) of three notions, those of general recursive function (Herbrand-Gödel), λ-definable function (Church-Kleene), and computable function (Turing-Post), which were proved to be equivalent, and were proposed (first by Church in 1936) as exact mathematical equivalents of the intuitive notion of an effectively calculable function.

Both the special and the general theory have developed in close association with applications to logic and the foundations of mathematics.

In line with Mrs. Péter's research interests, the special theory occupies about two-thirds of the present book, namely the first fifteen chapters, with the following titles (translated by the reviewer), and also a part of the last chapter. §1. The usual definition of number-theoretic functions by passage from n to $n+1$. §2. Recursive functions and relations. §3. Course-of-values recursion. §4. Simultaneous recursion. §5. Recursion in which substitutions take place for the parameter. §6. Recursion on several variables. §7. Reductions. §8. Elementary functions. §9. Example of a number-theoretic function which is not primitive recursive. §10. Nested recursion. §11. The diagonal procedure and the multiple recursions. §12. Transfinite recursions. §13. Recursions of higher order. §14. The normal form of the multiple recursions. §15. The "Gödelizing" of recursion of higher order.

The remaining third of the book is devoted to the general theory and applications, with chapter headings: §16. General recursive functions. §17. The explicit form of the general recursive functions. §18. Possibilities for the further simplification of the explicit form. §19. Example of a function which is not general recursive. §20. Computable functions. §21. History and applications. §22. Undecidability effectively of the question, which systems of equations define general recursive functions. §23. The question of the general decidability of the arithmetical problems. §24. Extension of the concept of recursiveness. Applications to analysis.

The aim of the book is primarily to give an elementary exposition of the existing theory, rather than to push into new territory. But the part dealing with the special theory is especially complete, and the latter chapters of this part contain material not covered or covered only summarily in the literature. It is of great value to have in this book for the first time a connected account of the special theory.

In the latter part of the book, the general recursive functions themselves (in the absolute sense) and their explicit form are treated very fully, but the associated notions and applications are treated less fully or only cited. In writing this book Mrs. Péter has carried out a considerable undertaking; and to go further would have constituted a still greater one, and required either a much larger book or a more compact style.

Only a minimum of knowledge of elementary number theory, analysis, and set theory including transfinite ordinals is presupposed and none of mathematical logic. Mrs. Péter aims to make the subject intelligible to the beginner by working out the treatment of many topics (particularly in the special theory) on an example, whence the reader can surmise how the treatment would go in general (or consult the literature). This method has both advantages and disadvantages. No student can complain that he has lost contact with the reality for want of concrete examples; but an unwary reader may be oppressed by the immense amount of detail involved in working out the examples and proofs.

S. C. KLEENE

Infinite matrices and sequence spaces. By R. G. Cooke. London, Macmillan, 1950. 14+347 pp. 42s.

This book, which might be considered a continuation of Chapter XII of Dienes' *Taylor series*, is a useful and welcome adjunct to the recent book by Hardy, *Divergent series*, Oxford, 1949. The overlap between these is slight since the present book is largely concerned with the study of general properties of classes of regular summability transformations.

Chapter 1 introduces several special classes of infinite matrices and certain of the special problems that arise in connection with their algebraic properties such as the ever-present need for the consideration of the validity of interchange of limit operations which leads, for example, to the failure of the associative law. Chapters 2 and 3 deal with the existence of left- and right-hand inverses and annihilators in rings of matrices, the notion of a "bound" (norm) and of weak convergence of sequences of matrices, and the special problem of solving the equations $AX=XD$ and $AX-XA=I$ for specified A and diagonal D. In Chapter 4, the class of K-matrices which transform convergent sequences into convergent sequences is characterized, as well as the subclass of Toeplitz T-matrices and the analogous classes of series to sequence transformations. The Steinhaus theorem and numerous variants are proved, together with others showing what may be achieved by a general T-matrix. Chapter 5 deals with the general question of consistency of two matrices A and B. Numerous results are obtained in the case of commutative A and B. The chapter concludes with a discussion of the relation between a matrix and its left or right translate, and between the iterative product and composition product of two matrices, as developed by Agnew. Chapter 6 is the best in the book and gives a very complete treatment of the behavior of the core of a sequence under T transformations. In particular, several new proofs and theorems due to A. Robinson are included. Chapters 7 and 8 discuss the efficiency of classes of T matrices as applied to the summability of power series, to bounded sequences, and to sequences of 0's and 1's. Chapter 9 contains a detailed account of the elementary theory of the Hilbert space l^2 and to the theory of bounded bilinear forms $A(x, y)$, similar in treatment to that contained in Hardy, Littlewood, and Pólya, *Inequalities*, Cambridge, 1934. In Chapter 10, certain linear spaces of sequences are discussed, together with their duals, in the spirit of Köthe-Toeplitz; various types of weak convergence are defined and their relations investigated, using results of H. S. Allen.

The book is clearly written with a profusion of details, and the bibliography is extensive, containing some papers as recent as 1950. The author is especially to be commended for the excellent sets of problems which follow each chapter. It is to be regretted that the book as a whole is so remote in point of view and method of treatment from the current trend of research in the fields most closely allied with the material presented. This extends not only to terminology (for example, the author's use of "field" to describe a possibly non-associative ring of matrices) but also to the entire approach. Chapters 2, 3, 4, 5, 9, and 10 would have been much improved if presented from the point of view of convex linear topological spaces, topological algebras, and their accustomed machinery of dual spaces of functionals, weak and strong topologies, and operators. It is noteworthy, for example, that the discussion of Hilbert space contains no mention of operators. The final chapter does contain a brief description of a Banach space and a proof of the Baire category theorem, and Chapter 3 contains a cursory discussion of the notion of a group algebra, but these are not germane to the rest of the book, and their full power is not used. Along more classical lines, the reviewer finds the author's remarks on the "right" value for the sum of a divergent series in

§8.7 (which is defined to be the Abel limit) extremely disturbing, especially in the light of the well known fact that Abel and Cesaro summability coincide for bounded sequences.

R. CREIGHTON BUCK

Randwertprobleme und andere Anwendungsgebiete der höheren Analysis für Physiker, Mathematiker und Ingenieure. By F. Schwank. Leipzig, Teubner, 1951. 6+406 pp. $5.47.

The book is a compendium of those portions of mathematical analysis beyond advanced calculus which are of great interest to engineers and physicists. It is an excellent book for physicists and engineers in that it provides a lucid introduction to a good selection of mathematical techniques and theories, with references for further reading, examples, practical applications, and an almost encyclopedic bibliography of applications. From the point of view of the mathematical reader the principal merit of Schwank's book is the wide range and amazingly large number of practical applications described or quoted in the various chapters.

In a preface, G. Hamel explains that Schwank's book is neither a textbook, nor a work of reference. Although the presentation is consequential and covers the ground thoroughly, the book is not as systematic as a textbook, and not as complete as a book of reference. It is written for the physicist or engineer with only a modest mathematical knowledge, and a desire to learn more about some of the more advanced mathematical techniques. The author aims at precision wherever it can be attained. In view of the readers for whom the book was written, it is quite clear that precision and rigour could not be maintained throughout the book, and examples and descriptions are called in when general formulations and proofs would seem out of place.

The material is organized in six chapters and a mathematical appendix.

Chapter I is an introduction to boundary value problems. The vibrations of a string are discussed in detail, and boundary conditions, normal modes of vibration, characteristic values, characteristic functions, orthogonality, Fourier expansion, and other relevant notions are introduced. D'Alembert's solution of the one-dimensional wave equation is also given. The last two sections of this chapter are devoted to the re-formulation of the problem of the vibrating string in terms of integral equations and calculus of variations respectively.

Chapter II is devoted to complex variables. It starts from the beginning with complex numbers, and proceeds to functions of a complex variable, Cauchy-Riemann equations, conformal mapping, complex integrals, power series expansions, singularities, residues. It concludes with an example of the evaluation of a real integral by means of contour integration.

Partial differential equations are discussed in Chapter III. This is the longest chapter in the book (141 p., about one third of the whole book); it was probably the most difficult chapter to plan and write, and quite possibly it is the least satisfactory one to read. Although the potential equation, wave equation, and diffusion equation all occur in this chapter, the classification of partial differential equations into elliptic, hyperbolic, and parabolic equations is not introduced, and characteristics are not mentioned at all. In view of the great difficulty of the subject this omission is understandable, but it is hardly justifiable. Quite apart from the vital importance, in certain branches of applied mathematics, of the distinction between elliptic and hyperbolic regions, and of the method of characteristics, it is a fact that the strikingly different nature of, say, Laplace's equation and the wave equation cannot be appreciated without knowing that they are representatives of two different types of partial differential equations. An extensive study of the potential equation, wave equation, and diffusion equation does not make sense as an introduction to partial differential equations unless one realizes that (roughly speaking) every (linear) partial differential equation behaves like one of the three types. Regarded as an introduction to potential, separation of variables, and Bessel and Legendre functions, the chapter seems adequate.

After a brief introduction, the study of potentials (both two- and three-dimensional) is taken up, the Laplace and Poisson equations are obtained, and some properties of harmonic functions are derived from Green's formula. Applications to fluid mechanics, electricity, and elasticity follow. Separation of variables in spherical polars leads to spherical harmonics and an introduction to Legendre functions (of the first kind).

Separation of variables in the two-dimensional wave equation in polar coordinates leads to an introduction to Bessel functions (all kinds). Separations in elliptic, or ellipsoidal, coordinates are shown to lead to Mathieu and Lamé functions, and transition to integral equations and variational problems are briefly mentioned.

Next follow several conduction of heat problems (mostly treated by separation of variables). In the (brief) section on the three-dimensional wave equation Kirchhoff's formula, Huygens' principle, retarded potentials, and similar matters are mentioned. The chapter concludes with a section on the partial differential equation $\Delta\Delta u = 0$ and its applications.

Integral equations are the subject of Chapter IV. After a brief introduction, Fredholm-type integral equations with degenerate ("polynomial") kernels are discussed. Neumann's expansion (in the general case), Fredholm's theory, and the theory of symmetric kernels, with the classical theorems, follow. The sections on estimates, approximations, and numerical methods will be particularly useful. The two final sections establish connections with boundary value problems and show a number of applications. Reading this chapter one wonders if some of its sections are not too theoretical and if it is justifiable to devote, in a book of this nature, to integral equations about twice as much space as to functions of a complex variable.

Chapter V is on the calculus of variations, and in spite of its brevity is amply illustrated by examples. Euler's differential equation and Legendre's condition are derived, and so is Jacobi's condition. The section on Ritz's method and applications is especially valuable for the readers for whom the book is designed.

Chapter VI is a brief chapter on linear difference equations with constant coefficients and systems of such equations. The section on applications includes an example to show how difference equations can be used for the approximate solutions of differential equations.

The Appendix is a brief summary of some topics, mostly belonging to advanced calculus. It is designed principally to refresh one's memory, although some of its parts could be used to fill in gaps in the mathematical education of the reader.

A. ERDÉLYI

The preparation of programs for an electronic digital computer. With special reference to the EDSAC and the use of a library of subroutines. By M. V. Wilkes, D. J. Wheeler, and S. Gill. Cambridge, Massachusetts, Addison-Wesley, 1951. 10+170 pp. $5.00.

The EDSAC, designed and constructed at the Cambridge University Mathematical Laboratory, was one of the earliest high speed automatic digital computing machines in operation. It has a mercury delay line storage for 1024 words of 17 binary digits with photo-electrically read teletype tape input and teleprinter output. It is a one-address machine with a multiplication time of about 7 milliseconds. It has been used as a true general purpose machine, and the problems handled seem to have been largely at the choice of the Laboratory. Accordingly, it has been possible to set up, without external interference, an organization for problem handling which has been tried out successfully on problems making progressively more demands on the machine and the numerical analysts.

One principle of the organization is the use of a thoroughly checked library of sub-routines. For instance the evaluation of an integral of the form

$$S(x, y) = \int_0^x e^{-y \sec \theta} d\theta$$

could be carried out by a program consisting mainly of the sub-routines for quadrature, for the exponential function, and for the secant. The actual assembly of these sub-routines into the main program can itself be effected by a special sub-routine.

A catalogue of (some of) the sub-routines used is given, together with detailed coding for some of these.

Another principle which is emphasized is the use of checking sub-routines. One of these indicates the progress of a computation by printing the operation symbols for each instruction as it is carried out, going to a new line at a branch order. This gives a compact representation which can help to indicate at what stage trouble develops.

This book will be of considerable value to those concerned with high-speed automatic digital computing machines. Unfortunately many ideas of general interest are only accessible after the assimilation of the EDSAC code. A use of flow diagrams and a sub-division of the book into three parts: one for the beginner, one for the expert, and one a hand-book for EDSAC, might have been helpful.

It is, for instance, worth pointing out that when carrying out a recursive process by means of a variable order, the end of the process can be determined by that order, instead of setting up a separate tally. Again the discussion of methods for the incorporation of "closed" sub-routines into a program, so that the control can pass from the main program, through them, and back to the main program whenever necessary, is valuable. These are matters of importance to all machine users.

It is worth observing that the objections raised to the use of the Gaussian approximate quadrature (namely, the trouble (on desk machines) of calculating the integrand at awkward arguments) no longer apply in the present circumstances. It is also worth recalling that there is a recurrence relation for the evaluation of a polynomial. But the detailed coding of such problems is of little interest save to the veriest beginner.

JOHN TODD

BRIEF MENTION

Programmgesteuerte digitale Rechengeräte (elektronische Rechenmaschinen). By H. Ruthishauser, A. Speiser, and E. Stiefel. (Mitteilungen aus dem Institut für angewandte Mathematik, no. 2.) Basel, Birkhäuser, 1951. 102 pp. 8.50 Swiss fr.

This booklet has been written after visits by the authors to various centers of high speed automatic digital computing in 1948–9. It is the first at all comprehensive report on its subject in the German language, and constitutes a very readable introduction. In addition to sections on history and bibliography there are careful discussions of the organization of machines, the representation of numbers, the handling of the elementary operations, possible address-systems, flow-diagrams, multiple-precision operations, checks. The concluding section discusses the physical realization of various components.

JOHN TODD

Colloque de topologie (Espaces fibrés). Centre Belge de Recherches Mathématiques. Liège, Georges Thone, and Paris, Masson, 1951. 129 pp. 1225 fr.

This booklet is a report of a colloquium on fibre spaces and fibre bundles held in Brussels in June, 1950. It contains the following papers: *Introduction à la théorie des espaces fibrés*, by H. Hopf; *Notions d'algèbre différentielle; applications aux groupes de Lie et aux variétés où opère un groupe de Lie*, by H. Cartan; *Les connexions infinitesimales dans un espace fibré différentiable*, by C. Ehresmann; *La transgression dans un groupe de Lie et dans un espace fibré principal*, by H. Cartan; *Sur un type d'algèbres différentielles en rapport avec la transgression*, by J. L. Koszul; *Espaces fibrés et homotopie*, by B. Eckmann; *Sur l'homologie des groupés de Lie, des espaces homogènes, et des espaces fibrés principaux*, by J. Leray; *Sur un formule de la théorie des espaces fibrés*, by H. Hopf; *Quelques relations entre l'homologie dans les espaces fibrés et les classes caractéristiques relative à un groupe de structure*, by G. Hirsch.

The first paper by Hopf is of an expository nature; it furnishes an introduction to the general theory of fibre spaces, giving examples of some of the problems treated and results obtained. The paper of Eckmann gives an excellent summary of the relationships and connections between homotopy theory and the theory of fibre spaces. The remaining papers are of a much more specialized nature, and are mainly concerned with announcing new methods and results which have not been published as yet. In many of them, the exposition is so condensed as to make reading difficult or impossible for all but those who are particularly familiar with the recent work of the author in question. To make matters even more difficult, some of the authors make their exposition depend heavily on results which, if published at all, have appeared only in the form of brief announcements, with no proofs or elaboration.

W. S. MASSEY

Tables of the error function and of its first twenty derivatives. By the Staff of the Computation Laboratory. Harvard University Press, 1952. 28+276 pp. $8.00.

With $\phi(x) = (2\pi)^{-1/2} \exp(-t^2/2)$, these tables give the integral of $\phi(x)$, $\phi(x)$ itself, and its first four derivatives at intervals of 0.004 from 0 to 6.468; the fifth through the tenth derivatives at intervals of 0.004 from 0 to 8.236; the eleventh through fifteenth derivatives at intervals of 0.002 from 0 to 9.610; the sixteenth through twentieth derivatives at intervals of 0.002 from 0 to 10.902.

Tables of $n!$ and $\Gamma(n+1/2)$ for the first thousand values of n. By H. E. Salzer. (National Bureau of Standards, Applied Mathematics Series, no. 16.) Washington, United States Government Printing Office, 1951. 6+10 pp. 15 cents.

The tables give $n!$ to 16 significant figures and $\Gamma(n+1/2)$ to eight.

Table of arctangents of rational numbers. By John Todd. (National Bureau of Standards, Applied Mathematics Series, no. 11.) Washington, United States Government Printing Office, 1951. 11+105 pp. $1.50.

The volume includes two tables: The first gives m^2+n^2, the principal value of $\tan^{-1}m/n$ and $\cot^{-1}m/n$ (in radians to 12 decimals) for $0<m<n\leqq 100$, and the complete reductions of $\tan^{-1}m/n$; the second gives the complete reductions of $\tan^{-1}n$ for those integers less than 2090 which are reducible. A reduction is a representation of the

NOTES

The University of Georgia has purchased the Library of the American Mathematical Society and has combined it with its own mathematical collection to form an unusually complete library for research in mathematics. The combined library is particularly strong in long runs of periodicals. It is being kept up to date and none of the titles previously received by the American Mathematical Society is being dropped.

The collection of doctors' dissertations, both American and foreign, is unique. In the belief that an outstanding collection of dissertations can be administered for the benefit of mathematicians everywhere, the University of Goergia is making every effort to maintain the completeness of this collection by obtaining copies of all new dissertations. The Librarian of the University of Georgia would appreciate the cooperation of department chairmen and authors of dissertations in this undertaking.

The University of Georgia hopes that all mathematicians will feel free to utilize any item in the entire collection. It is the desire of the University to organize its library for the benefit of all competent users, and rules for administering the mathematical collection are being framed with this in mind. It is particularly hoped that the location of this library at Athens will benefit mathematicians in the Southeast.

The Institute for the Unity of Science is offering a prize of $500 for the best essay on the theme *Mathematical logic as a tool of analysis: its uses and achievements in the sciences and philosophy*. Two additional prizes of $200 each will be given for the next best two essays. This is an international contest, and is open to everyone. Essays must not exceed 25,000 words. They may be written in English, French, or German, and must be submitted before January 1, 1953.

Further information on the contest can be obtained from the Institute for the Unity of Science, American Academy of Arts and Sciences, 28 Newbury Street, Boston 16, Massachusetts.

A new journal, Journal of Rational Mechanics and Analysis, is being published by the Graduate Institute for Applied Mathematics, Indiana University, beginning January 1952.

The National Science Foundation has prepared "A Guide for the Submission of Research Proposals" to assist investigators in applying to the Foundation for research grants. Copies may be obtained from The National Science Foundation, Washington 25, D. C.

The Association for Computing Machinery Meeting will be at Mellon Institute, Pittsburgh, Pennsylvania, May 2 and 3, 1952.

Dr. E. J. Gumbel of Stanford University has been elected a Fellow of the American Statistical Association, in recognition of his work on the statistical theory of extreme values.

Dr. Violet B. Haas has been awarded an American Association of University Women postdoctoral fellowship.

Assistant Professor L. U. Albers of Western Reserve University has accepted a position as staff member with the Laboratory of the National Advisory Committee for Aeronautics, Cleveland, Ohio.

Mr. H. H. Barnett has been appointed to an acting assistant professorship at Florida State University.

Associate Professor R. E. Bellman of Stanford University is on leave of absence and has accepted a position at the Forrestal Research Center, Princeton University.

Mr. Jonas Beraru of the Rand Corporation has accepted a position as Design Engineer with the Ford Instrument Company, Long Island City, New York.

Professor Emeritus H. E. Buchanan of Tulane University of Louisiana has been appointed Acting President of the College of the Ozarks.

Professor E. A. Cameron of the University of North Carolina is on leave of absence and is visiting a number of universities under the provisions of an Educational Foundation Fund grant.

Professor H. H. Conwell of Beloit College has retired from his position as Dean of the college.

Dr. Philip Davis has been appointed a staff member at Massachusetts Institute of Technology.

Dr. K. A. Fowler of the University of Michigan has been appointed to an assistant professorship at the University of Arizona. (This is a correction of a note published in January, 1952, p. 111.)

Mr. I. E. Gaskill of Johns Hopkins University has accepted a position as mathematician with the Army Map Service, Washington, D. C.

Associate Professor Abe Gelbart of Syracuse University is on leave of absence and has been appointed a Fulbright Lecturer at the Institute of Technology, Trondheim, Norway.

Mr. J. C. Gibson of Rensselaer Polytechnic Institute has accepted a position as a Sales Trainee with the Applied Science Department, International Business Machines Corporation, Inc., New York, New York.

Assistant Professor C. H. Gordon of Williams College has accepted a position as research mathematician with the Boeing Aircraft Company, Seattle, Washington.

Mr. R. L. Graves of Harvard University has accepted a position as associate staff engineer with the Standard Oil Company of Indiana, Whiting, Indiana.

Mr. H. S. Heaps of the University of Toronto has been appointed to an assistant professorship at Nova Scotia Technical College.

Professor Vaclav Hlavatý of Indiana University has been appointed to the staff of the Graduate Institute for Applied Mathematics, Indiana University.

Dr. W. H. Ito of the University of Minnesota has accepted a position as Senior Project Analytical Engineer with the Chance Vought Aircraft, Dallas, Texas.

Professor Evelyn K. Kinney of Ferrum Junior College has been appointed to an assistant professorship at Illinois Wesleyan University.

Associate Professor Fred Kiokemeister of Mount Holyoke College is on leave of absence and has been appointed to a visiting professorship at the University of Chicago.

Acting Assistant Professor E. G. Kundert of the University of Tennessee has been appointed to an assistant professorship at Louisiana State University.

Professor A. E. Landry of Catholic University has retired with the title emeritus.

Mr. G. F. Leger of the University of Illinois has accepted a position as mathematician with the Bell Aircraft Corporation, Buffalo, New York.

Mr. E. L. Lehmann of Columbia University has been appointed to an associate professorship in the Statistical Laboratory at the University of California, Berkeley.

Dr. D. H. Lehmer of the University of California has been appointed Director of Research of the Institute for Numerical Analysis, National Bureau of Standards, Los Angeles, California.

Assistant Professor W. J. LeVeque of the University of Michigan is on leave of absence and is studying at the University of Manchester, Manchester, England.

Dr. J. A. Lewis has accepted a position as a member of the technical staff with the Bell Telephone Laboratories, Inc., Murray Hill, New Jersey.

Mr. Eric Liban of Indiana University has accepted a position as associate mathematician with Reeves Instrument Corporation, New York, New York.

Professor Charles Loewner of Syracuse University has been appointed to a professorship at Stanford University.

Mr. T. J. McNamara of Park College has accepted a position as group sales assistant with the Business Men's Assurance Company of America, Independence, Missouri.

Professor D. S. Morse of Union College has been appointed to the recently established Marie Louise Bailey Professorship in Mathematics.

Dr. T. S. Motzkin of the Hebrew University and the National Bureau of Standards has been appointed to a visiting professorship at the University of California, Los Angeles.

Dr. T. E. Oberbeck of Massachusetts Institute of Technology has been appointed to an associate professorship at the United States Naval Postgraduate School, Monterey, California.

Mr. S. T. Paine of the National Bureau of Standards has accepted a position as mathematician with the Department of Meteorology, University of California, Los Angeles.

Mr. W. D. Peeples of the University of Georgia has been appointed to an assistant professorship at Howard University.

Dr. C. R. Putnam of the Institute for Advanced Study has been appointed to an assistant professorship at Purdue University.

Professor W. T. Reid of Northwestern University is on leave of absence and has accepted a position as staff member at the Sandia Corporation.

Mr. R. E. Roberson of the Naval Research Laboratory has accepted a position as research engineer with North American Aviation, Downey, California.

Mr. Max Rosenberg of the National Bureau of Standards has accepted a position as mathematician with the Picatinny Arsenal, Dover, New Jersey.

Mr. R. H. Spohn of Lehigh University has accepted a position as mathematician with The Vitro Corporation of America, Orange, New Jersey.

Mr. W. C. Stone has been appointed to an assistant professorship at Union College.

Assistant Professor M. C. Waddell of Western Reserve University has accepted a position with the Applied Physics Laboratory, Johns Hopkins University, Silver Spring, Maryland.

Professor L. T. Wilson of the United States Naval Academy has retired with the title emeritus.

Mr. N. Z. Wolfsohn of Harvard University has been appointed to an assistant professorship at the University of Maryland.

Dr. D. M. Young of Harvard University has accepted a position as mathematician at Aberdeen Proving Ground, Maryland.

The following promotions are announced:

M. R. Demers, University of Nevada, to an assistant professorship.

J. C. Harden, Clemson Agricultural College, to an assistant professorship.

Nola A. Haynes, University of Missouri, to an associate professorship.

J. D. Matheson, United States Military Academy, to an assistant professorship.

J. R. Mayor, University of Wisconsin, to a professorship.

E. J. Mickle, Ohio State University, to a professorship.

R. W. Moller, Catholic University of America, to an associate professorship.

Kathryn A. Morgan, Syracuse University, to an assistant professorship.

David Moskovitz, Carnegie Institute of Technology, to a professorship.

E. N. Nilson, Trinity College, to an associate professorship.

E. G. Olds, Carnegie Institute of Technology, to a professorship.

T. G. Ostrom, Montana State University, to an associate professorship.

Dean Emeritus J. R. Overman, College of Liberal Arts, Bowling Green State University, to Dean of Faculties.

K. L. Palmquist, United States Naval Academy, to an associate professorship.

Abraham Schwartz, City College of New York, to an assistant professorship.

C. B. Smith, University of Florida, to a professorship.

H. E. Taylor, Florida State University, to an assistant professorship.

The following appointments to instructorships are announced: DePaul University: Dr. L. M. Weiner; Hamilton College: Mr. C. F. Standish; The Johns Hopkins University: Dr. J. M. Cook; Lamar State College of Technology: Mr. H. A. Morris; Syracuse University: Mr. D. G. Austin; University of Tennessee: Miss Louise M. Knifley; Yale University: Dr. John Wermer.

Dr. C. O. Lampland of the Lowell Observatory died on December 14, 1951 at the age of seventy-eight years. He had been a member of the Society for twenty-nine years.

Professor L. C. Mathewson of Dartmouth College died on October 27, 1951. He had been a member of the Society for thirty-six years.

NEW PUBLICATIONS

Becker, O. *Einfuhrung in die Logistik, vorzuglich in den Modalkalkül.* Meisenheim Am Glan, Hain, 1951. 92 pp.

Bocheński, I. M. *Ancient formal logic.* Amsterdam, North-Holland, 1951. 9+122 pp. 12 florins.

Carnot, N. L. S. *Reflexions sur la puissance motrice du feu et sur les machines propres à developper cette puissance.* Reprint of the ed. of 1824, to which have been added notes later written by the author. Paris, Blanchard, 1952. 152 pp. 550 fr.

Curry, H. B. *Outlines of a formalist philosophy of mathematics.* Amsterdam, North-Holland, 1951. 8+75 pp. 7.50 florins.

Destouches-Février, P. *La structure des théories physiques.* Paris, Presses Universitaires de France, 1951. 11+423 pp.

Dürr, K. *The propositional logic of Boethius.* Amsterdam, North-Holland, 1951. 10+79 pp. 8 florins.

Freytag gen. Löringhoff, B. *Philosophical problems of mathematics.* New York, Philosophical Library, 1951. 4+88 pp. $2.75.

Gauss, K. F. *Recherches arithmétiques.* Trans. by Poullet-Delisle. Reprint of the 1st French ed. of the Disquisitiones Arithmeticae. Paris, Blanchard, 1952. 524 pp. 5500 fr.

Hamermesh, M. See Landau, L.

Landau, L., and Lifschitz, E. *The classical theory of fields.* Trans. from the Russian by M. Hamermesh. Cambridge, Addison-Wesley, 1951. 9+354 pp. $7.00.

Legendre, A. M. *Théorie des nombres.* 4th ed. Paris, Blanchard, 1952. 2 vols., 944 pp. unbd. 2500 fr.

Lifshitz, E. See Landau, L.

Mammana, F. See Picone, M.

Ostrowski, A. *Vorlesungen uber Differential- und Integralrechnung.* Vol. 2. *Differentialrechnung auf dem Gebiete mehrerer Variablen.* Basel, Birkhäuser, 1951. 482 pp. 67 Swiss fr.

Petrovskiĭ, I. G. *Lekcii po teorii integral'nyh uravnenii.* 2d ed. Moscow, Gosudarstvennoe Izdatel'stvo Tehniko-Teoreticeskoi Literatury, 1951. 127 pp.

Picone, M. *Teoria moderna dell'integrazione delle funzioni.* Lezioni d'analisi tenute nell'anno accademico 1945–46 raccolte dal dott. F. Mammana. Pisa, Goliardica, n.d. 271 pp. 600 lire.

Poullet-Delisle. See Gauss, K. F.

Proceedings of a second symposium on large-scale digital calculating machinery, 1949. Cambridge, Harvard University Press. 38+393 pp. $8.00.

Proceedings of the second Berkeley symposium on mathematical statistics and probability, 1950. Berkeley, University of California Press, 1951. 11+666 pp. $11.00.

Schouten, J. A. *Tensor analysis for physicists.* Oxford, Clarendon, 1951. 10+275 pp. $6.00.

Sokolnikoff, I. S. *Tensor analysis. Theory and applications.* New York, Wiley, 1951. 9+335 pp. $6.00.

Sperner, E. *Einfuhrung in die analytische geometrie und algebra.* Vol. 2. Göttingen, Vandenhoeck and Ruprecht, 1951. 389 pp. 43 DM.

Whittaker, E. *Eddington's principle in the philosophy of science.* Cambridge University Press, 1951. 5+35 pp.

von Wright, G. H. *An essay in modal logic.* Amsterdam, North-Holland, 1951. 7+90 pp. 9 florins.

Journals Published by the

American Mathematical Society

Bulletin of the American Mathematical Society

This journal is the official organ of the Society. It reports official acts of the Society and the details of its meetings, with abstracts of all research papers presented It contains some of the officially invited addresses presented before the Society. It contains also announcements and notes on matters of current interest, and reviews of advanced mathematical books.

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THE RIEMANN ZETA AND ALLIED FUNCTIONS

SARVADAMAN CHOWLA

1. **On $\zeta(s)$.** Let s be a complex variable, $s=\sigma+it$, where σ and t are real. We define

$$\zeta(s) = \sum_{1}^{\infty} n^{-s} \qquad (\sigma > 1). \tag{1}$$

For $\sigma>1$ it is easily deduced from (1) that

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) - \frac{1}{s(s-1)} = \int_{1}^{\infty} (x^{s/2-1} + x^{(1-s)/2-1})\Theta(x)dx, \tag{2}$$

where

$$\Theta(x) = \sum_{n=1}^{\infty} e^{-n^2\pi x}.$$

To prove (2) we use Jacobi's relation

$$1 + 2\Theta(x) = \frac{1}{x^{1/2}}\left\{1 + 2\Theta\left(\frac{1}{x}\right)\right\}.$$

Since the right-hand side of (2) is an analytic function of s for all complex values of s, we use (2) to define $\zeta(s)$ over the whole s-plane.

It easily follows from (2) that

$$\zeta(s) - \frac{1}{s-1} \text{ is an integral function of } s; \tag{3}$$

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-(1-s)/2}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s). \tag{4}$$

We have

$$\zeta(s) = \prod_{p} (1 - p^{-s})^{-1} \qquad [\sigma > 1], \tag{5}$$

where p runs through all primes. Equation (5) follows immediately from the so-called unique factorization theorem of arithmetic, which states that a positive integer can be decomposed into a product of primes in one and only one way.

An address delivered before the Annual Meeting of the Society at New York on December 28, 1949 by invitation of the Committee to Select Hour Speakers for Annual and Summer Meetings; received by the editors December 23, 1950 and, in revised form, December 5, 1951.

Equation (4) is the "functional equation" of the zeta function due to Riemann. From (3), (4), and (5) it easily follows that except for the "trivial" zeros of $\zeta(s)$ at $s=-2, -4, -6, \cdots$, all the remaining zeros of $\zeta(s)$ lie in the critical strip $0<\sigma<1$. In fact it also follows from the Hadamard theory of integral functions that there is an infinity of complex zeros in this strip.

The "Riemann hypothesis," one of the most famous and difficult of unsolved problems today, states that all the nonreal zeros s of $\zeta(s)$ satisfy $\sigma=1/2$. It can also be stated as follows:

If $1^{-s}-2^{-s}+3^{-s}-4^{-s}+-\cdots=0$ $(0<\sigma<1)$, then $\sigma=1/2$. This follows from the fact that

$$(1-2^{1-s})\zeta(s) = 1^{-s}-2^{-s}+3^{-s}-4^{-s}+-\cdots$$

is clearly true for $\sigma>1$, and hence for $\sigma>0$.

Riemann also found that the zero s of $\zeta(s)$ with the smallest ordinate $t=\alpha_1$ satisfies

$$s = 1/2 + 14.1386\cdots i = 1/2+\alpha_1 i.$$

A later calculation by Gram showed that $\alpha_1=14.1347\cdots$. Let α_n run through all the solutions α of $\zeta(1/2+i\alpha)=0$; then (assuming the Riemann hypothesis) we have

$$\sum_{n=1}^{\infty}\left(\alpha_n^2+\frac{1}{4}\right)^{-1} = 1+\frac{1}{2}C-\frac{1}{2}\log\pi-\log 2$$
$$= .02\ 309\ 570\ 896\ 612\ 103\ 381$$

according to a calculation of Riemann. Further zeros of $\zeta(s)$ have been calculated by Backlund, Hutchinson, and Titchmarsh. In particular Titchmarsh calculated all complex zeros s with $|t|\leqq 1468$. They all had $\sigma=1/2$, in confirmation of the Riemann hypothesis. Rademacher and Lehmer hope to substantially extend the range of these calculations. Should these or any authors find a zero off the line $\sigma=1/2$, such a discovery would certainly be sensational. It would, however, be a sad thing for mathematics, as it would destroy in one cruel blow the beautiful and harmonious structure which has been built on the assumption of the Riemann hypothesis.

2. **On $\zeta(s)$, continued.** We owe to A. Selberg the sharpest known estimate on the number of zeros s of $\zeta(s)$ with $\sigma=1/2$, $0\leqq t\leqq T$. Call the number of such zeros $N_0(T)$. Hardy was the first to prove in 1915 that

$$N_0(T)\to\infty \qquad [T\to\infty].$$

Later Hardy and Littlewood sharpened this estimate to

$$N_0(T) > cT,$$

where c is a positive constant independent of T. Selberg proved that

$$N_0(T) > cT \log T.$$

Denote by $N(T)$ the number of zeros s of $\zeta(s)$ with $0<\sigma<1$, $0\leqq t\leqq T$. H. von Mangoldt proved Riemann's conjecture that

$$N(T) \sim (1/2\pi)\, T \log T.$$

From the last two relations

$$\frac{N_0(T)}{N(T)} > c.$$

In his 1943 paper A. Selberg proved more; namely, he proved the following:

Let $U \geqq T^a$, where $a>1/2$; then there is an $A=A(a)>0$ and a $T_0=T_0(a)$ such that

$$N_0(T+U) - N_0(T) > AU \log T \qquad (T > T_0)$$

(p. 46 of Selberg).

Let p_n denote the nth prime. Erdös has conjectured that

$$\sum_{p_n \leqq x} (p_n - p_{n-1})^2 = O(x \log x).$$

This would imply

$$\sum_{p_n \leqq x} \frac{(p_n - p_{n-1})^2}{p_n} = O(\log^2 x).$$

On the Riemann hypothesis A. Selberg proved that

$$\sum_{p_n \leqq x} \frac{(p_n - p_{n-1})^2}{p_n} = O(\log^3 x).$$

Denote by $N(T)$ the number of zeros $\beta+i\gamma$ of $\zeta(s)$ for which $0<\gamma<T$, if T is not equal to any γ; otherwise we put

$$N(T) = \lim_{\epsilon\to 0} \frac{1}{2}\left\{N(T+\epsilon) + N(T-\epsilon)\right\}.$$

It is well known that

$$N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + \frac{7}{8} + S(T) + O(T^{-1}),$$

where

$$S(T) = \frac{1}{\pi} \operatorname{am} \zeta\left(\frac{1}{2} + iT\right),$$

the amplitude being obtained by continuous variation along the straight lines joining 2, $2+iT$, and $1/2+iT$, starting with the value 0. The variation of $S(T)$ is thus closely related to the distribution of the imaginary parts of the zeros. We know a little about $S(T)$; namely,

$$S(T) = O(\log T); \quad \int_0^T S(t)dt = O(\log T); \quad S(t) \neq o(1).$$

Selberg proved the following Ω result for $S(T)$:

$$S(t) = \Omega_{\pm}\left\{(\log t)^{1/3}(\log \log t)^{-7/3}\right\}.$$

Selberg proved some remarkable theorems about the distribution of the zeros of the functions $L(s, \chi)$ where $\chi(n)$ is a character (mod k).

If $|t_1| \leqq k^{1/4-\epsilon}$ and $A = A(\epsilon)$ is chosen sufficiently great, at least $k/2$ of the $k-2$ different functions $L(s, \chi)$ (here k is a prime, and the character χ is nonprincipal) have a zero in the interval described by

$$\sigma = \frac{1}{2}, \qquad t_1 < t < t_1 + \frac{A}{\log k}.$$

Also the total number of zeros of the functions $L(s, \chi)$ in the rectangle $0<\sigma<1$, $t_1<t<t_1+A/\log k$ is $O(k)$.

3. **On $\zeta(s)$, continued.** In some unedited papers left by Riemann (these were in a very untidy state—often with only one side of a formula), Siegel recently found the following result (the proofs were not clear and many details were supplied by Siegel):

$$\zeta(s) = \sum_{l=1}^{m} l^{-s} + \frac{(2\pi)^s}{2\Gamma(s)\cos(s\pi/2)} \sum_{l=1}^{m} l^{s-1}$$
$$+ (-1)^{m-1} \frac{(2\pi)^{(s+1)/2}}{\Gamma(s)} t^{(s-1)/2} e^{\pi i s/2 - it/2 - \pi i/8} S,$$

with

$$S = \sum_{0 \leq 2r \leq k \leq n-1} \frac{2^{-k} i^{r-k} k!}{r!(k-2r)!} a_k F^{(k-2r)}(\delta),$$

where

$$n \leqq 2.10^{-8}\, t, \qquad m = \left[\left(\frac{t}{2\pi}\right)^{1/2}\right], \quad \delta = t^{1/2} - \left(m + \frac{1}{2}\right)(2\pi)^{1/2},$$

$$F(u) = \frac{\cos\,(u^2 + 3\pi/8)}{\cos\,((2\pi)^{1/2}u)},$$

and the coefficients a_k are fixed by the recursion formula

$$(n+1)\tau a_{n+1} = -\,(n+1-\sigma)a_n + ia_{n-2} \qquad [a_{-2} = a_{-1} = 0].$$

Here

$$\tau = t^{1/2}, \qquad a_n = O(t^{-n/2+[n/3]}).$$

This formula enables one (theoretically at least!) to obtain rapid approximations to $\zeta(s)$, for example when $\sigma \doteq 1/2$ and t is large. The principal term of the semi-convergent series for $\zeta(s)$, namely,

$$\sum_{l=1}^{m} l^{-s} + \frac{(2\pi)^s}{\pi} \sin \frac{s\pi}{2}\Gamma(1-s)\sum_{l=1}^{m} l^{s-1} \qquad \left(m = \left[\left(\frac{t}{2\pi}\right)^{1/2}\right]\right),$$

was also found by Hardy and Littlewood. In place of Riemann's development of S, they gave only an upper bound of the absolute value of the "Rest." Siegel says, "Wie stark Riemanns analytische Technik war, geht besonders deutlich aus seiner Ableitung und Uniformung der semikonvergenten Reihe für $\zeta(s)$ hervor."

4. **On $\zeta(s)$, continued.** We shall now make a few remarks on "Gram's law." These remarks are due to Atle Selberg. Define

$$\vartheta = \arg\left\{\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\right\},$$

where

$$s = \frac{1}{2} + it.$$

The function $\vartheta(t)$, thus defined, first decreases to a minimum between -2π and $-\pi$ (when t increases from 0 to ∞) and is from this point onward steadily increasing. For large t we have

$$\vartheta = \frac{t}{2}\log\frac{t}{2\pi} - \frac{t}{2} - \frac{\pi}{8} + O(t^{-1}).$$

We now for $\nu = 0, 1, 2, \cdots$ determine the positive number t_ν by the conditions $\vartheta'(t_\nu) > 0$ and

$$\vartheta(t_\nu) = (\nu - 1)\pi.$$

Then

$$\zeta\left(\frac{1}{2} + it_\nu\right) = (-1)^{\nu-1}X(t_\nu);$$

so that $\zeta(1/2+it_\nu)$ is real.

The Riemann-Siegel formula

$$X(t) = \sum_{n \leq (t/2\pi)^{1/2}} \frac{\cos(\vartheta - t\log n)}{n^{1/2}} + O(t^{-1/4})$$

gives

$$\zeta\left(\frac{1}{2} + it_\nu\right) = 1 + \sum_{1<n<(t_\nu/2\pi)^{1/2}} \frac{\cos(t_\nu \log n)}{n^{1/2}} + O(t_\nu^{-1/4}),$$

which seems to suggest that $\zeta(1/2+it_\nu)$ is generally positive, and that $X(t_{\nu-1})$ and $X(t_\nu)$ should generally be of opposite sign, so that $\zeta(1/2+it)$ should generally have at least one real zero in the interval $(t_{\nu-1}, t_\nu)$. The numerical results of Gram and Backlund verified that for $t_\nu<200$, $\zeta(1/2+it_\nu)$ were always positive and that always $t_{\nu-1}<\gamma<t_\nu$, so that the γ's were separated by the t_ν's. It is this property which is referred to as Gram's law.

See A. Selberg, Proceedings of the Tenth Scandinavian Congress, Copenhagen, August, 1946, pp. 197–200. Titchmarsh proved that "Gram's law" has infinitely many exceptions, and Selberg proved that the exceptions even have a positive density; and for this reason Selberg viewed the "Riemann hypothesis" with suspicion.

5. On $\pi(x)$. The equation (5) is fundamental for all estimates of $\pi(x)$, the number of primes not exceeding x.

Euclid proved that

$$\pi(x) \to \infty \qquad [x \to \infty].$$

Legendre and Gauss independently (the latter at a very early age) surmised that

$$\pi(x) \sim \frac{x}{\log_e x}. \tag{6}$$

The sign of asymptotic equality

$$f(x) \sim g(x)$$

expresses that

$$\lim_{x\to\infty}\frac{f(x)}{g(x)}=1.$$

Riemann, apparently, developed his whole theory of the function $\zeta(s)$ in an attempt to prove the beautiful relation (6), now known as the Prime Number Theorem. He did not succeed, and it was left to Hadamard and de la Vallée Poussin (independently) to give complete and rigorous proofs of (6). These proofs leaned heavily on properties of $\zeta(s)$. Landau simplified these proofs considerably, using essentially Cauchy's theorem and the fact that $\zeta(s)\neq 0$ on the line $\sigma=1$ (or in a region slightly to the left of this line). Wiener proved (6) without using Cauchy's theorem, but used $\zeta(s)\neq 0$ for $\sigma=1$.

It was reserved to Atle Selberg [33] to give a really elementary proof of (6)—a proof which mentions neither the zeta function nor the complex variable. This proof was a great achievement. It rests on Selberg's fundamental lemma, namely,

$$\sum_{p\leq x}\log^2 p+\sum_{p\leq x}\log p\,\vartheta\left(\frac{x}{p}\right)\sim 2x\log x,$$

where

$$\vartheta(x)=\sum_{p\leq x}\log p.$$

This result can be proved very simply. From it one can deduce without much difficulty that

$$\vartheta(x)\sim x,$$

which is equivalent to (6).

Another proof of (6), also based on Selberg's fundamental lemma, was published by Paul Erdös.

6. **On $\pi(x)$, continued.** Gauss, as well as subsequent workers, noticed that

$$\pi(x)<\int_0^x\frac{du}{\log u}$$

as far as existing tables went.

Littlewood [25] proved the following deep and beautiful result: Writing

$$P(x)=\pi(x)-\int_0^x\frac{du}{\log u},$$

we have

$$P(x) = \Omega_{\pm}\left(\frac{x^{1/2}\log\log\log x}{\log x}\right).$$

Thus $P(x)$ changes sign infinitely often as $x \to \infty$. Littlewood's method, which used the Phragmen-Lindelöf principle, does not give an explicit value of x_0 such that $P(x_0) > 0$. Skewes [45] found a method which determines an upper bound for x_0 such that $P(x_0) > 0$, namely,

$$x_0 < 10^{10^{10^{34}}}.$$

In this connection see also Littlewood [25] and Turán [51] and Ingham [22].

On the assumption of the Riemann hypothesis one can show that

$$P(x) = O(x^{1/2+\epsilon})$$

for every positive ϵ; and conversely, if this relation is true, then the Riemann hypothesis is true.

W. H. Mills [26] found for the first time in 1947 a prime-representing function; he proved the existence of an $A > 1$ such that

$$[A^{3^x}]$$

is a prime for all positive integers x. Here $[x]$ denotes the greatest integer in x. This Mills deduced in a trivial way from the fact that there exists a prime between

$$x^3 \quad \text{and} \quad (x+1)^3$$

for all large x, a result prove by Ingham. Hoheisel was the first to find a number θ such that

$$\sum_{x^\theta < p < (x+1)^\theta} 1 > 0$$

for $\theta = 33{,}000$. His θ was improved successively by Heilbronn, Tchudakoff, Ingham. Whether the above inequality is true for $\theta = 2$ is still unknown.

7. **Functions allied to $\zeta(s)$.** Dirichlet found the remarkable theorem that every arithmetical progression $ax+b$ $(x = 1, 2, 3, \cdots)$ contains an infinity of primes if a and b have no common factor. To prove this classical result he introduced what are known as Dirichlet's L-functions. We first define $\chi(n)$, a character (mod k), as follows:

$$\chi(m) = \chi(n) \qquad \text{if } m \equiv n \pmod{k};$$

$$\begin{aligned} \chi(mn) &= \chi(m)\chi(n); \\ \chi(n) &= 0 \qquad\qquad \text{if } (n, k) > 1; \\ \chi(1) &\neq 0. \end{aligned}$$

Then for $\sigma > 1$,

$$L_k(s) = L(s, \chi) = \sum_1^\infty \chi(n) n^{-s}.$$

As an example $k = 5$ of L-functions we take

$$\begin{aligned} L_5(s) = L(s, \chi) = 1^{-s} &+ i\cdot 2^{-s} - i\cdot 3^{-s} - 4^{-s} + 0\cdot 5^{-s} \\ &+ 6^{-s} + i\cdot 7^{-s} - i\cdot 8^{-s} - 9^{-s} + 0\cdot 10^{-s} \\ &+ 11^{-s} + i\cdot 12^{-s} - i\cdot 13^{-s} - 14^{-s} + 0\cdot 15^{-s} \\ &+ \cdots . \end{aligned}$$

In fact the series on the right is convergent for $\sigma > 0$. One easily proves

$$L(s, \chi) = \prod_p \{1 - \chi(p) p^{-s}\}^{-1} \qquad (\sigma > 1).$$

For the proof of Dirichlet's theorem it is fundamental that

$$L(1) = \sum_1^\infty \chi(n) n^{-1} \neq 0,$$

where $\chi(n)$ is a character (mod k) such that the series is convergent.

The L-functions have many properties similar to the zeta function. They can be continued over the whole s-plane; they possess functional equations; and the Riemann hypothesis for the L-functions simply states that if $L(s, \chi) = 0$ for $0 < \sigma < 1$, then $\sigma = 1/2$.

It is noteworthy that if one assumes the "extended Riemann hypothesis" (the assumption that the Riemann hypothesis is true for all Dirichlet's L-functions), one proves, as Hardy and Littlewood did, that all large odd numbers are expressible as a sum of three odd primes. Vinogradoff proved this result in 1937 without any assumption.

Even the statement

$$(7) \qquad L(s, \chi) > 0 \qquad (0 < s < 1)$$

(here $\chi \neq \chi_0$; χ_0 is the so-called principal character which is such that $\chi_0(n) = 1$ or 0 for all n) is still unproved.

Siegel proved that

$$\sum_{n=1}^{\infty} \chi(n) n^{-1} > k^{-\epsilon} \qquad [k > k_0(\epsilon)],$$

where $\chi(n)$ is a nonprincipal character (mod k), and this shows that $L(s, \chi) > 0$ for $1 - A(\epsilon)k^{-\epsilon} < s \leqq 1$. Thus the hypothesis (7) is proved only for an infinitesimal portion of the line $0 \leqq s \leqq 1$.

Recently Estermann and Chowla [12] gave new simplified proofs of this result.

Chowla [10] found a method (greatly improved by Rosser [30]) of proving for real nonprincipal characters

$$L_k(s) > 0 \qquad (0 < s < 1)$$

for a large number of special values of k. The most difficult case up to $k = 200$ was $k = 163$. Chowla and Selberg [14] proved that

$$L_{163}(1/2) > 0$$

and Rosser [31] proved that

$$L_{163}(s) > 0 \qquad (s > 0).$$

Showalter verified that

$$L_{409}(s) > 0 \qquad (s > 0).$$

Regarding $L(1)$ nothing better than the trivial result

$$L(1) = O(\log k)$$

is known. This is proved as follows:

$$\begin{aligned} L(1) &= \sum_{1}^{\infty} \frac{\chi(n)}{n} = \sum_{1}^{k} \frac{\chi(n)}{n} + \sum_{k+1}^{\infty} \frac{\chi(n)}{n} \\ &= \sum\nolimits_1 + \sum\nolimits_2. \end{aligned}$$

Since $|\chi(n)| = 1$ or 0,

$$\sum\nolimits_1 = O(\log k).$$

By partial summation using

$$\left| \sum_{1}^{t} \chi(n) \right| \leqq k,$$

we get

$$\sum\nolimits_2 = O(1).$$

Littlewood proved that if $\chi(n)$ is a primitive character (mod k), then

$$L(1) = O(\log \log k)$$

on the assumption of the extended Riemann hypothesis. He also proved (again under the assumption of the extended Riemann hypothesis) that there are infinitely many k such that

$$L(1) < \frac{b}{\log \log k}$$

where $\chi(n)$ is a real primitive character (mod k).

While Linnik and Walfisz proved independently that there are infinitely many k such that

$$L(1) < \frac{b}{(\log \log k)^{1/2}},$$

Chowla [11] proved Littlewood's result without any assumption.

Bateman, Chowla, and Erdős [6] proved that there are infinitely many primes p for which

$$L(1) = \sum_{1}^{\infty} \left(\frac{n}{p}\right) n^{-1} < \frac{b}{\log \log p},$$

where b is a certain suitable positive constant independent of p.

8. **Functions allied to $\zeta(s)$, continued.** Given an algebraic number-field K we define

$$\zeta_K(s) = \prod_P \{1 - N(P)^{-s}\}^{-1},$$

where P runs over all different prime ideals in K, and $N(P)$ denotes the norm of the prime ideal P in K.

This function was introduced by Dedekind. Hecke in 1917 proved that

$$(s - 1)\zeta_K(s)$$

can be continued analytically over the whole s-plane, and that it is an entire function of s.

Further the function $\zeta_K(s)$ satisfies a functional equation similar to that satisfied by $\zeta(s)$. See also Siegel [41].

An intriguing conjecture about $\zeta_K(s)$ is that the ratio $\zeta_K(s)/\zeta(s)$ is an entire function of s for all complex values of s. See Artin [5]. The conjecture is known from the work of Takagi to be true when K is an Abelian field over the rationals. Some other cases of the conjecture—non-Abelian cases—were proved by Dedekind and more recently in the brilliant work of Artin and R. Brauer. Dedekind found for $K = R(2^{1/3})$ that

$$\zeta_K(s) = \zeta(s)\left\{ \sum{}'(x^2 + 27y^2)^{-s} - \sum{}'(4x^2 + 2xy + 7y^2)^{-s} \right\}$$

where the summations are for all pairs of integers x, y excluding $x = y = 0$.

The two expressions in the braces are examples of the so-called Epstein zeta function.

Dedekind's (1900) result had its origin in the following remarks of Gauss. If p is a prime $\equiv 1$ (3) and if 2 is a cubic residue of p, then $p = x^2 + 27y^2$. If 2 is not a cubic residue of p, then $p = 4x^2 + 2xy + 7y^2$. Gauss stated similar results when 2 is replaced by 3 or 5.

Hecke (and after him Landau) defined "Abelian L-series" as follows. Let $\mathfrak{F}$ be a fixed ideal. Then two ideals $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are equivalent (mod $\mathfrak{F}$) if there exist algebraic integers α_1, α_2 in K, such that

$$(\alpha_1)\mathfrak{M}_1 = (\alpha_2)\mathfrak{M}_2,$$
$$\alpha_1 \equiv \alpha_2 \pmod{\mathfrak{F}},$$

provided $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are prime to $\mathfrak{F}$. It is not difficult to see that all ideals of K fall into a finite number of nonequivalent classes (mod $\mathfrak{F}$), and that these nonequivalent classes form a finite multiplicative Abelian group. Let χ be any character of this group. Then for $\sigma > 1$ we define

$$L(s, \chi) = \prod \left\{1 - \chi(P)N(P)^{-s}\right\}^{-1},$$

where P runs over all different prime ideals of K. These Abelian L-series can also be continued over the whole s-plane. In fact, if K is the cubic field $R(d^{1/3})$, d an integer $\neq m^3$, a striking result of Artin asserts

$$\zeta_K(s) = \zeta(s)(L_1L_2)^{1/2},$$

where L_1, L_2 are certain Abelian L-series in the field $R((-3)^{1/2})$. Artin also obtains relations between the zeta functions of certain fields and their subfields. Further the Abelian L-series do not vanish for $\sigma = 1$ as Hecke found. Further they have a functional equation. They enable us to extend Dirichlet's theorem on primes in an arithmetic progression to the following:

If $(\rho, \mathfrak{F}) = 1$, then there exist infinitely many integers π such that

$$\pi \equiv \rho \pmod{\mathfrak{F}},$$

where (π) is a prime ideal.

9. Functions allied to $\zeta(s)$, continued. We shall now consider an entirely different type of L-series whose introduction into number-theory has had astounding results. For an account of these see Davenport [15].

Let $f_1(x), \cdots, f_r(x)$ be any polynomials with integer coefficients, p a prime. We define a normalized polynomial as a polynomial with integer coefficients whose highest coefficient is 1. Further (f, g), the resultant of f and g, is defined as

$$\prod f(\phi),$$

where ϕ runs over the roots of $g(x)=0$.

We then define

$$L(s) = 1 + \frac{\sigma_1}{p^s} + \frac{\sigma_2}{p^{2s}} + \cdots + \frac{\sigma_{K-1}}{p^{(K-1)s}},$$

where

$$\sigma_\nu = \sum_g \chi_1((f_1, g)) \cdots \chi_r((f_r, g)),$$

where the χ's are characters (mod p), K = sum of degrees of the f's which are irreducible (mod p), and the summation is over all normalized polynomials of degree ν (in the definition of σ_ν above).

These L-functions also have a functional equation and a Riemann hypothesis which André Weil proved. However, the proof is exceedingly difficult, and assumes a knowledge of the recent treatment of algebraic geometry which is due to André Weil himself.

In the special case $r=1$, $f_1=x^3+ax+b$, we obtain

$$L(s) = 1 + \frac{\sigma_1}{p^s} + \frac{p}{p^{2s}},$$

where

$$\sigma_1 = \sum_{x=1}^{p} \left(\frac{x^3 + ax + b}{p}\right).$$

Here $\left(\frac{n}{p}\right)$ is Legendre's symbol defined as 0 if $n \equiv 0$ (p), while if $n \not\equiv 0$ (mod p), $\left(\frac{n}{p}\right)=1$ or -1 according as the quadratic congruence $x^2 \equiv n$ (mod p) is or is not soluble.

Thus $L(s)=0$ implies

$$p^{-s} = \frac{\sigma_1 \pm (\sigma_1^2 - 4p)^{1/2}}{2p}.$$

Thus $R(s) = 1/2$ if and only if

$$|\sigma_1| < 2p^{1/2}.$$

Artin conjectured this in 1924 and verified it in about 40 numerical instances. A proof was given by Hasse in 1934. It uses the complex multiplication of elliptic functions. See also Deuring [17] and Chowla [13].

10. **Some unsolved problems.** In this section we collect together a few problems of interest whose solution depends on the theory of different types of zeta functions.

(1) Let k be a positve integer; let $f(n)$ have the period k, i.e.,

$$f(m) = f(n) \qquad [m \equiv n \pmod{k}];$$

let $f(x)$ be a number-theoretic function; suppose further that not all $f(n)$ are 0; then

$$\sum_{n=1}^{\infty} \frac{f(n)}{n} \neq 0$$

whenever the series is convergent.

This problem was suggested by Paul Erdös. If true, it naturally includes a result like

$$\sum_{1}^{\infty} \left(\frac{n}{p}\right) n^{-1} \neq 0$$

due to Dirichlet. Here $\left(\frac{n}{p}\right)$ is Legendre's symbol. I proved this under the assumptions $f(x) = -f(-x)$, $k =$ a prime, $(k-1)/2 =$ a prime. Professor Siegel, to whom I showed my proof, proved the result under only the first of the above three assumptions, namely, whenever $f(x) = -f(-x)$ is true for all x.

Remark. The condition for the convergence of our series is of course

$$\sum_{1}^{k} f(n) = 0.$$

(2) Let $h(d)$ denote the class-number of the quadratic field $R(d^{1/2})$. It is known that there are nine negative d for which $h(d) = 1$,

$$-d = 3, 4, 7, 8, 11, 19, 43, 67, 163.$$

It is also known (Heilbronn and Linfoot) that there is at most one more negative d with $h(d) = 1$. This $-d$ must necessarily be a prime p

of the form $8k+3$. Chowla and Selberg [14] proved that for such a prime p we would have

$$\sum_{1}^{\infty}\left(\frac{n}{p}\right)n^{-1/2} < 0;$$

so that the "extended Riemann hypothesis" would certainly be false!

Another interesting way of stating the conjecture that $h(d)=1$ has no tenth solution with $d<0$ is the following (which I have not seen before).

If p is a prime $\equiv 3$ (8), greater than 163, then we can write

$$p = x^2 + y^2 + z^2,$$

where x, y, z are all *different* positive integers.

(3) *The cosine problem.* Let K be an arbitrary positive integer. Then can we find $s_0=s_0(K)$ such that

$$\min_{0\leq x<2\pi}\left\{\cos a_1x + \cos a_2x + \cdots + \cos a_sx\right\} < -K$$

for all $s>s_0$ where $a_1, \cdots, a_s$ is any set of s different positive integers?

This problem is due to Ankeny and Chowla. It has been communicated to several mathematicians, and it appears that the answer is probably in the affirmative, but a proof seems difficult. The problem actually arose in a question concerning zeta functions.

(4) *There are infinitely many primes which do not divide a given irreducible polynomial of degree $n\geq 2$ with integer coefficients.*

Professor Siegel, to whom I showed this problem, gave a proof which depends on the theory of the Dedekind zeta function. It does not seem easy to give an "elementary" proof, in particular one which avoids infinite series.

(5) If $f(x)$ is a polynomial with integer coefficients, what is the number of values which $f(x)$ assumes (mod p)?

For example, it is likely that for fixed integers a, $b(ab\neq 0)$

$$x^n + ax + b \qquad (n \geq 3)$$

assumes more than $p/2$ values (mod p) when p is large enough.

This I have been unable to prove. Professor Davenport, to whom I communicated the problem, obtained the following results:

The number of values assumed by $f(x)=x^4+ax^2+bx+c$ (mod p) is $\sim 5p/8$.

(6) If $f(x)=a_nx^n+\cdots+a_0$, where the a's are integers and $a_n\not\equiv 0$ (p), André Weil (see Mordell, *Thoughts on number theory*, but a detailed proof is still unpublished) proved that

$$\left| \sum_{x=1}^{p} e^{2\pi i f(x)/p} \right| \leqq (n-1)p^{1/2};$$

for $n=2$, there is equality (by Gauss).

(7) Is $\zeta(3)$ irrational? This is a well known unsolved problem. For $\zeta(3)$ here one can in general substitute $\zeta(2n+1)$ where n is any positive integer.

(8) Heuristically one can argue (as Artin did; see Bilharz [7]) that the primes for which 2 is a primitive root have an exact positive density. However, even assuming any of the Riemann hypotheses one has not been able to prove this.

(9) While the usual expression for the class-number of a real quadratic field contains infinite series or transcendental functions, the following expression (due to Ankeny, Artin, and Chowla; see [1]) for the class-number of a real quadratic field is purely arithmetical:

$$2h\frac{u}{t} \equiv \frac{A+B}{p} \pmod{p}.$$

Here h is the class-number of the real quadratic field $R(p^{1/2})$, where p is a prime $\equiv 1 \pmod 4$,

$$\epsilon = (t + up^{1/2})/2$$

is its fundamental unit; A stands for the product of the least positive quadratic residues of p, B for the corresponding product of non-residues.

Let q be a prime $\equiv 7 \pmod{12}$, H the class-number of the imaginary quadratic field $R((-q)^{1/2})$, h the class-number of the real quadratic field $R((3q)^{1/2})$, $\epsilon = (t+u(3q)^{1/2})/2$ the fundamental unit of the latter field.

Then (see [1])

$$H \equiv -\frac{u}{t}h \pmod{3}.$$

(10) Is

$$RL(1) > 0,$$

where

$$L(1) = \sum_{1}^{\infty} \chi(n)n^{-1}$$

and $\chi(n)$ is a complex character (mod k)? This problem was raised by Chowla and verified by Showalter in many cases.

(11) *Pillai's problem*. If p_n is the nth prime, is it true that

$$\left|\sum_{n\leq x}(-1)^{n-1}p_n\right| \sim \frac{1}{2}\,p_x$$

where $x\to\infty$ through positive integral values?

(12) Is it true that

$$\sum_{1}^{t}\chi(n)n^{-1} > 0$$

if $\chi(n)$ is a real nonprincipal character (mod k), and t is an arbitrary number?

(13) If $1^{-s}+2^{-s}+\cdots+n^{-s}\neq 0$ $(\sigma>1,\ n>n_0)$ then the "Riemann hypothesis" is true. This result is due to Turán [52].

(14) If $h(d)$ is the class-number of the quadratic field $R(d^{1/2})$, Gauss conjectured that

$$h(d) = 1$$

has infinitely many solutions with $d>0$. This problem remains far from solution. See K. Schaffstein, Math. Ann. vol. 98 (1928). In particular Schaffstein finds primes $p>10^6$ with $h(p)=1$.

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University of Kansas

MEASURE AND AREA

HERBERT FEDERER

1. **Introduction.** For a long time the efforts of many mathematicians have been directed toward the creation of a complete theory which extends the quantitative concepts of Euclidean geometry to general subsets of Euclidean n-space, E_n, and to mappings into E_n. The first and fundamental steps in the modern development of this subject were taken by Lebesgue and Carathéodory with the invention of the general theory of measure. Subsequently the main interest has shifted naturally to the specific geometric problems which arise when the new concepts are applied in the extension of classical analytic formalisms and in the structural study of sets and mappings.

Measures are functions defined on classes of point sets; areas are functions defined on classes of mappings.

The first four sections of this paper are mainly expository. They contain the definitions of certain measures of geometric interest; a discussion of the rectifiability, tangent planes, densities and projections of point sets; and a description of a very general form of the Gauss-Green Theorem which illustrates the scope of the theory and its applicability to classical problems. Pivotal in this whole structure is the relationship between the purely metric notion of Hausdorff measure and the group-theoretic concepts of integral geometry.

The remaining four sections of this paper serve to establish some new results concerning the area of mappings. The basic issue motivating these investigations is the validity of the principle that every area defined by any reasonable method is naturally representable as an integral of a multiplicity function with respect to a measure. This raises a difficult problem because Lebesgue, in introducing his particular area which has been used very successfully by many workers in the field, was guided entirely by the analytic requirement of semi-continuity; his definition appears to have no immediate connection with the geometric measure theory which was created later by Carathéodory. This difficulty has now been overcome for the case of prime classical interest: mappings of a two-cell into three-space.

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The principal new results are contained in the theorems 8.17, 8.15, and 6.9.

2. Definitions of some k-dimensional measures over n-space. Suppose $k \leqq n$ are positive integers, $\mathcal{L}_k$ is the k-dimensional Lebesgue measure over E_k, and

$$\alpha(k) = \mathcal{L}_k(E_k \cap \{x \mid |x| < 1\})$$

is the volume of the k-dimensional unit sphere.

Let $A \subset E_n$. With each positive number r we associate the infimum of all numbers of the form

$$\sum_{i=1}^{\infty} 2^{-k}\alpha(k) \text{ (diameter } B_i)^k$$

where

$$A \subset \bigcup_{i=1}^{\infty} B_i,$$

$$\text{diameter } B_i < r \qquad \text{for } i = 1, 2, 3, \cdots.$$

As r decreases toward zero, this infimum never decreases; it approaches the limit

$$\mathcal{H}_n^k(A),$$

the k-dimensional Hausdorff measure of the subset A of E_n.

If $k = n$, then $\mathcal{H}_n^k = \mathcal{L}_n$ and the above infimum is actually independent of r. Carathéodory was the first to recognize the necessity of the double limiting process in case $k < n$. By attaching to each set B_i a constant times the kth power of its diameter, Hausdorff modified the earlier definition of Carathéodory, who had required B_i to be convex and had associated with it the supremum of the $\mathcal{L}_k$ measures of its orthogonal projections into E_k. The measures of Carathéodory and Hausdorff are not identical, but their ratio is bounded [MEF 1]. This implies that the structure theorems for sets of finite Hausdorff measure, described in the following section, apply equally to sets of finite Carathéodory measure. However Hausdorff measure appears to be usually easier to work with and it has the advantage, exploited in §6, that its definition remains meaningful for arbitrary metric spaces.

The simplest orthogonal projection of E_n onto E_k is the function p_n^k on E_n such that

$$p_n^k(x) = (x_1, \cdots, x_k) \in E_k \qquad \text{for } x = (x_1, \cdots, x_n) \in E_n.$$

Every orthogonal projection of E_n onto E_k is a superposition of the form

$$p_n^k \circ R,$$

where R is an element of the group G_n of orthogonal transformations of E_n.

For each function f, each set X, and each point y, let

$$N(f, X, y)$$

be the number (possibly ∞) of elements of the set

$$X \cap \{x \mid f(x) = y\}.$$

Now let A be an analytic subset of E_n. For each $R \in G_n$ the integral

$$\int_{E_k} N(A, p_n^k \circ R, y) d\mathcal{L}_k y$$

indicates the size of the corresponding orthogonal projection of A, counting each point in E_k with its multiplicity of projection. Using the Haar measure ϕ_n over G_n for which $\phi_n(G_n) = 1$ and the constant

$$\beta(n, k) = \frac{\alpha(k) \cdot \alpha(n-k)}{\alpha(n) \cdot \binom{n}{k}},$$

we define the *integralgeometric Favard measure*

$$\mathcal{I}_n^k(A) = \beta(n, k)^{-1} \cdot \int_{G_n} \int_{E_k} N(p_n^k \circ R, A, y) d\mathcal{L}_k y d\phi_n R$$

as a kind of average projection size of A. In his note in the Comptes Rendus Favard attributes the idea for this definition—more precisely, a trivially equivalent definition—to a suggestion by Lebesgue. After that Favard never did anything to exploit this concept, whose fundamental importance will become clear in the next section.

The assumption that A be an analytic set was made in order to assure the measurability of the integrand in the definition of $\mathcal{I}_n^k(A)$. The measure may be extended to arbitrary subsets B of E_n in such a way that $\mathcal{I}_n^k(B)$ is the infimum of $\mathcal{I}_n^k(A)$ for all analytic sets A containing B.

The functions $\mathcal{H}_n^k$ and $\mathcal{I}_n^k$ are (Carathéodory outer) measures over E_n. Every closed subset of E_n is both $\mathcal{H}_n^k$ and $\mathcal{I}_n^k$ measurable (in the sense of Carathéodory). Every subset of E_n is contained in a G_δ set of equal $\mathcal{H}_n^k$ measure and also in an analytic set of equal $\mathcal{I}_n^k$ measure.

Every $\mathcal{H}_n^k$ measurable subset of finite $\mathcal{H}_n^k$-measure contains an F_σ set of equal $\mathcal{H}_n^k$ measure. Every analytic set of finite $\mathcal{I}_n^k$ measure contains an F_σ set of equal $\mathcal{I}_n^k$ measure.

If a function g on E_n to $\{t \mid -\infty \leqq t \leqq \infty\}$ is analytically measurable, which means that the g counterimage of every open subset of the extended real number system is an analytic subset of E_n, then

$$\int_{E_n} g(x)\,d\mathcal{I}_n^k x = \beta(n,\,k)^{-1} \int_{G_n} \int_{E_k} \sum_{(p_n^k \circ R)(x) = y} g(x)\,d\mathcal{L}_k y\,d\phi_n R.$$

In view of the obvious additivity and convergence closure properties of the set of all functions g for which this formula holds it is sufficient to observe that the formula is trivially true in the special case in which g is the characteristic function of an analytic subset of E_n.

3. The structure of sets whose Hausdorff measure is finite.

A function f on E_k to E_n is said to be *Lipschitzian* if and only if there is a number M such that

$$|f(x) - f(x')| \leqq M \cdot |x - x'| \qquad \text{for } x \in E_k \text{ and } x' \in E_k.$$

According to Rademacher, such a function f has at $\mathcal{L}_k$ almost every point x of E_k a total (Fréchet) differential which is represented by a matrix of k columns and n rows of partial derivatives; the square root of the sum of the squares of the determinants of the k-rowed minors of this matrix is the Jacobian $Jf(x)$. The connection between these analytic concepts and the measure theory of the preceding section is established by the formula

$$\int_X Jf(x)\,d\mathcal{L}_k x = \int_{E_n} N(f,\,X,\,y)\,d\mathcal{H}_n^k y$$
$$= \int_{E_n} N(f,\,X,\,y)\,d\mathcal{I}_n^k y,$$

which is valid for each Lipschitzian function on E_k to E_n and each $\mathcal{L}_k$ measurable set X (see [F 1, 3]). The two integrals on the right are equal to the countable sums

$$\sum_{i=1}^{\infty} i \cdot \mathcal{H}_n^k(\{y \mid N(f,\,X,\,y) = i\}) + \infty \cdot \mathcal{H}_n^k(\{y \mid N(f,\,X,\,y) = \infty\})$$

and

$$\sum_{i=1}^{\infty} i \cdot \mathcal{I}_n^k(\{y \mid N(f,\,X,\,y) = i\}) + \infty \cdot \mathcal{I}_n^k(\{y \mid N(f,\,X,\,y) = \infty\}).$$

We follow the usual convention of measure theory that $\infty \cdot 0 = 0$. In particular, if f maps X univalently onto Y, then

$$\int_X Jf(x)\,d\mathcal{L}_k x = \mathcal{H}_n^k(Y) = \mathcal{J}_n^k(Y).$$

In this connection it is useful to know that every $\mathcal{L}_k$ measurable set X contains an $\mathcal{L}_k$ measurable set X' such that f is univalent on X' and such that the sets X and X' have the same f image ([MF]).

This leads us to consider those sets which are almost representable as Lipschitzian images of subsets of E_k. More precisely we say that a subset A of E_n is *Hausdorff k rectifiable* if and only if there is a Lipschitzian function f on E_k to E_n such that

$$\mathcal{H}_n^k(A - \text{range}\, f) = 0.$$

Every subset of a Hausdorff k rectifiable set is Hausdorff k rectifiable. A countable union of Hausdorff k rectifiable sets is Hausdorff k rectifiable.

Before entering into a deeper discussion of Hausdorff k rectifiable sets we must introduce some of the concepts which help to describe the local behavior of sets of finite Hausdorff measure. For $a \in E_n$ and $r > 0$ we let

$$K(a, r) = \{x \mid |x - a| < r\}$$

be the open sphere with center a and radius r. If A is a regularly embedded k-dimensional manifold through a, then $\mathcal{H}_n^k[A \cap K(a, r)]$ is asymptotically equal to $\alpha(k) \cdot r^k$ as r approaches zero. This suggests the definition, for arbitrary $A \subset E_n$ and $a \in E_n$, of the *upper and lower Hausdorff k densities*

$$\overline{\mathcal{O}}_n^k(A, a) = \limsup_{r \to 0+} \frac{\mathcal{H}_n^k[A \cap K(a, r)]}{\alpha(k) r^k},$$

$$\underline{\mathcal{O}}_n^k(A, a) = \liminf_{r \to 0+} \frac{\mathcal{H}_n^k[A \cap K(a, r)]}{\alpha(k) r^k}.$$

If A is any set for which $\mathcal{H}_n^k(A) < \infty$, then

$$2^{-k} \leq \overline{\mathcal{O}}_n^k(A, a) \leq 1 \text{ for } \mathcal{H}_n^k \text{ almost all } a \text{ in } A.$$

If, in addition, the set A is $\mathcal{H}_n^k$ measurable, then

$$\overline{\mathcal{O}}_n^k(A, a) = 0 \text{ for } \mathcal{H}_n^k \text{ almost all } a \text{ in } E_n - A.$$

There exist $\mathcal{H}^k_n$ measurable sets A for which $0<\mathcal{H}^k_n(A)<\infty$ and $\underline{\odot}^k_n(A, a)=0$ for all $a\in E_n$.

Turning now to the subject of approximate tangent planes, we first observe that for each $R\in G_n$ the orthogonal projection $(p^k_n \circ R)$ maps the k-plane

$$\{x \mid |x| = |(p^k_n \circ R)(x)|\}$$

isometrically onto E_k. The k-plane parallel to the above and through the point $a\in E_n$ is

$$\square^k_n(R, a) = \{x \mid |x-a| = |(p^k_n \circ R)(x-a)|\}.$$

By saying that the k-plane $\square^k_n(R, a)$ is an approximate tangent plane of the set $A\subset E_n$ at the point a we mean roughly that most of those points of A which are close to a—say in $K(a, r)$—lie in a suitably preassigned neighborhood of $\square^k_n(R, a)-\{a\}$. For the latter we choose the sets

$$\diamondsuit^k_n(R, \epsilon, a) = \{x \mid |x-a| < (1+\epsilon^2)^{1/2} |(p^k_n \circ R)(x-a)|\}$$

corresponding to $\epsilon>0$. A point x of E_n is in $\diamondsuit^k_n(R, \epsilon, a)$ if and only if the distance from x to the $(n-k)$-plane through a and perpendicular to $\square^k_n(R, a)$ is less than ϵ times the distance from x to $\square^k_n(R, a)$. Requiring of course that A should appreciably enter into arbitrarily small neighborhoods of a, we phrase the precise definition as follows: The set $A\subset E_n$ has at the point $a\in E_n$ the *Hausdorff k (approximate) tangent plane* $\square^k_n(R, a)$ if and only if

$$\bar{\odot}^k_n(A, a) > 0$$

and

$$\bar{\odot}^k_n[A - \diamondsuit^k_n(R, \epsilon, a), a] = 0 \text{ for every } \epsilon > 0.$$

Under these circumstances we shall refer to $(p^k_n \circ R)$ as a *tangential projection of A at a*. It is clear that R is not uniquely determined by $\square^k_n(R, a)$ and serves only as a convenient parameter, nor is the tangential projection uniquely determined by $\square^k_n(R, a)$. However the k-plane $\square^k_n(R, a)$ is usually unique in cases of interest to us, for if $\mathcal{H}^k_n(A)<\infty$, then A has at $\mathcal{H}^k_n$ almost all points of E_n *at most one* Hausdorff k tangent plane, though it need not have any.

Fixing $\epsilon>0$, we observe that

$$\bar{\odot}^k_n[A - \diamondsuit^k_n(R, \epsilon, a), a] = \limsup_{r\to 0+} \frac{\mathcal{H}^k_n[A \cap K(a, r) - \diamondsuit^k_n(R, \epsilon, a)]}{\alpha(k) r^k}.$$

If A is a regularly embedded k-dimensional manifold through a whose tangent plane at a in the classical sense is $\square_n^k(R, a)$, then the set whose Hausdorff measure occurs in the numerator is vacuous whenever r is sufficiently small. The notion of an approximate tangent plane is therefore an asymptotic generalization of the corresponding classical concept.

We have now filled in the background necessary for the understanding of the following basic theorem describing the structure of sets whose Hausdorff measure is finite.

Suppose A is an $\mathcal{H}_n^k$ measurable set for which

$$\mathcal{H}_n^k(A) < \infty,$$

B is the set of all those points of A at which A has a Hausdorff k tangent plane, and

$$C = A - B.$$

Then:

(1) *B and C are $\mathcal{H}_n^k$ measurable sets.*

(2) *B is Hausdorff k rectifiable.*

(3) *C contains no Hausdorff k rectifiable set of positive $\mathcal{H}_n^k$ measure.*

(4) $\mathcal{I}_n^k(B) = \mathcal{H}_n^k(B)$.

(5) $\mathcal{I}_n^k(C) = 0$; *for ϕ_n almost all R in G_n the $(p_n^k \circ R)$ image of C has $\mathcal{L}_k$ measure zero.*

(6) $\mathcal{H}_n^k(A) = \mathcal{H}_n^k(B) + \mathcal{H}_n^k(C) = \mathcal{I}_n^k(B) + \mathcal{H}_n^k(C) = \mathcal{I}_n^k(A) + \mathcal{H}_n^k(C)$.

(7) $\mathcal{I}_n^k(A) \leq \mathcal{H}_n^k(A)$.

(8) *The following three conditions are equivalent:*
  (i) $\mathcal{I}_n^k(A) = \mathcal{H}_n^k(A)$.
  (ii) *A is Hausdorff k rectifiable.*
  (iii) *A has Hausdorff k tangent planes at $\mathcal{H}_n^k$ almost all of its points.*

(9) *The following three conditions are equivalent:*
  (i) $\mathcal{I}_n^k(A) = 0$.
  (ii) *A contains no Hausdorff k rectifiable set of positive $\mathcal{H}_n^k$ measure.*
  (iii) *A has Hausdorff k tangent planes at $\mathcal{H}_n^k$ almost none of its points.*

For the special case in which $k = 1$ and $n = 2$ these results are substantially due to A. S. Besicovitch, A. P. Morse, and J. F. Randolph ([B 1], [MR]). The extension to general dimensions was made by the writer ([F 4]).

Intimately related to this structure theory, there has been a great deal of research into the subject of Hausdorff densities. It is easy to prove that

$$\overline{\Theta}_n^k(A, a) = \underline{\Theta}_n^k(A, a) = 1 \text{ for } \mathcal{H}_n^k \text{ almost all } a \text{ in } B,$$

but the behavior of the densities on C presents a difficult problem. For the special case in which $k=1$ it is known that

$$\overset{1}{\mathfrak{S}}_n(A, a) \geqq 1.01\,\overset{1}{\mathfrak{Q}}_n(A, a) \text{ for } \overset{1}{\mathcal{H}}_n \text{ almost all } a \text{ in } C.$$

This intriguing estimate was obtained by A. P. Morse and J. F. Randolph for $n=2$ and extended by E. F. Moore to all positive integers n (See [MR], [MEF 2]).

4. **The Gauss-Green Theorem.** The classical formula bearing this name asserts that the integral of a partial derivative of a function over an open subset of E_n equals the integral of the function itself, multiplied by a component of the exterior normal, over the boundary of the open set. Giving precise measure-theoretic interpretations to the intuitive concepts which enter into this statement, we are able to give it a completely natural and general form.

Suppose U is a bounded open subset of E_n, A is the boundary of U, and $\mathcal{H}_n^{n-1}(A) < \infty$.

We shall integrate with respect to $\mathcal{L}_n$ over U and with respect to $\mathcal{H}_n^{n-1}$ over A. Our definition of the term "exterior normal" is as follows: Suppose $a \in A$ and w is a unit vector, that is, $w \in E_n$ and $|w| = 1$. For each $r > 0$ we consider the solid hemispheres

$$s(r) = K(a, r) \cap \{x \mid (x-a) \bullet w \geqq 0\},$$
$$t(r) = K(a, r) \cap \{x \mid (x-a) \bullet w \leqq 0\}$$

into which the sphere $K(a, r)$ is divided by the $(n-1)$-plane through a and perpendicular to w. The vector w points from a into $s(r)$, and $-w$ points into $t(r)$. By saying that *w is an exterior normal of U at a* we mean roughly that for small r the hemisphere $t(r)$ is filled mostly with points of U while $s(r)$ contains only few points of U; precisely we mean that

$$\lim_{r \to 0+} \frac{\mathcal{L}_n[t(r) \cap U]}{\mathcal{L}_n[t(r)]} = 1$$

and

$$\lim_{r \to 0+} \frac{\mathcal{L}_n[s(r) \cap U]}{\mathcal{L}_n[s(r)]} = 0.$$

The set U has at each point a *at most one* exterior normal, but it may have none. We define

$$\nu(U, a)$$

as the exterior normal of U at a whenever that exists, and as the zero vector otherwise. We also let $\nu_j(U, a)$ be the jth component of the vector $\nu(U, a)$ for $j=1, 2, \cdots, n$.

From our discussion of the geometric properties of the domains of integration we now turn to the assumptions about the function which is to occur in our formula.

Suppose f is a numerically-valued function, continuous on $U\cup A$, and absolutely continuous in the sense of Tonelli on U.

The last part of this hypothesis means that for $j=1, 2, \cdots, n$ the partial derivative

$$D_j f(x)$$

of f in the direction of the jth base vector exists for $\mathcal{L}_n$ almost all x; that

$$\int_U | D_j f(x) | \, d\mathcal{L}_n x < \infty ;$$

and that for $\mathcal{L}_{n-1}$ almost all y in E_{n-1} the function

$$f(y_1, \cdots, y_{j-1}, z, y_{j+1}, \cdots, y_{n-1})$$

of one variable z is absolutely continuous in the classical sense on every interval I for which

$$\{ (y_1, \cdots, y_{j-1}, z, y_{j+1}, \cdots, y_{n-1}) \mid z \in I \} \subset U.$$

We observe that for all this to hold it would be sufficient, but not necessary, to assume that the first partial derivatives of f are continuous and bounded on U. Under the foregoing assumptions the writer has proved (in [F 2, 3]) the *Gauss-Green formula*:

$$\int_U D_j f(x) d\mathcal{L}_n x = \int_A f(a) \nu_j(U, a) d \, \mathcal{H}_n^{n-1} a \quad \text{for } j = 1, 2, \cdots, n.$$

Inasmuch as all earlier definitions of exterior normals utilized tangent planes of the boundary set, it is interesting to note that at $\mathcal{H}_n^{n-1}$ almost all of those points a of A at which U has an exterior normal the $(n-1)$-plane

$$\{ x \mid (x - a) \bullet \nu(A, a) = 0 \}$$

is a Hausdorff $n-1$ tangent plane of A at a. The converse is false, as may be seen from the simple example in which $n=2$ and U is an open disc minus a radius; at each point of this radius the set A has a Hausdorff 1 tangent plane, but U has no exterior normal.

The set of those points of A at which U has no exterior normal contributes nothing to the integral on the right, because $\nu_j(U, a)=0$ for all such points a. However this point set and, worse than that, the set of those points of A at which A has no Hausdorff $n-1$ tangent plane may very well have positive $\mathcal{H}_n^{n-1}$ measure. This situation may occur even in the relatively simple case in which $n=3$ and A is homeomorphic with a two sphere. Significant in this connection is the fact that, while every connected set of finite $\mathcal{H}_n^1$ measure is Hausdorff 1 rectifiable, the intrinsic topological properties of a set of finite $\mathcal{H}_n^k$ measure have no bearing whatever on its rectifiability in case $k>1$.

To our version of the Gauss-Green formula corresponds the following form of the *Cauchy Theorem*:

If U is a bounded set of complex numbers, A is the boundary of U, $\mathcal{H}_2^1(A)<\infty$, and if the complex-valued function f is continuous on $U\cup A$ and analytic on U, then

$$\int_A f(a)\nu(U, a)\,d\mathcal{H}_2^1 a = 0.$$

This reduces to the classical formula in case A is a simple closed rectifiable curve in the plane.

5. Multiplicity functions and areas. *Assuming that X is a locally compact, locally connected, separable subset of a triangulable k-dimensional manifold* we let

$$C_n(X)$$

be the set of all continuous functions on X to E_n. We wish to associate with each mapping $f\in C_n(X)$ a non-negative number (possibly ∞) indicating the k-dimensional extent of the point set occupied by the values of f as well as the multiply overlapping manner in which the various parts of this point set may be covered by f. If such an association has invariance properties consistent with the Euclidean geometry of E_n, and if it relates to relatively simple (differentiable) mappings the same numbers as do the classical integral formulae, then we shall refer to it as a *k-dimensional area on $C_n(X)$*. No precise definition of this general concept of area, which would complement Carathéodory's axiomatization of the notion of measure, has yet been devised. Up to now all the work in this field has been directed toward the study and comparison of certain particular functions which belong to the still vague category of all areas. In the remainder of the present paper we shall continue these special investigations, but we shall point out some desirable general properties of areas as they occur in our work.

Most areas are defined as integrals of multiplicity functions by the following scheme: Suppose μ is such a function that, for $f \in C_n(X)$ and $y \in E_n$,

$$\mu(f, X, y)$$

is either a non-negative integer or ∞ and gives a sensible appraisal of the multiplicity with which the mapping f assumes the value y. Suppose further that ψ is a measure over E_n and is reasonably indicative of k-dimensional Euclidean extent. Then the function which associates

$$\int_{E_n} \mu(f, X, y) d\psi y$$

with $f \in C_n(X)$ is the *area on $C_n(X)$ corresponding to the multiplicity function μ and the measure ψ.*

In §2 we have already described the elementary multiplicity function N which simply counts the points x in X for which $f(x) = y$. Used in conjunction with the measure $\mathcal{H}_n^k$ it yields the *k-dimensional Hausdorff N area*

$$\int_{E_n} N(f, X, y) d\mathcal{H}_n^k y,$$

and with $\mathcal{I}_n^k$ the *k-dimensional integralgeometric N area*

$$\int_{E_n} N(f, X, y) d\mathcal{I}_n^k y,$$

of the mapping $f \in C_n(X)$. According to §3 these areas agree with the classical integral formula in case X is an $\mathcal{L}_k$ measurable subset of E_k and f is a Lipschitzian mapping. For any X and any $f \in C_n(X)$ our structure theorem concerning sets of finite Hausdorff measure implies that

$$\int_{E_n} N(f, X, y) d\mathcal{H}_n^k y \geqq \int_{E_n} N(f, X, y) d\mathcal{I}_n^k y$$

and that, in case the k-dimensional Hausdorff area of f is finite, equality holds if and only if the range of f is a Hausdorff k rectifiable set. A comprehensive list of such inequalities, containing all known results of this type except for those derived in the following sections of this paper, may be found in the writer's recent note [F 6].

Assuming that ψ is such a measure over E_n that all closed subsets of E_n are ψ measurable, the ψ N area of any $f \in C_n(X)$ may (according to [F 1]) be computed as follows: Let a *partition P of X* be such a countable disjointed family of analytic subsets of X that

$$\bigcup_{W \in P} W = X$$

and let

$$\sigma(\psi, f, P) = \sum_{W \in P} \psi[f(W).]$$

Then the ψN area

$$\int_{E_n} N(f, X, y) d\psi y$$

of f equals the supremum of $\sigma(\psi, f, P)$ for all partitions P of X; it also equals

$$\lim_{i \to \infty} \sigma(\psi, f, P_i)$$

whenever $P_1, P_2, P_3, \cdots$ are such partitions of X that

$$\lim_{i \to \infty} \sup_{W \in P_i} \text{diameter } W = 0.$$

These alternate methods for the computation of the N area embody the general measure-theoretic principle underlying the classical theorem which states that the length of a continuous curve, defined as the supremum of the lengths of all inscribed polygons, equals the limit of the lengths of the terms of any sequence of inscribed polygons with vertices corresponding to finer and finer subdivisions of the interval on which the curve is parametrically represented. In this case ψ is $\mathcal{H}_n^1$ or $\mathcal{J}_n^1$, X is a closed interval, and only finite partitions of X into subintervals need be considered. The classical length of any curve $f \in C_n(X)$ is found to be equal to the one-dimensional Hausdorff N area of f and also to the one-dimensional integral-geometric N area of f.

Among the elementary properties of the multiplicity function N the following three appear to have some general significance:

(i) *Localizability of N. If $X' \subset X$ and*

$$X \cap \{x \mid f(x) = y\} \subset X',$$

then

$$N(f, X, y) = N(f, X', y).$$

(ii) *Additivity of N. If P is such a disjointed family of subsets of X that*

$$\bigcup_{W \in P} W = X,$$

then

$$N(f, X, y) = \sum_{W \in P} N(f, W, y).$$

(iii) *Superposition formula for N. If*

$$Y_i = \{y \mid N(f, X, y) \geqq i\} \qquad \text{for } i = 1, 2, 3, \cdots,$$

then

$$N(g \circ f, X, z) = \sum_{i=1}^{\infty} N(g, Y_i, z).$$

Analogous, somewhat weaker properties are shared by all those functions to which we vaguely refer in this paper as "multiplicity functions," and will certainly play a role in any future precise definition of that term.

The superposition formula for N may be used to show that the k-dimensional integralgeometric N area of $f \in C_n(X)$ equals

$$\beta(n, k)^{-1} \cdot \int_{G_n} \int_{E_k} N(p_n^k \circ R \circ f, X, z) d\mathcal{L}_k z \, d\phi_n R,$$

a kind of average over G_n of the areas of the orthogonal projections $p_n^k \circ R \circ f$ of f. In case X is k-dimensional and $k < n$ these projections, which map X into E_k, are much easier to study than the mapping f of X into E_n. In this respect the integralgeometric area is simpler than the Hausdorff area, which cannot be represented as an average of this type.

There are several natural multiplicity functions other than N. Among these the function S is connected with the concept of stability. We recall that y is said to be a stable value of f if and only if y is in the range of every mapping sufficiently close to f. The underlying topology on $C_n(X)$ is based on the neighborhoods

$$C_n(X) \cap \{g \mid |g(x) - f(x)| < \epsilon \text{ for } x \in X\}$$

of f corresponding to $\epsilon > 0$. Convergence in this topology is uniform on X. Introducing (as in [R 3]) the *stable multiplicity*

$$S(f, X, y) = \liminf_{g\to f} N(g, X, y),$$

we note that y is a stable value of f if and only if $S(f, X, y) > 0$. We also observe that

$$\limsup_{g\to f} N(g, X, y) = \infty$$

whenever X is dense in itself and y is in the closure of the range of f. Since X is at most k-dimensional, $S(f, X, y) = 0$ in case $k < n$. Therefore the function S is directly useful only in case $k = n$, where it gives rise to the $\mathcal{L}_k S$ area on $C_k(X)$. However the alternate formula for the integralgeometric N area, which was discussed in the preceding paragraph, suggests the indirect definition of the ***k*-dimensional integralgeometric stable area** on $C_n(X)$ associating

$$\beta(n, k)^{-1} \int_{G_n} \int_{E_k} S(p_n^k \circ R \circ f, X, z)\, d\mathcal{L}_k z\, d\phi_n R$$

with $f \in C_n(X)$ for $k \leq n$. In case X is a subset of E_k whose boundary has $\mathcal{L}_k$ measure zero and f is a Lipschitzian mapping this stable area of f agrees (according to [F 3]) with the classical Jacobian integral formula, because each projection $p_n^k \circ R \circ f$ is a Lipschitzian mapping in $C_k(X)$ and because

$$S(h, X, z) = N(h, X, z) \text{ for } \mathcal{L}_k \text{ almost all } z \text{ in } E_k$$

whenever h is a Lipschitzian mapping in $C_k(X)$. Furthermore we have

$$S(h, X, z) \leq N(h, X, z) \text{ for } h \in C_k(X) \text{ and } z \in E_k,$$

whence we conclude that the k-dimensional integralgeometric stable area of every $f \in C_n(X)$ is less than or equal to the k-dimensional integralgeometric N area of f.

From the point of view of approximation theory the function N, which accounts only for the relatively accidental behavior of an individual mapping, is less appropriate than the stable multiplicity function S. Indeed $S(f, X, y)$ is computable by the Čech cohomology theory which approximates X and f by complexes and simplicial mappings [F 7].

For $f \in C_k(X)$, $y \in E_k$, and $r > 0$ the components of the open set

$$X \cap \{x \mid |f(x) - y| < r\}$$

are connected open sets. We shall refer to them as *canonical regions of* (f, X, y, r). Furthermore we shall call F a *canonical family of* (f, X, y, ϵ) if and only if each element of F is a canonical region of

(f, X, y, r) for some $r \leq \epsilon$, F is disjointed, F covers the closed set

$$X \cap \{x \mid f(x) = y\},$$

and every point of X has a neighborhood which meets only finitely many elements of F.

If $f \in C_k(X)$, $y \in E_k$, *and* $\epsilon > 0$, *then there exists a canonical family* F *of* (f, X, y, ϵ).

To prove this we let

$$\begin{aligned} A &= \{x \mid f(x) = y\}, \\ G(r) &= \text{the set of all canonical regions of } (f, X, y, r) \end{aligned}$$

whenever $r > 0$, choose compact subsets

$$0 = B_0 \subset B_1 \subset B_2 \subset \cdots$$

of X for which

$$X = \bigcup_{i=1}^{\infty} \text{Interior } B_i,$$

and inductively define the sequences

$$r_0, r_1, r_2, \cdots ;\ H_0, H_1, H_2, \cdots ;\ U_0, U_1, U_2, \cdots$$

in such a way that

$$r_0 = \epsilon, \qquad H_0 = 0, \qquad U_0 = 0$$

and

$$\begin{aligned} r_i &= \inf\left(\{r_{i-1}\} \cup \{\, |f(x) - y| \mid x \in B_{i-1} - U_{i-1}\}\right), \\ H_i &= G(r_i) \cap \{V \mid V \cap A \cap B_i \not\subset U_{i-1}\}, \\ U_i &= U_{i-1} \cup \bigcup_{V \in H_i} V \end{aligned}$$

for $i = 1, 2, 3, \cdots$. We shall prove that

$$F = \bigcup_{i=1}^{\infty} H_i$$

is a canonical family of (f, X, y, ϵ).

First we verify by induction that

$$0 < r_i \leq \epsilon \text{ and } (A \cap B_i) \subset U_i$$

for every non-negative integer i. The statement holds if $i = 0$. If it holds for $i = j - 1$, then

$$(A \cap B_{j-1}) \subset U_{j-1}, \qquad (B_{j-1} - U_{j-1}) \cap A = 0,$$

$$|f(x) - y| > 0 \text{ for } x \in B_{j-1} - U_{j-1},$$

$$B_{j-1} - U_{j-1} \text{ is compact, } 0 < r_j \leqq r_{j-1},$$

$$A \subset \{x \mid |f(x) - y| < r_j\} \subset \bigcup_{V \in G(r_j)} V,$$

$$(A \cap B_j) - U_{j-1} \subset \bigcup_{V \in H_j} V, \qquad (A \cap B_j) \subset U_j,$$

whence the statement holds for $i = j$.

We infer that each element of F is a canonical region of (f, X, y, r_i) for some $r_i \leqq \epsilon$, and that F covers A.

Since $G(r)$ is disjointed for $r > 0$, H_i is disjointed for $i = 1, 2, 3, \cdots$. If F were not disjointed, there would exist $i < j$, $V \in H_i$, $W \in H_j$ with $V \cap W \neq 0$; hence $r_j \leqq r_i$, $V \in G(r_i)$, $W \in G(r_j)$, $W \subset V \subset U_i \subset U_{j-1}$, $W \notin H_j$. It follows that F is disjointed.

For $i = 1, 2, 3, \cdots$ the set H_i is a disjointed family of open sets each of which meets the compact set B_i; hence H_i is finite. In view of this and of the fact that every point of X is interior to some set B_j, we may complete the proof by checking that

$$F \cap \{V \mid V \cap B_j \neq 0\} \subset \bigcup_{i=1}^{j} H_i \text{ for } j = 1, 2, 3, \cdots.$$

To do this suppose $i < j$, $V \in H_j$, $x \in V \cap B_i$. Then

$$x \in V \in G(r_j), \qquad |f(x) - y| < r_j,$$

$$V \cap U_i = 0, \qquad x \in B_i - U_i, \qquad |f(x) - y| \geqq r_{i+1},$$

whence $r_{i+1} < r_j$, contrary to the fact that $i + 1 \leqq j$ and $r_{i+1} \geqq r_j$.

The proof is complete.

A canonical region V of (f, X, y, r) is said to be ***inessential*** if and only if there exists a mapping

$$v \in C_k(\text{Closure } V)$$

such that

$$v(x) = f(x) \text{ for } x \in \text{Boundary } V,$$

$$|v(x) - y| \neq r \text{ for } x \in V.$$

Otherwise V is ***essential***; in this case there exists a mapping

$$v \in C_k(\text{Closure } V)$$

such that

$$v(x) = f(x) \text{ for } x \in \text{Boundary } V,$$

$$\begin{aligned} | v(x) - y | &\leqq r \text{ for } x \in V, \\ N(v, V, y) &= 1. \end{aligned}$$

Corresponding to each canonical family F of (f, X, y, ϵ) we construct a function g by choosing for each region $V \in F$ a mapping $v \in C_k$ (Closure V) as above, letting $g(x) = v(x)$ for $x \in V$, and letting

$$g(x) = f(x) \text{ for } x \in X - \bigcup_{V \in F} V.$$

The locally finite character of F assures us that $g \in C_k(X)$. Furthermore

$$\begin{aligned} | g(x) - f(x) | &\leqq 2\epsilon \text{ for } x \in X, \\ N(g, X, y) &= \text{the number of essential regions in } F. \end{aligned}$$

With this machinery at hand we proceed to the proof of the following proposition:

If $f \in C_k(X)$, $y \in E_k$, *and*

$$s(r) = \textit{the number of essential canonical regions of } (f, X, y, r)$$

whenever $r > 0$, *then*

$$\lim_{r \to 0+} s(r) = S(f, X, y).$$

The above limit exists because $r > r' > 0$ implies that every essential canonical region of (f, X, y, r) contains an essential canonical region of (f, X, y, r'), whence $s(r) \leqq s(r')$.

If $g \in C_k(X)$, $r > 0$, and

$$| g(x) - f(x) | < r \text{ for } x \in X,$$

then $N(g, V, y) \geqq 1$ for every essential canonical region V of (f, X, y, r), because f and g define homotopic mappings of Boundary V into $E_k - \{y\}$; it follows that $N(g, X, r) \geqq s(r)$. Consequently

$$S(f, X, y) \geqq \lim_{r \to 0+} s(r).$$

In proving the opposite inequality we may assume that there exist a positive number ρ and an integer i such that

$$s(r) = i \text{ for } 0 < r \leqq \rho.$$

Suppose $\epsilon > 0$ and let F be a canonical family of (f, X, y, ϵ). If $V_1, V_2, \cdots, V_j$ are essential regions in F, let $r_1, r_2, \cdots, r_j$ and r be such numbers that

$$0 < r \leqq \rho, r \leqq r_t,$$

and

$$V_t \text{ is a canonical region of } (f, X, y, r_t) \text{ for } t = 1, 2, \cdots, j.$$

Then each V_t contains an essential canonical region of (f, X, y, r), whence $j \leqq i$. Constructing g corresponding to F as previously indicated, we find that $g \in C_k(X)$,

$$| g(x) - f(x) | \leqq 2\epsilon \text{ for } x \in X,$$
$$N(g, X, y) \leqq i.$$

It follows that $S(f, X, y) \leqq i$.

The proposition whose proof we have just completed reduces the computation of the stable multiplicity to the counting of essential canonical regions. The Hopf extension theorem [D] of combinatorial topology provides an apt criterion to decide whether a canonical region V of (f, X, y, r) is essential. Since f maps Closure V into E_k and Boundary V into $E_k \cap \{z \mid |z-y| \geqq r\}$, f induces a homomorphism

$$f^*: H^k(E_k, E_k \cap \{z \mid | z - y | \geqq r\}) \to H^k(\text{Closure } V, \text{Boundary } V)$$

of the k-dimensional Čech cohomology groups with integer coefficients and based on locally finite coverings. A necessary and sufficient condition for V to be inessential is that f^* be a trivial homomorphism (whose range has only one element).

The preceding condition is certainly satisfied whenever H^k (Closure V, Boundary V) is a trivial group (with only one element). This is actually the case unless V is a k-dimensional manifold and Closure V is compact. Consequently:

If $f \in C_k(X)$ and $r > 0$, then every essential canonical region of (f, X, y, r) is a k-dimensional manifold whose closure is compact.

If $f \in C_k(X)$, then $S(f, X, y)$ is the supremum of the set of all integers i such that there are nonvacuous k-dimensional manifolds $V_1, V_2, \cdots, V_i$ whose closures are disjoint compact subsets of X and for which y is a stable value of each of the mappings $(f| V_1), (f| V_2), \cdots, (f| V_i)$.

From this alternate characterization of the multiplicity function S we infer that though S was originally defined in terms of the strong "uniform" topology on $C_k(X)$, it may equally well be defined in terms of the weaker "compact-open" topology on $C_k(X)$ based on the neighborhoods

$$C_k(X) \cap \{g \mid | g(x) - f(x) | < \epsilon \text{ for } x \in A\}$$

of f corresponding to each $\epsilon > 0$ and each compact subset A of X.

Convergence in this topology is uniform on every compact subset of X.

Whenever $f\in C_k(X)$ and X' is a locally compact, locally connected subset of X, then $(f|X')\in C_k(X')$ and we agree that

$$S(f, X', y) = S(f|X', X', y) \qquad \text{for } y \in E_k.$$

We are now ready to state five basic properties of S:

(i) *Localizability of S: If $f\in C_k(X)$, X' is a locally compact, locally connected subset of X, $y\in E_k$, and*

$$X \cap \{x \mid f(x) = y\} \subset \text{Interior } X',$$

then

$$S(f, X, y) = S(f, X', y).$$

(ii) *Additivity of S: If $f\in C_k(X)$, $y\in E_k$, and P is such a disjointed family of open subsets of X that*

$$\bigcup_{W\in P} W = X,$$

then

$$S(f, X, y) = \sum_{W\in P} S(f, W, y).$$

(iii) *Superposition formula for S: If $f\in C_k(X)$, Y is a locally compact, locally connected subset of E_k,* range $f\subset Y$, *$g\in C_k(Y)$, $z\in E_k$, and*

$$Y_i = \{y \mid S(f, X, y) \geq i\} \quad \textit{for } i = 1, 2, 3, \cdots,$$

then

$$S(g\circ f, X, y) \leq \sum_{i=1}^{\infty} S(g, Y_i, z).$$

Sufficient for equality are the conditions that X be embedded in an orientable k-dimensional manifold and that the set

$$\{y \mid g(y) = z\}$$

be totally disconnected.

(iv) *Compact origin of S: If $f\in C_k(X)$, $y\in E_k$, and F is the set of all compact, locally connected subsets of X, then*

$$S(f, X, y) = \sup_{X'\in F} S(f, X', y).$$

(v) *Lower semi-continuity of S: $S(f, X, y)$ is lower semi-continuous with respect to (f, y) on the cartesian product $C_k(X)\times E_k$ with the "com-*

pact-open" topology on $C_k(X)$ and the usual topology on E_k.

The sets Y_t in (iii) are open by virtue of (v). The strict inequality may hold in the conclusion of (iii) even if $k=1$ and X and Y are closed intervals.

Combining (v) with Fatou's lemma which states that the lower limit of the integrals of non-negative measurable functions is greater than or equal to the integral of the lower limit of these functions, we find that *the k-dimensional integralgeometric stable area on $C_n(X)$ is lower semi-continuous with respect to the "compact-open" topology on $C_n(X)$.*

Now suppose that the space X admits finite triangulations. A mapping $g \in C_n(X)$ is *polyhedral* if and only if X has such a finite triangulation T that g maps each simplex of T baricentrically onto some rectilinear simplex of E_n. The k-dimensional area of g, defined according to any one of the methods which we have described, equals

$$\sum_{s \in T_k} \mathcal{H}_n^k[g(s)],$$

where T_k is the set of all k-dimensional simplices of T. The Hausdorff measures occurring in this finite sum are computable in elementary fashion by determinants.

The set P of all polyhedral mappings in $C_n(X)$ is dense in $C_n(X)$. For any mapping $f \in C_n(X)$ we have

$$S(p_n^k \circ R \circ f, X, z) = \liminf_{g \to f,\, g \in P} N(p_n^k \circ R \circ g, X, z)$$

whenever $R \in G_n$ and $z \in E_k$; hence the k-dimensional integralgeometric stable area of f equals

$$\beta(n, k)^{-1} \int_{G_n} \int_{E_k} \liminf_{g \to f,\, g \in P} N(p_n^k \circ R \circ g, X, z)\, d\mathcal{L}_k z\, d\phi_n R.$$

True to the tradition of analysis we wonder what happens if we put the "lim inf" ahead of both integral signs. In this way we get the function L_k on $C_n(X)$ such that

$$L_k(f) = \liminf_{g \to f,\, g \in P} (\text{the } k\text{-dimensional area of } g)$$

for any $f \in C_n(X)$. *The function L_k is the k-dimensional Lebesgue area on $C_n(X)$.* We note that, if X is k-dimensional, the corresponding upper limit is infinite for all $f \in C_n(X)$.

Another way of thinking of the Lebesgue area is the following: The k-dimensional area is lower semi-continuous on the dense subset P

of $C_n(X)$, because it may be obtained by confining the lower semi-continuous k-dimensional integralgeometric stable area on $C_n(X)$ to P. Hence the k-dimensional area on P can be extended in infinitely many ways to a lower semi-continuous function on $C_n(X)$. The numerically largest such extension is the k-dimensional Lebesgue area on $C_n(X)$.

For any mapping $f \in C_n(X)$ the proposition

$$L_k(f) < \infty$$

has far-reaching analytic and geometric consequences. It implies that f can be uniformly approximated by polyhedral mappings whose areas tend to the Lebesgue area of f and that many properties, such as being quasiconformal, which may be possessed by these polyhedral mappings are then inherited by f. For these reasons the Lebesgue area has been used successfully ([R 1], [DJ], [MCS]) in solving Plateau's problem: to inscribe a surface of least 2-dimensional area in a given simple closed curve in E_3. It is true that the solution is necessarily a saddle surface for which all the definitions of area described here agree with each other, so that the answer to the problem of Plateau is really independent of the particular definition of area used in stating it precisely. However the Lebesgue area is a powerful tool in proving the existence of solutions of this and other problems in the calculus of variations.

Inasmuch as the k-dimensional integralgeometric stable area is less than or equal to $L_k(f)$ for all $f \in C_n(X)$, it is natural to wonder under what conditions on k, n, and X these two areas of f are actually equal for all $f \in C_n(X)$. It is known that if X is a k-dimensional cell or sphere, then the equality holds in case $1 = k \leq n$ and in case $2 = k \leq n = 3$, but fails in case $3 \leq k \leq n$ ([R 5], [F 3, 6]).

We now recall the fundamental principle with which we started our whole discussion of areas: *Most areas can be defined as integrals of multiplicity functions.* Indeed we constructed the Hausdorff N area and the integralgeometric N area according to this scheme, but we varied it somewhat in introducing the integralgeometric stable area, and abandoned it completely in defining the Lebesgue area. This raises the question: *Can the Lebesgue area be represented as an integral of a multiplicity function?* Our present state of knowledge does not allow a complete answer to this question, but several special cases have been studied thoroughly and in each of these cases the answer is affirmative.

For $k = 1$ the Lebesgue area equals the Hausdorff N area.

For $k = n$ the problem was solved in [F 6], though with the unnecessary restriction that the manifold in which X is embedded be orientable, by the discovery of such a new multiplicity function M that the Lebesgue area equals the Hausdorff M area. We shall presently describe M, without the assumption of orientability.

In the later parts of this paper we shall answer the question for the case in which $k=2$, $n=3$, and X is a 2-cell.

Our characterization of the stable multiplicity S in terms of the Čech cohomology theory suggests the following definition of the new function M: Suppose X is a locally compact, locally connected, separable subset of a k-dimensional manifold (not necessarily admitting finite triangulations), $f\in C_k(X)$, and $y\in E_k$. For $r>0$ and each canonical region V of (f, X, y, r) we consider the induced homomorphism

$$f^*: H^k(E_k, E_k\cap\{z \mid |z-y|\geq r\})\to H^k \text{ (Closure } V\text{, Boundary } V)$$

of the k-dimensional Čech cohomology group with integer coefficients and based on locally finite coverings. The first group is infinite cyclic. According to the three possible isomorphism types of the second group we define the non-negative integer

$$D(f, r, V)$$

in three alternate steps:

(i) If V is an *orientable* k-dimensional manifold whose closure is compact, then the second group is infinite cyclic, f^* maps a generator of the first group onto an integral multiple of a generator of the second group, and $D(f, r, V)$ is the absolute value of the multiplier.

(ii) If V is a *nonorientable* k-dimensional manifold whose closure is compact, the second group is cyclic of order two and $D(f, r, V)$ equals 0 or 1 depending on whether f^* is trivial or nontrivial.

(iii) Otherwise the second group is trivial and $D(f, r, V)=0$.

Letting $F(r)$ be the set of all canonical regions of (f, X, y, r) whenever $r>0$, we see that

$$\sum_{V\in F(r)} D(f, r, V)$$

does not decrease with r. The limit, as $r\to 0+$, of this sum is the multiplicity

$$M(f, X, y).$$

Proceeding as in [F 6] we find that the function M is connected with the Lebesgue area by the following theorem:

If X admits finite triangulations and $f\in C_k(X)$, then

$$L_k(f) = \int_{E_k} M(f, X, y) d\mathcal{L}_k y.$$

The triangulation hypothesis can be removed by a suitable natural extension of the concept of Lebesgue area (see §6).

The functions S and M are related by the inequality

$$S(f, X, y) \leq M(f, X, y) \text{ for } f \in C_k(X),\ y \in E_k,$$

which can be strict for a set of points y with positive $\mathcal{L}_k$ measure. However these two multiplicities vanish simultaneously and are simultaneously infinite. Furthermore the five listed basic properties (localizability, additivity, superposition formula, compact origin, lower semi-continuity) of S hold also for M.

Returning to the general case in which $k \leq n$ we conclude this section by discussing two properties of L_k on $C_n(X)$ which will be of later use:

(1) *If X is finitely triangulable, $f \in C_n(X)$, and G is the set of all components of X, then*

$$L_k(f) = \sum_{z \in G} L_k(f \mid Z).$$

(2) *If X is finitely triangulable, $f \in C_n(X)$, and Z is a finitely triangulable subset of X, then*

$$L_k(f) \geq L_k(f \mid Z).$$

The first of these two propositions is obvious, but the second seems worthy of further discussion.

Suppose $\epsilon > 0$ and $g \in C_n(X)$ is a polyhedral mapping, with a corresponding finite triangulation T of X, such that

$$\left| g(x) - f(x) \right| < \epsilon \text{ for } x \in X,$$
$$L_k(g) \leq L_k(f) + \epsilon.$$

Letting ρ be any distance function metrizing the compact space X, we next choose $\delta > 0$ so that

$$\left| g(x) - g(x') \right| < \epsilon \text{ whenever } \rho(x, x') < 2\delta.$$

From T we obtain by successive barycentric subdivisions a finite triangulation T' of X such that the star of each vertex of T' has a diameter less than δ. Then the mapping g is polyhedral with respect to T'. Let P be the set of all k simplices of T' and let

$$Q = P \cap \{s \mid s \subset Z\}.$$

We next choose a sufficiently fine finite triangulation U of Z and a function q on Q to the set of all k simplices of U such that

$$q(s) \subset s \text{ for } s \in Q.$$

Applying the modification procedure of [F 6] to the identity map of Z into X we obtain a map u of Z into X with the following properties:

$$u \text{ maps } q(s) \text{ barycentrally onto } s \text{ for } s \in Q,$$
$$u[s - q(s)] \subset \text{Boundary } s \text{ for } s \in Q,$$
$$u(s \cap Z) \subset \text{Boundary } s \text{ for } s \in P - Q.$$

It follows that

$$\rho[u(x), x] < \delta \text{ for } x \in Z.$$

From U and u we obtain by successive barycentric subdivisions and simplicial approximation a finite triangulation U' of Z and a map u' of Z into X such that:

$$u' \text{ is simplicial with respect to } U' \text{ and } T',$$
$$\rho[u'(x), u(x)] < \delta \text{ for } x \in Z,$$
$$u' \mid q(s) = u \mid q(s) \text{ for } s \in Q,$$
$$u \text{ maps } Z - \bigcup_{s \in Q} q(s) \text{ into the } k - 1 \text{ skeleton of } T'.$$

We conclude that the mapping $(g \circ u') \in C_n(Z)$ is polyhedral with

$$\rho[u'(x), x] < 2\delta, \qquad |(g \circ u')(x) - g(x)| < \epsilon,$$
$$|(g \circ u')x - (f \mid Z)(x)| < 2\epsilon \text{ for } x \in Z$$

and that

$$L_k(g \circ u') = \sum_{s \in Q} \mathcal{H}_n^k[g(s)] \leqq L_k(g) \leqq L_k(f) + \epsilon.$$

This concludes the proof of the Proposition (2).

6. **Integration over the middle space.** The results of the preceding section suggest that among all the open subsets of X the canonical regions associated with a mapping $f \in C_n(X)$ are most important in computing those multiplicities of f which are connected with the Lebesgue area of f. For $x \in X$ and $r > 0$ we accordingly consider the neighborhood

$$\Delta_f(x, r)$$

of x, which is defined as the canonical region U of $(f, X, f(x), r)$

such that $x \in U$.

These neighborhoods do not always form a basis for the topology of X, for it may happen that

$$\Delta_f(x, r) = \Delta_f(x'r) \text{ for all } r > 0$$

even though x and x' are distinct points of X. Therefore we consider the equivalence relation

$$(X \times X) \cap \{(x, x') \mid \Delta_f(x, r) = \Delta_f(x', r) \text{ for all } r > 0\}$$

and the corresponding partition

$$\mathfrak{M}_f$$

of X into equivalence classes, called the ***middle space** of f*. Each $x \in X$ is an element of the equivalence class

$$m_f(x) = \bigcap_{r>0} \Delta_f(x, r) \in \mathfrak{M}_f.$$

The function m_f has domain X and range $\mathfrak{M}_f$.

Assuming that X is connected we metrize the set $\mathfrak{M}_f$ (as in [B 2]) by means of the distance function d_f as follows: If $A \in \mathfrak{M}_f$ and $B \in \mathfrak{M}_f$, then

$$d_f(A, B) = \inf_{W \in F} \text{diameter } f(W)$$

where F is the set of all those compact connected subsets of X which meet both A and B.

The connectedness of X is essential only in proving that $d_f(A, B) < \infty$ for all $A, B \in \mathfrak{M}_f$. If we wanted to drop the assumption that X is connected, we could replace $d_f(A, B)$ by $d_f(A, B)/[1+d_f(A, B)]$ in case $d_f(A, B) < \infty$ and by 1 in case $d_f(A, B) = \infty$. All subsequent results would remain substantially unchanged.

The metric d_f induces such a topology on $\mathfrak{M}_f$ that the function m_f is continuous, the m_f counterimage of every connected open subset of $\mathfrak{M}_f$ is a connected subset of X, and the space $\mathfrak{M}_f$ is locally connected.

If $A \in \mathfrak{M}_f$ and $r > 0$, then $\Delta_f(x, r)$ is the same for all $x \in A$. It will be denoted by

$$\Delta_f(A, r).$$

The elements of $\mathfrak{M}_f$ are closed subsets of X but need be neither compact nor connected. If $A \in \mathfrak{M}_f$, then A is a compact subset of X if and only if the sets $\Delta_f(A, r)$ have compact closures for all sufficiently small positive numbers r. Every compact element of $\mathfrak{M}_f$ is connected. The set of all compact elements of $\mathfrak{M}_f$ is an open locally compact subset of $\mathfrak{M}_f$. The whole space $\mathfrak{M}_f$ need not be locally compact.

If X is compact, then the elements of $\mathfrak{M}_f$ are the maximal continua of constancy of f and the mapping $\mathbf{m}_f$ is monotone. However, if X is not compact, then $\mathfrak{M}_f$ and $\mathbf{m}_f$ need not have these properties.

In verifying these properties of $\mathfrak{M}_f$, $\mathbf{m}_f$, and d_f it is helpful to note that

$$\{A \mid d_f[A, \mathbf{m}_f(x)] < r\} \subset \mathbf{m}_f[\Delta_f(x, r)] \subset \{A \mid d_f[A, \mathbf{m}_f(x)] < 2r\},$$
$$\{z \mid \mathbf{m}_f(z) \in \mathbf{m}_f[\Delta_f(x, r)]\} = \Delta_f(x, r) \qquad \text{for } x \in X,\ r > 0.$$

Since f is constant on each equivalence class in $\mathfrak{M}_f$, there is a unique function l_f on $\mathfrak{M}_f$ to E_n such that

$$f = l_f \circ \mathbf{m}_f.$$

l_f is continuous; in fact it satisfies the Lipschitz condition

$$|l_f(A) - l_f(B)| \leq d_f(A, B) \text{ for } A, B \in \mathfrak{M}_f.$$

If Y is any subset of E_n and Z is any component of the l_f counterimage of Y, then the diameter of Z is not greater than the diameter of Y. It follows that the l_f counterimage of each point of E_n is totally disconnected, so that l_f is a light mapping.

Recalling the manner in which $\mathcal{H}_n^k$ was defined in terms of the metric of E_n we analogously define *the k-dimensional Hausdorff measure*

$$H_f^k$$

over $\mathfrak{M}_f$ in terms of the metric d_f. The Lipschitz condition satisfied by l_f implies that

$$H_f^k(Q) \geq \mathcal{H}_n^k[l_f(Q)] \qquad \text{whenever } Q \subset \mathfrak{M}_f.$$

It follows that

$$\int_{\mathfrak{M}_f} g(A)\,dH_f^k A \geq \int_{E_n} \sum_{l_f(A)=y} g(A)\,d\mathcal{H}_n^k y$$

for every non-negative analytically measurable function g on $\mathfrak{M}_f$.

To prove this we observe that the set of all those non-negative analytically measurable functions g for which the above inequality holds is closed to addition, to multiplication by positive numbers, and to monotone convergence. Hence it is sufficient to prove that this set contains all characteristic functions of analytic subsets of $\mathfrak{M}_f$. However, if g is the characteristic function of the analytic set $Q\subset\mathfrak{M}_f$, then the inequality reduces to the formula

$$H_f^k(Q) \geqq \int_{E_n} N(l_f, Q, y)d\mathcal{H}_n^k y$$

which can be proved by the method of [F 1, 4.4].

If $Q\subset\mathfrak{M}_f$ and $H_f^k(Q) < \infty$, then

$$2^{-k} \leqq \limsup_{r\to 0+} \frac{H_f^k(Q \cap m_f[\Delta_f(A, r)])}{\alpha(k)r^k} \leqq 1$$

for H_f^k almost all A in Q. If, in addition, the set Q is H_f^k measurable, then

$$\lim_{r\to 0+} \frac{H_f^k(Q \cap m_f[\Delta_f(A, r)])}{\alpha(k)r^k} = 0$$

for H_f^k almost all A in $\mathfrak{M}_f - Q$.

These density properties of H_f^k can be proved just like the corresponding properties of $\mathcal{H}_n^k$ established in [F 4].

An arbitrary subset V of X may fail to be finitely triangulable and the k-dimensional Lebesgue area of the mapping $f| V$ may not be well defined. We therefore define the substitute

$$\lambda_k(f, V)$$

as the supremum of $L_k(f| W)$ for all those subsets W of V which possess finite triangulations. It is clear that

$$\lambda_k(f, V) = L_k(f| V) \text{ for every finitely triangulable } V \subset X,$$

$$\lambda_k(f, V) \geqq \sum_{i=1}^{\infty} \lambda_k(f, V_i) \quad \text{whenever } V_1, V_2, V_3, \cdots \text{ are disjoint subsets of } V \subset X.$$

If $A\in\mathfrak{M}_f$ and $r>0$, then

$$f[\Delta_f(A, r)] \subset K[l_f(A), r],$$

and it is natural to compare $\lambda_k[f, \Delta_f(A, r)]$ with $\alpha(k)r^k$, the $\mathcal{L}_k$ measure of any orthogonal projection of $K[l_f(A), r]$ into E_k. This leads us to consider the upper and lower limits

$$L_k^*(f, A) = \limsup_{r\to 0+} \frac{\lambda_k[f, \Delta_f(A, r)]}{\alpha(k)r^k},$$

$$L_{*k}(f, A) = \liminf_{r \to 0+} \frac{\lambda_k[f, \Delta_f(A, r)]}{\alpha(k) r^k},$$

which we think of as the upper and lower densities of the k-dimensional Lebesgue area of f at A. Using integration with respect to H_f^k we shall establish formulae which allow us to regard these densities as derivatives of the Lebesgue area. Furthermore the densities are related to multiplicity functions connected with the Lebesgue area.

We note that $\lambda_k[f, \Delta_f(A, r)]$ is lower semi-continuous with respect to (A, r) on the cartesian product space $\mathfrak{M}_f \times \{r \mid r > 0\}$. It follows that $L_k^*(f, A)$ and $L_{*k}(f, A)$ are analytically measurable with respect to A on $\mathfrak{M}_f$.

LEMMA 6.1. *If* $f \in C_n(X)$, V *is an open subset of* $\mathfrak{M}_f$, $W \subset V$, $t > 0$, *and*

$$L_k^*(f, A) > t \qquad \text{for } A \in W,$$

then

$$\lambda_k(f, \{x \mid m_f(x) \in V\}) \geqq t \cdot H_f^k(W).$$

PROOF. We may assume that

$$\lambda_k(f, \{x \mid m_f(x) \in V\}) < \infty.$$

Let $\epsilon > 0$. Defining

$$u(A, r) = \text{Closure}\, m_f[\Delta_f(A, r)] \text{ for } A \in \mathfrak{M}_f \text{ and } r > 0,$$

$$F = \{u(A, r) \mid A \in \mathfrak{M}_f,\ 0 < r < \epsilon/10,\ u(A, r) \subset V \text{ and } \lambda_k[f, \Delta_f(A, r)] > t\alpha(k) r^k\},$$

$$\hat{Z} = \{A \mid d_f(A, B) \leqq 2 \operatorname{diam} Z \text{ for some } B \in Z\} \text{ for } Z \subset \mathfrak{M}_f,$$

we note that F covers W in the sense of Vitali and apply a covering theorem of A. P. Morse [M, 3.10] to obtain such a disjointed subfamily G of F that

$$W \subset \bigcup_{z \in G} Z \cup \bigcup_{z \in H} \hat{Z}$$

whenever $H \subset G$ and $G - H$ is finite. For each $Z \in G$ we choose A_Z and r_Z so that

$$Z = u(A_Z, r_Z), \qquad A_Z \in \mathfrak{M}_f, \qquad 0 < r_Z < \epsilon/10,$$
$$\lambda_k[f, \Delta_f(A_Z, r_Z)] > t\alpha(k)(r_Z)^k$$

and observe that

$$\text{diameter } Z \leqq 2r_Z < \epsilon, \text{ diameter } \widehat{Z} \leqq 5 \text{ diameter } Z \leqq 10r_Z < \epsilon.$$

If Z and Z' are distinct elements of G, then $\Delta_f(A_Z, r_Z)$ and $\Delta_f(A_{Z'}, r_{Z'})$ are disjoint nonvacuous open subsets of X. It follows that G is countable.

If $H \subset G$ and $G - H$ is finite, then

$$G \cup \{\widehat{Z} \mid Z \in H\}$$

is a countable covering of W by sets whose diameter is less than ϵ and

$$\begin{aligned}\sum_{Z \in G} 2^{-k}\alpha(k)(\text{diameter } Z)^k + \sum_{Z \in H} 2^{-k}\alpha(k)(\text{diameter } \widehat{Z})^k \\ \leqq \sum_{Z \in G} \alpha(k)(r_Z)^k + 5^k \sum_{Z \in H} \alpha(k)(r_Z)^k \\ \leqq t^{-1}\Big(\sum_{Z \in G} \lambda_k[f, \Delta_f(A_Z, r_Z)] + \sum_{Z \in H} \lambda_k[f, \Delta_f(A_Z, r_Z)]\Big).\end{aligned}$$

Inasmuch as

$$\sum_{Z \in G} \lambda_k[f, \Delta_f, (A_Z, r_Z)] \leqq \lambda_k(f, \{x \mid m_f(x) \in V\}),$$

we can choose H so that

$$\sum_{Z \in H} \lambda_k[f, \Delta_f(A_Z, r_Z)]$$

is arbitrarily small. Then the last member of our string of inequalities exceeds

$$t^{-1}\lambda_k(f, \{x \mid m_f(x) \in V\})$$

by arbitrarily little.

Letting ϵ approach 0 we conclude that

$$H_f^k(W) \leqq t^{-1}\lambda_k(f, \{x \mid m(x) \in V\}).$$

THEOREM 6.2. *If $f \in C_k(X)$ and V is an open subset of $\mathfrak{M}_f$, then*

$$\lambda_k(f, \{x \mid m_f(x) \in V\}) \geqq \int_V L_k^*(f, A)dH_f^k A.$$

PROOF. We may assume that

$$\lambda_k(f, \{x \mid m_f(x) \in V\}) < \infty.$$

Let $1 < t < \infty$. For each integer i we define

$$U_i = V \cap \{A \mid t^i < L_k^*(f, A) \leqq t^{i+1}\},$$

infer from the preceding lemma that

$$H_f^k(U_i) \leqq t^{-i}\lambda_k[f, \{x \mid m_f(x) \in V\}] < \infty,$$

and choose a closed subset W_i of U_i such that

$$H_f^k(U_i) \leqq tH_f^k(W_i).$$

Since the sets W_i are disjoint closed subsets of V, we can choose disjoint open subsets V_i of V such that $W_i \subset V_i$ for every positive integer i.

Applying the preceding lemma again, we conclude that

$$\begin{aligned}\int_V L_k^*(f, A)dH_f^kA &= \sum_{i=-\infty}^{\infty} \int_{U_i} L_k^*(f, A)dH_f^kA \\ &\leqq \sum_{i=-\infty}^{\infty} t^{i+1}\cdot H_f^k(U_i) \leqq t^2 \sum_{i=-\infty}^{\infty} t^i\cdot H_f^k(W_i) \\ &\leqq t^2 \sum_{i=-\infty}^{\infty} \lambda_k(f, \{x \mid m_f(x) \in V_i\}) \\ &\leqq t^2\lambda_k(f, \{x \mid m_f(x) \in V\})\end{aligned}$$

and let t approach 1 to complete the proof.

The preceding theorem is complemented by the inequality

$$\int_V L_k^*(f, A)dH_f^kA \geqq \int_{E_n} \sum_{A\in V, l_f(A)=y} L_k^*(f, A)d\mathcal{H}_n^k y$$

which is valid whenever $f \in C_n(X)$ and V is an analytic subset of $\mathfrak{M}_f$.

It is natural to ask:

Does equality hold in these inequalities?

Is $L_k^*(f, A) = L_{*k}(f, A)$ for H_f^k almost all A in $\mathfrak{M}_f$?

No counterexamples are known. Moreover we shall prove the affirmative answers for some special cases. The problem is trivial for $k = 1 \leqq n$. The case in which $k = n$, the case in which S is a finitely connected subset of E_2 and $n \geqq 2$, and in particular the case in which X is a 2-cell and $n = 3$ will be discussed in detail. Complete answers will be obtained in the first and last of these three cases. In general the solution of these problems seems to depend on the answer to the following question:

What geometric properties of a point $A \in \mathfrak{M}_f$ and which topological and metric invariants of the mappings $f|\Delta_f(a, r)$, for $r > 0$, are significant *for the densities* $L_k^*(f, A)$ *and* $L_{*k}(f, A)$?

One such property of A is that of being i-fold essential in dimension k, which we shall define later in this section. In some special cases the densities can be completely described in terms of this property. However the geometry underlying the behavior of the densities is not yet fully understood in the general case.

In the remainder of this section we compile the fragmentary results known to us for the general case in which $k \leqq n$ as well as the complete results for the case in which $k = n$. The case in which X is a finitely connected compact subset of E_2 is taken up in the later sections.

THEOREM 6.3. *If* $f \in C_k(X)$, V *is an open subset of* $\mathfrak{M}_f$,

$$\lambda_k(f, \{x \mid m_f(x) \in V\}) = \int_V L_k^*(f, A)\,dH_f^k A < \infty,$$

and V' *is an open subset of* V, *then*

$$\lambda_k(f, \{x \mid m_f(x) \in V'\}) = \int_{V'} L_k^*(f, A)\,dH_f^k A.$$

PROOF. Otherwise there is a finitely triangulable set W such that

$$W \subset \{x \mid m_f(x) \in V'\}, L_k(f \mid W) > \int_{V'} L_k^*(f, A)\,dH_f^k A.$$

Then W and $m_f(W)$ are compact, $V - m_f(W)$ is open, W and $\{x \mid m_f(x) \in V - m_f(W)\}$ are disjoint subsets of $\{x \mid m_f(x) \in V\}$, and $m_f(W) \subset V'$. It follows that

$$\begin{aligned}\lambda_k(f, \{x \mid m_f(x) \in V\}) &\geqq L_k(f \mid W) + \lambda_k(f, \{x \mid m_f(x) \in V - m_f(W)\}) \\ &> \int_{V'} L_k^*(f, A)\,dH_f^k A + \int_{V - m_f(W)} L_k^*(f, A)\,dH_f^k A \\ &\geqq \int_V L_k^*(f, A)\,dH_f^k A = \lambda_k(f, \{x \mid m_f(x) \in V\}),\end{aligned}$$

which is false.

LEMMA 6.4. *If* $f \in C_k(X)$ *and* U *is an open subset of* X, *then*

$$\lambda_k(f, U) = \int_{E_k} M(f, U, y)\,d\mathcal{L}_k y.$$

PROOF. If W is any finitely triangulable subset of U, then

$$L_k(f\,|\,W) = \int_{E_k} M(f, W, y)\,d\mathcal{L}_k y,$$

$$M(f, W, y) \leqq M(f, U, y) \text{ for } y \in E_k.$$

It follows that

$$\lambda_k(f, U) \leqq \int_{E_k} M(f, U, y)\,d\mathcal{L}_k y.$$

On the other hand U has finitely triangulable subsets $W_1 \subset W_2 \subset W_3 \subset \cdots$ with the property that each point of U which has a k-cell neighborhood in U is an interior point of W_i for some positive integer i. We infer that

$$M(f, W_i, y) \leqq M(f, W_{i+1}, y) \quad \text{for } y \in E_k \text{ and } i = 1, 2, 3, \cdots,$$

$$M(f, U, y) = \lim_{i \to \infty} M(f, W_i, y) \text{ for } y \in E_k,$$

$$\begin{aligned} \int_{E_k} M(f, U, y)\,d\mathcal{L}_k y &= \lim_{i \to \infty} \int_{E_k} M(f, W_i, y)\,d\mathcal{L}_k y \\ &= \lim_{i \to \infty} L_k(f\,|\,W_i) \leqq \lambda_k(f, U). \end{aligned}$$

THEOREM 6.5. *If* $f \in C_n(X)$, V *is an open subset of* $\mathfrak{M}_f$,

$$\lambda_k(f, \{x \mid m_f(x) \in V\}) = \int_V L_k^*(f, A)\,dH_f^k A < \infty,$$

$$P = V \cap \{A \mid L_k^*(f, A) > 0\}, \qquad R \in G_n,$$

$$Z = \{z \mid S(p_n^k \circ R \circ f, \{x \mid m_f(x) \in V\}, z) > 0\},$$

then

$$\mathcal{L}_k[Z - (p_n^k \circ R \circ l_f)(P)] = 0.$$

PROOF. Otherwise there exist a closed set F and open sets $U_1 \supset U_2 \supset U_3 \supset \cdots$ contained in E_k such that

$$\mathcal{L}_k(F) > 0 \quad \text{and} \quad \bigcap_{i=1}^{\infty} U_i = F \subset Z - (p_n^k \circ R \circ l_f)(P).$$

Defining

$$F' = V \cap \{A \mid (p_n^k \circ R \circ l_f)(A) \in F\},$$

$$U_i' = V \cap \{A \mid (p_n^k \circ R \circ l_f)(A) \in U_i\} \text{ for } i = 1, 2, 3, \cdots,$$

we infer that

$$V \supset U_1' \supset U_2' \supset U_3' \supset \cdots \quad \text{and} \quad \bigcap_{i=1}^{\infty} U_i' = F' \subset V - P,$$

whence

$$\lim_{i \to \infty} \int_{U_i'} L_k^*(f, A)\, dH_f^k A = \int_{F'} L_k^*(f, A)\, dH_f^k A = 0$$

and we choose a positive integer i for which

$$\int_{U_i'} L_k^*(f, A)\, dH_f^k A < \mathcal{L}_k(F).$$

However

$$\begin{aligned}
\mathcal{L}_k(F) &\leqq \int_F S(p_n^k \circ R \circ f, \{x \mid m_f(x) \in V\}, z)\, d\mathcal{L}_k z \\
&\leqq \int_{U_i} M(p_n^k \circ R \circ f, \{x \mid m_f(x) \in V\}, z)\, d\mathcal{L}_k z \\
&= \int_{U_i} M(p_n^k \circ R \circ f, \{x \mid m_f(x) \in U_i'\}, z)\, d\mathcal{L}_k z \\
&= \int_{E_k} M(p_n^k \circ R \circ f, \{x \mid m_f(x) \in U_i'\}, z)\, d\mathcal{L}_k z \\
&= \lambda_k(p_n^k \circ R \circ f, \{x \mid m_f(x) \in U_i'\}) \leqq \lambda_k(f, \{x \mid m_f(x) \in U_i'\}) \\
&= \int_{U_i'} L_k^*(f, A)\, dH_f^k A.
\end{aligned}$$

THEOREM 6.6. *If* $f \in C_n(X)$, V *is an open subset of* $\mathfrak{M}_f$,

$$\lambda_k(f, \{x \mid m_f(x) \in V\}) = \int_V L_k^*(f, A)\, dH_f^k A < \infty,$$

and $R \in G_n$, *then*

$$S(p_n^k \circ R \circ f, \{x \mid m_f(x) \in V\}, z) \leqq N(p_n^k \circ R \circ l_f, V \cap \{A \mid L_k^*(f, A) > 0\}, z)$$

for $\mathcal{L}_k$ *almost all* z *in* E_k.

PROOF. We choose a countable basis B for the topology of E_k, define

$$F_U = \{W \mid W \text{ is a component of } V \cap \{A \mid (p_n^k \circ R \circ l_f)(A) \in U\}\}$$

for each $U \in B$ and let

$$Z_W = \{z \mid S(p_n^k \circ R \circ f, \{x \mid m_f(x) \in W\}, z) > 0\}$$

whenever $W \in F_U$ for some $U \in B$. Then

$$\bigcup_{U \in B} F_U \text{ is countable}$$

and we infer from Theorem 6.5 that

$$\mathcal{L}_k\left(\bigcup_{U \in B} \bigcup_{W \in F_U} [Z_W - (p_n^k \circ R \circ l_f)(W \cap P)]\right) = 0,$$

where

$$P = \{A \mid L_k^*(f, A) > 0\}.$$

Now suppose

$$z \in E_k - \bigcup_{U \in B} \bigcup_{W \in F_U} [Z_W - (p_n^k \circ R \circ l_f)(W \cap P)],$$

m is a positive integer, and

$$S(p_n^k \circ R \circ f, \{x \mid m_f(x) \in V\}, z) \geqq m.$$

Then there exists a $U \in B$ such that the set

$$G = F_U \cap \{W \mid z \in Z_W\}$$

has at least m elements. Recalling our choice of z we find that

$$z \in (p_n^k \circ R \circ l_f)(W \cap P) \text{ for } W \in G$$

and use the disjointedness of G to conclude that

$$N(p_n^k \circ R \circ l_f, V \cap P, z) \geqq m.$$

The proof is complete.

We say that

A is i-fold essential for f in dimension k if and only if $f \in C_n(X)$, $A \in \mathfrak{M}_f$, i is an integer, and there exists an $R \in G_n$ such that

$$\lim_{r \to 0+} \frac{\mathcal{L}_k(\{z \mid M(p_n^k \circ R \circ f, \Delta_f(A, r), z) \geqq i\})}{\alpha(k) r^k} = 1.$$

We refer to $p_n^k \circ R$ as an *i-fold essential projection for f at A in dimension k.*

Furthermore we define

$$\mathcal{E}_k(f, A) = \sup \{ i \mid A \text{ is } i\text{-fold essential for } f \text{ in dimension } k \}.$$

We note that $M(p_n^k \circ R \circ f, \Delta_f(A, r), z)$ is lower semi-continuous with respect to (A, R, r, z) on the cartesian product space $\mathfrak{M}_f \times G_n \times \{r \mid r > 0\} \times E_k$, and that

$$\mathcal{L}_k(\{z \mid M(p_n^k \circ R \circ f, \Delta_f(A, r), z) \geqq i\})$$

is lower semi-continuous with respect to (A, R, r) on the cartesian product space $\mathfrak{M}_f \times G_n \times \{r \mid r > 0\}$ for each integer i. It follows that $\mathcal{E}_k(f, A)$ is analytically measurable with respect to A on $\mathfrak{M}_f$.

THEOREM 6.7. *$L_{*k}(f, A) \geqq \mathcal{E}_k(f, A)$ for $f \in C_n(X)$ and $A \in \mathfrak{M}_f$.*

PROOF. If $R \in G_n$, then

$$\begin{aligned} \lambda_k[f, \Delta_f(A, r)] &\geqq \lambda_k[p_n^k \circ R \circ f, \Delta_f(A, r)] \\ &= \int_{E_k} M[p_n^k \circ R \circ f, \Delta_f(A, r)] d\mathcal{L}_k z \\ &\geqq i\mathcal{L}_k(\{z \mid M(p_n^k \circ R \circ f, \Delta_f(A, r), z) \geqq i\}). \end{aligned}$$

THEOREM 6.8. *If $p_n^k \circ R$ is an i-fold essential projection for f at A in dimension k, $\epsilon > 0$, and*

$$T(r) = \Delta_f(A, r) \cap \{x \mid f(x) \in \diamond_n^k[R, \epsilon, l_f(A)]\} \quad \textit{for } r > 0,$$

then

$$\lim_{r \to 0+} \frac{\mathcal{L}_k(\{z \mid M(p_n^k \circ R \circ f, T(r), z) \geqq i\})}{\alpha(k) r^k} = 1.$$

PROOF. Suppose $t < 1$. Let

$$u = (1 + \epsilon^2)^{-1/2}, \qquad v = u^k + t(1 - u^k),$$

note that $u < 1$ and $v < 1$, and choose $\delta > 0$ so that

$$\mathcal{L}_k(\{z \mid M(p_n^k \circ R \circ f, \Delta_f(A, r), z) \geqq i\}) \geqq v\alpha(k) r^k$$

whenever $0 < r < \delta$.

Suppose $0 < r < \delta$. For $j = 1, 2, 3, \cdots$ we define

$$Z_j = E_k \cap \{z \mid ru^j < |z - (p_n^k \circ R \circ l_f)(A)| < ru^{j-1}\},$$

$$U_j = \{x \mid (p_n^k \circ R \circ f)(x) \in Z_j\},$$

and easily verify that $U_j \cap \Delta_f(A, ru^{j-1}) \subset U_j \cap T(r)$. Hence $z \in Z_j$ implies

$$\begin{aligned} M(p_n^k \circ R \circ f, \Delta_f(A, ru^{j-1}), z) &= M(p_n^k \circ R \circ f, U_j \cap \Delta_f(A, ru^{j-1}), z) \\ &\leqq M(p_n^k \circ R \circ f, U_j \cap T(r), z) \\ &= M(p_n^k \circ R \circ f, T(r), z), \end{aligned}$$

and we infer that

$$\begin{aligned} &Z_j \cap \{z \mid M(p_n^k \circ R \circ f, T(r), z) \geqq i\} \\ &\quad \supset \{z \mid M(p_n^k \circ R \circ f, \Delta_f(A, ru^{j-1}), z) \geqq i\} - K[(p_n^k \circ R \circ l_f)(A), ru^j], \\ &\mathcal{L}_k(Z_j \cap \{z \mid M(p_n^k \circ R \circ f, T(r), z) \geqq i\}) \\ &\quad \geqq v\alpha(k)(ru^{j-1})^k - \alpha(k)(ru^j)^k = (u^k)^{j-1}(v - u^k)\alpha(k)r^k. \end{aligned}$$

Since the sets Z_j are disjoint and $\mathcal{L}_k$ measurable, summation with respect to j yields the inequality

$$\mathcal{L}_k(\{z \mid M(p_n^k \circ R \circ f, T(r), z) \geqq i\}) \geqq \frac{(v - u^k)\alpha(k)r^k}{1 - u^k} = t\alpha(k)r^k.$$

THEOREM 6.9. *Suppose*

$$f \in C_n(X), \quad V \textit{ is an open subset of } \mathfrak{M}_f,$$

$$\lambda_k(f, \{x \mid m_f(x) \in V\}) = \int_V L_k^*(f, A)\, dH_f^k A < \infty,$$

$$P = V \cap \{A \mid L_k^*(f, A) > 0\}, \qquad H_f^k(P) < \infty,$$
$$Q = V \cap \{A \mid \mathcal{C}_k(f, A) > 0\}.$$

Then:

(1) *P and Q are analytic subsets of* $\mathfrak{M}_f$, *and* $Q \subset P$.

(2) $$\lim_{r \to 0+} \frac{H_f^k(Q \cap m_f[\Delta_f(A, r)])}{\alpha(k)r^k} = 1 \qquad \textit{for } H_f^k \textit{ almost all } A \textit{ in } Q.$$

(3) $$H_f^k(W) = \int_{E_n} N(l_f, W, y)\, d\mathcal{H}_n^k y \qquad \textit{for every analytic set } W \subset Q.$$

(4) $$\int_Q g(A)\, dH_f^k A = \int_{E_n} \sum_{A \in Q,\, l_f(A) = y} g(A)\, d\mathcal{H}_n^k y$$

for every analytically measurable function g on $\mathfrak{M}_f$.

(5) $\qquad L_k^*(f, A) = L_{*k}(f, A)$ *for* H_f^k *almost all* A *in* Q.

(6) *For* H_f^k *almost all* A *in* Q *the following implication holds:*
If $i>0$, $R \in G_n$, *and* $p_n^k \circ R$ *is an* i*-fold essential projection for* f *at* A *in dimension* k, *then* $\square_n^k[R, l_f(A)]$ *is a Hausdorff* k *tangent plane of* $l_f(Q)$ *at* $l_f(A)$.

(7) $l_f(Q)$ *is a Hausdorff* k *rectifiable subset of* E_n.

Proof of (2). Since $H_f^k(Q) < \infty$ and $H_f^k(P-Q) < \infty$, we have

$$\limsup_{r\to 0+} \frac{H_f^k(Q \cap \boldsymbol{m}_f[\Delta_f(A, r)])}{\alpha(k)r^k} \leqq 1$$

and

$$\lim_{r\to 0+} \frac{H_f^k(P \cap \boldsymbol{m}_f[\Delta_f(A, r)] - Q)}{\alpha(k)r^k} = 0$$

for H_f^k almost all A in Q.

On the other hand if A is any point of Q, then there exist $i>0$ and $R \in G_n$ such that $p_n^k \circ R$ is an i-fold essential projection for f at A in dimension k. For any positive number r which is so small that

$$\boldsymbol{m}_f[\Delta_f(A, r)] \subset V$$

we infer with the help of the theorems 6.3 and 6.5 that

$$\begin{aligned} &\mathcal{L}_k(\{z \mid M(p_n^k \circ R \circ f, \Delta_f(A, r), z) \geqq i\} \\ &\qquad\qquad - (p_n^k \circ R \circ l_f)(P \cap \boldsymbol{m}_f[\Delta_f(A, r)])) = 0, \\ &H_f^k(P \cap \boldsymbol{m}_f[\Delta_f(A, r)]) \geqq \mathcal{L}_k(p_n^k \circ R \circ l_f)(P \cap \boldsymbol{m}_f[\Delta_f(A, r)]) \\ &\qquad\qquad \geqq \mathcal{L}_k(\{z \mid M(p_n^k \circ R \circ f, \Delta_f(A, r), z) \geqq i\}). \end{aligned}$$

It follows that

$$\liminf_{r\to 0+} \frac{H_f^k(P \cap \boldsymbol{m}_f[\Delta_f(A, r)])}{\alpha(k)r^k} \geqq 1.$$

Proof of (3). Inasmuch as we already know that

$$\int_{E_k} N(l_f, W, y)\, d\mathcal{H}_n^k y \leqq H_f^k(W),$$

it is sufficient to prove the following statement:

If $0<t<1$ and $\epsilon>0$, then there is a partition F of W such that

$$\sup_{Z\in F} \text{diameter } Z < \epsilon,$$

$$\sum_{T\in F} \mathcal{H}_n^k[l_f(T)] \geqq t^2 H_f^k(W) - \epsilon.$$

To prove this we choose an open subset U of $\mathfrak{M}_f$ for which

$$W \subset U \subset V, \qquad H_f^k(U \cap P - W) < \epsilon$$

and let G be the family of all sets Z such that

$$\text{Closure } m_f[\Delta_f(A, r)] = Z \subset U, \qquad A \in W, \qquad 0 < 2r < \epsilon,$$
$$t\alpha(k)r^k < \mathcal{H}_n^k[l_f(Z \cap P)] \leqq H_f^k(Z \cap P) < t^{-1}\alpha(k)r^k$$

for some A and r. Then G covers H_f^k almost all of the W in the sense of Vitali, and the Vitali covering theorem [M, 4.1] yields a countable disjointed subfamily H of G such that

$$H_f^k\left(W - \bigcup_{Z\in H} Z\right) = 0.$$

We let

$$F_1 = \{Z \cap W \mid Z \in H\},$$

choose such a partition F_2 of $W - \bigcup_{Z\in H} Z$ that

$$\sup_{T\in F_2} \text{diameter } T < \epsilon,$$

and define

$$F = F_1 \cup F_2.$$

It follows that

$$\begin{aligned} t^2 H_f^k(W) &\leqq \sum_{Z\in H} t^2 H_f^k(Z \cap P) \leqq \sum_{Z\in H} \mathcal{H}_n^k[l_f(Z \cap P)] \\ &\leqq \sum_{Z\in H} (\mathcal{H}_n^k[l_f(Z \cap W)] + H_f^k(Z \cap P - W)) \\ &\leqq \sum_{T\in F} \mathcal{H}_n^k[l_f(T)] + H_f^k(U \cap P - W) \\ &\leqq \sum_{T\in F} \mathcal{H}_n^k[l_f(T)] + \epsilon. \end{aligned}$$

PROOF OF (5). By virtue of (2) the usual theorem on the differentiation of an indefinite integral is valid on Q. We infer, for H_f^k almost all A in Q, that

$$L_k^*(f, A) = \lim_{r\to 0+} \frac{\int_{m_f[\Delta_f(A,r)]} L_k^*(f, A)\,dH_f^k A}{H_f^k(Q \cap m_f[\Delta_f(A, r)])}$$

and use Theorem 6.5 and (2) to conclude that the limit on the right is equal to

$$\lim_{r\to 0+} \frac{\lambda_k[f, \Delta_f(A, r)]}{\alpha(k) r^k}\,.$$

PROOF OF (6). Defining

$$\begin{aligned} Y &= \{y \mid 2^{-k} \leqq \Theta_n^k[l_f(Q), y] \leqq \Theta_n^k[l_f(P), y] \leqq 1\}, \\ W &= Q \cap \{A \mid l_f(A) \in Y\}, \end{aligned}$$

we find that

$$\mathcal{H}_n^k[l_f(Q)] \leqq \mathcal{H}_n^k[l_f(P)] < \infty, \qquad \mathcal{H}_n^k[l_f(Q) - Y] = 0,$$
$$H_f^k[Q - W] = \int_{E_n} N(l_f, Q - W, y)\,d\mathcal{H}_n^k y = 0.$$

Now suppose $A \in W$, i is a positive integer, and the hypotheses of Theorem 6.8 hold. Applying the Theorems 6.3 and 6.5 as in the proof of (2) we conclude, for all sufficiently small positive numbers r, that

$$\begin{aligned} &\mathcal{L}_k(\{z \mid M(p_n^k \circ R \circ f, T(r), z) \geqq i\}) \\ &\qquad \leqq \mathcal{L}_k[(p_n^k \circ R \circ l_f)(P \cap m_f[T(r)])] \\ &\qquad \leqq \mathcal{H}_n^k[l_f(P \cap m_f[T(r)])] \leqq \mathcal{H}_n^k(l_f(P) \cap f[T(r)]) \\ &\qquad \leqq \mathcal{H}_n^k(l_f(P) \cap \diamond_n^k[R, \epsilon, l_f(A)] \cap K[l_f(A), r]). \end{aligned}$$

It follows that

$$\begin{aligned} &\Theta_n^k[l_f(P) \cap \diamond_n^k[R, \epsilon, l_f(A)], l_f(A)] \geqq 1, \\ &\Theta_n^k[l_f(P) - \diamond_n^k[R, \epsilon, l_f(A)], l_f(A)] = 0. \end{aligned}$$

THEOREM 6.10. *Suppose $f \in C_k(X)$ and U is the union of all those open subsets V of $\mathfrak{M}_f$ for which*

$$\lambda_k(f, \{x \mid m_f(x) \in V\}) < \infty.$$

Then:

(1) $$L_k^*(f, A) \geqq \limsup_{r\to 0+} D[f, r, \Delta_f(A, r)] \quad \text{for } A \in M_f.$$

(2) $$\mathcal{E}_k(f, A) \geqq \liminf_{r\to 0+} D[f, r, \Delta_f(A, r)] \quad \text{for } A \in M_f.$$

(3) *If W is an open subset of $\mathfrak{M}_f$, $y\in E_k$, $r>0$, and Z is an essential canonical region of $(f, \{x \mid m_f(x)\in W\}, y, r)$, then there is an $A\in W$ for which $l_f(A)=y$, $\Delta_f(A, r)=Z$, and*

$$\limsup_{t\to 0+} D[f, t, \Delta_f(A, t)] \geqq 1.$$

(4) *If W is an open subset of $\mathfrak{M}_f$, $A\in W$, and*

$$\liminf_{r\to 0+} D[f, r, \Delta_f(A, r)] < \limsup_{r\to 0+} D[f, r, \Delta_f(A, r)],$$

then $M(f, \{x \mid m_f(x)\in W\}, l_f(A)) = \infty$.

(5) *If W is an open subset of $\mathfrak{M}_f$ and $y\in E_k$, then*

$$\sum_{A\in W, l_f(A)=y} L_k^*(f, A) \geqq M(f, \{x \mid m_f(x) \in W\}, y).$$

(6) *If W is an open subset of $\mathfrak{M}_f$, $y\in E_k$, and*

$$M(f, \{x \mid m_f(x) \in W\}, y) < \infty,$$

then

$$M(f, \{x \mid m_f(x) \in W\}, y) = \sum_{A\in W, l_f(A)=y} \lim_{r\to 0+} D[f, r, \Delta_f(A, r)].$$

(7) *If W is an open subset of $\mathfrak{M}_f$, then*

$$\begin{aligned}\lambda_k(f, \{x \mid m_f(x) \in W\}) &= \int_W L_k^*(f, A)\,dH_f^k A \\ &= \int_{E_k} \sum_{A\in W, l_f(A)=y} L_k^*(f, y)\,d\mathcal{L}_k y \\ &= \int_{E_k} M(f, \{x \mid m_f(x) \in W\}, y)\,d\mathcal{L}_k y.\end{aligned}$$

(8) *For $\mathcal{L}_k$ almost all y in E_k the following implication holds: If W is an open subset of U, then*

$$M(f, \{x \mid m_f(x) \in W\}, y) = \sum_{A\in W, l_f(A)=y} L_k^*(f, A).$$

(9) $$L_k^*(f, A) = L_{*k}(f, A) = \mathcal{E}_k(f, A) = \lim_{r \to 0+} D[f, r, \Delta_f(A, r)] < \infty$$

for H_f^k almost all A in U.

(10) $$L_k^*(f, A) = L_{*k}(f, A) = \infty \text{ for } A \in \mathfrak{M}_f - U.$$

PROOF OF (1) AND (2). If $A \in \mathfrak{M}_f$, $r > 0$, and

$$D[f, r, \Delta_f(A, r)] \geqq i,$$

then

$$M[f, \Delta_f(A, r), y] \geqq i \text{ for } y \in K[l_f(A), r],$$
$$\lambda_k[f, \Delta_f(A, r)] \geqq i\mathcal{L}_k(\{y \mid M[f, \Delta_f(A, r), y] \geqq i\}) = i\alpha(k) r^k.$$

PROOF OF (3). We define $Z_0, Z_1, Z_2, Z_3, \cdots$ inductively in such a way that $Z_0 = Z$ and Z_i is an essential canonical region of $(f, \{x \mid m_f(x) \in W\}, y, r2^{-i})$ with $Z_i \subset Z_{i-1}$ for every positive integer i. We infer that Closure Z_i is a compact subset of Z_{i-1} for every positive integer i, and choose

$$a \in \bigcap_{i=0}^{\infty} \text{Closure}\, Z_i = \bigcap_{i=0}^{\infty} Z_i, \qquad A = m_f(a).$$

It follows that

$$a \in Z \subset \{x \mid m_f(x) \in W\}, \qquad A \in W,$$
$$l_f(A) = f(a) = y, \qquad \Delta_f(A, r) = \Delta_f(a, r) = Z,$$
$$\Delta_f(A, r2^{-i}) = Z_i \text{ for } i = 1, 2, 3, \cdots.$$

PROOF OF (4). There exist positive numbers

$$r_0 > r_1 > r_2 > r_3 > \cdots$$

such that

$$D[f, r_{i-1}, \Delta_f(A, r_{i-1})] \neq D[f, r_i, \Delta_f(A, r_i)]$$

for $i = 1, 2, 3, \cdots$. Hence we can associate with each positive integer i an essential canonical region Z_i of $(f, \{x \mid m_f(x) \in W\}, l_f(A), r_i)$ such that

$$Z_i \subset \Delta_f(A, r_{i-1}) - \Delta_f(A, r_i).$$

Since $Z_1, Z_2, Z_3, \cdots$ are disjoint subsets of $\{x \mid m_f(x) \in W\}$ and

$$S[f, Z_i, l_f(A)] \geqq 1 \qquad \text{for } i = 1, 2, 3, \cdots,$$

we conclude that

$$M[f, \{x \mid m_f(x) \in W\}, l_f(A)] \geqq S[f, \{x \mid m_f(x) \in W\}, l_f(A)] = \infty.$$

PROOF OF (5) AND (6). By virtue of (3) we have

$$M(f, \{x \mid m_f(x) \in W\}, y) = \lim_{r\to 0+} \sum_{A \in W, l_f(A) = y} D[f, r, \Delta_f(A, r)]$$

whenever W is an open subset of $\mathfrak{M}_f$ and $y \in E_k$.

Hence (1) implies (5), and (4) implies (6).

PROOF OF (7). Consider the statement obtained from (7) upon replacing "$=$" by "$\geqq$" in front of each of the three integral signs. This is a true statement by virtue of Theorem 6.2, the remark following that theorem, and Proposition (5) of the present theorem. However the first and last term in this string of inequalities are equal by Lemma 6.4.

PROOF OF (8). Let B be such a countable basis for the topology of U that

$$\lambda_k(f, \{x \mid m_f(x) \in V\}) < \infty \text{ for } V \in B,$$

and let

$$Z_V = \left\{y \,\middle|\, M(f, \{x \mid m_f(x) \in V\}, y) < \sum_{A \in V, l_f(A) = y} L_k^*(f, A)\right\}$$

for $V \in B$. From (5) and (7) we infer that

$$\mathcal{L}_k(Z_V) = 0 \text{ for } V \in B,$$

$$\mathcal{L}_k\left(\bigcup_{V \in B} Z_V\right) = 0.$$

Now suppose

$$y \in E_k - \bigcup_{V \in B} Z_V$$

and W is any open subset of $\mathfrak{M}_f$.

If $A_1, A_2, \cdots, A_m$ are distinct elements of W such that $l_f(A_i) = y$ for $i = 1, 2, \cdots, m$, then there exist disjoint elements $V_1, V_2, \cdots, V_m$ of B such that $A_i \in V_i$ for $i = 1, 2, \cdots, m$. It follows that

$$\sum_{i=1}^{m} L_k^*(f, A_i) \leqq \sum_{i=1}^{m} \sum_{A \in V_i, l_f(A) = y} L_k^*(f, A)$$

$$= \sum_{i=1}^{m} M(f, \{x \mid m_f(x) \in V_i\}, y)$$

$$\leqq M(f, \{x \mid m_f(x) \in W\}, y).$$

Proof of (9). We see from (1), (2), and Theorem 6.7 that

$$\liminf_{r\to 0+} D[f, r, \Delta_f(A, r)] \leqq \limsup_{r\to 0+} D[f, r, \Delta_f(A, r)] \leqq L_k^*(f, A),$$
$$\liminf_{r\to 0+} D[f, r, \Delta_f(A, r)] \leqq \mathcal{E}_k(f, A) \leqq L_{*k}(f, A) \leqq L_k^*(f, A)$$

whenever $A \in \mathfrak{M}_f$. Choosing B as in the proof of (8) we shall complete the proof by showing that

$$\int_V \liminf_{r\to 0+} D[f, r, \Delta_f(A, r)] dH_f^k A \geqq \int_V L_k^*(f, A) dH_f^k A$$

whenever $V \in B$.

Suppose $V \in B$. By (6) and (7) we conclude that

$$\begin{aligned}\int_V \liminf_{r\to 0+} & D[f, r, \Delta_f(A, r)] dH_f^k A \\ &\geqq \int_{B_k} \sum_{A\in V,\, l_f(A)=y} \liminf_{r\to 0+} D[f, r, \Delta_f(A, r)] d\mathcal{L}_k y \\ &= \int_{B_k} M(f, \{x \mid m_f(x) \in V\}, y) d\mathcal{L}_k y = \int_V L_k^*(f, A) dH_f^k A.\end{aligned}$$

7. **Fréchet equivalence.** If $f \in C_n(X)$ and T is a homeomorphism of X onto X, then f and $f \circ T$ behave identically with respect to the geometric properties discussed in the two preceding sections. These mappings have the same multiplicities, the same areas, and isometric middle spaces. However $f \circ T$ may be preferable to f on account of differentiability properties which are not invariant under homeomorphisms of X.

This passage from a mapping to a geometrically equivalent one, which may have superior analytic properties, can be extended by a limiting process. If $f \in C_n(X)$ and $T_1, T_2, T_3, \cdots$ are homeomorphisms of X onto X such that $f \circ T_i$ converges as i tends to infinity, then the limit mapping is geometrically very similar to f but may be smoother and easier to compute with than f. An example is the familiar process of reparametrization by arc length which replaces an arbitrary curve of finite length by one satisfying a Lipschitz condition.

The formal definition follows:

Suppose $f \in C_n(X)$ and $g \in C_n(X)$. We say that

f and g are Fréchet equivalent

if and only if there exist homeomorphisms $T_1, T_2, T_3, \cdots$ of X onto X such that for each compact subset Q of X

$$(f \circ T_i)(x) \to g(x) \text{ uniformly for } x \in Q \text{ as } i \to \infty,$$

$$(g \circ \overset{-1}{T_i})(x) \to f(x) \text{ uniformly for } x \in Q \text{ as } i \to \infty,$$

$$\text{Closure } \bigcup_{i=1}^{\infty} [T_i(Q) \cup T_i^{-1}(Q)] \text{ is compact.}$$

It is easy to verify that we have indeed defined an equivalence relation on $C_n(X)$. However, most important for the theory of area is the following proposition:

If f and g are Fréchet equivalent elements of $C_n(X)$, then there exist homeomorphisms $T_1, T_2, T_3, \cdots$ of X onto X and an isometry u of $\mathfrak{M}_g$ onto $\mathfrak{M}_f$ such that

$$l_f \circ u = l_g$$

and such that for each compact subset Q of X

$$(m_f \circ T_i)(x) \to (u \circ m_g)(x) \textit{ uniformly for } x \in Q \textit{ as } i \to \infty,$$

$$(m_g \circ \overset{-1}{T_i})(x) \to (\overset{-1}{u} \circ m_f)(x) \textit{ uniformly for } x \in Q \textit{ as } i \to \infty.$$

To prove this we start with a sequence of homeomorphisms $T_1, T_2, T_3, \cdots$ which have the properties stated in the definition of Fréchet equivalence. Later we shall replace this sequence by a subsequence.

If $a \in X$, $a' \in X'$, and $\epsilon > 0$, then there exist a positive integer j and neighborhoods V and V' of a and a' such that

$$d_f[(m_f \circ T_i)(x), (m_f \circ T_i)(x')] \leq d_g[m_g(x), m_g(x')] + \epsilon$$

whenever $x \in V$, $x' \in V'$, i is an integer, and $i > j$.

To check this we choose a continuum $W \subset X$ for which

$$a \in W,\ a' \in W, \text{ diameter } g(W) < d_g[m_g(a), m_g(a')] + \epsilon/5$$

and secure a positive integer j and open connected subsets V and V' of X such that

$$a \in V,\ a' \in V', \text{ Closure } V \text{ and Closure } V' \text{ are compact,}$$

$$d_g[m_g(x), m_g(a)] < \epsilon/5 \text{ and } |(f \circ T_i)(x) - g(a)| < \epsilon/5 \text{ for } x \in V \text{ and } i > j,$$

$$d_g[m_g(x'), m_g(a')] < \epsilon/5 \text{ and } |(f \circ T_i)(x') - g(a')| < \epsilon/5 \text{ for } x' \in V' \text{ and } i > j,$$

$$|(f \circ T_i)(x) - g(x)| < \epsilon/5 \text{ for } x \in W \text{ and } i > j.$$

Letting

$$Z = W \cup \text{Closure } V \cup \text{Closure } V',$$

we conclude for $x\in V$, $x'\in V'$ and $i>j$ that

$$\begin{aligned} &d_f[(\boldsymbol{m}_f\circ T_i)(x), (\boldsymbol{m}_f\circ T_i)(x')] \\ &\qquad \leqq \text{diameter } (f\circ T_i)(Z) \leqq \text{diameter } g(Z) + 2\epsilon/5 \\ &\qquad \leqq d_g[\boldsymbol{m}_g(a), \boldsymbol{m}_g(a')] + 3\epsilon/5 \leqq d_g[\boldsymbol{m}_g(x), \boldsymbol{m}_g(x')] + \epsilon. \end{aligned}$$

The preceding equicontinuity property implies the following:

If Q is a compact subset of X and $\epsilon>0$, then there exists a positive integer j such that

$$d_f[(\boldsymbol{m}_f\circ T_i)(x), (\boldsymbol{m}_f\circ T_i)(x')] < d_g[\boldsymbol{m}_g(x), \boldsymbol{m}_g(x')] + \epsilon$$

whenever $x\in Q$, $x'\in Q$, i is an integer, and $i>j$.

In view of this and also of the fact that

$$\text{Closure } \{(\boldsymbol{m}_f\circ T_i)(x) \mid i = 1, 2, 3, \cdots\} \text{ is compact for } x\in X$$

we may just as well assume, after passage to a subsequence, that there is a mapping U of X into M_f such that

$$(\boldsymbol{m}_f\circ T_i)(x) \to U(x) \text{ uniformly for } x\in Q \text{ as } i\to\infty$$

whenever Q is a compact subset of X.

It follows that

$$d_f[U(x), U(x')] \leqq d_g[\boldsymbol{m}_g(x), \boldsymbol{m}_g(x')] \text{ for } x\in X,\ x'\in X$$

and that there is a function $\boldsymbol{u}$ on $\mathfrak{M}_g$ to $\mathfrak{M}_f$ such that

$$U = \boldsymbol{u}\circ\boldsymbol{m}_g,$$

$$d_f[\boldsymbol{u}(A), \boldsymbol{u}(A')] \leqq d_g(A, A') \text{ for } A\in\mathfrak{M}_g,\ A'\in\mathfrak{M}_g.$$

Furthermore

$$\begin{aligned} (l_f\circ\boldsymbol{u})[\boldsymbol{m}_g(x)] &= l_f[(\boldsymbol{u}\circ\boldsymbol{m}_g)(x)] = l_f[\lim_{i\to\infty} (\boldsymbol{m}_f\circ T_i)(x)] \\ &= \lim_{i\to\infty} (f\circ T_i)(x) = g(x) = l_g[\boldsymbol{m}_g(x)] \end{aligned}$$

for $x\in X$, whence $l_f\circ\boldsymbol{u} = l_g$.

By symmetry we may also assume, after passage to another subsequence, that there is a function $\boldsymbol{v}$ on $\mathfrak{M}_f$ to $\mathfrak{M}_g$ such that

$$(\boldsymbol{m}_g\circ T_i^{-1})(x) \to (\boldsymbol{v}\circ\boldsymbol{m}_f)(x) \text{ uniformly for } x\in Q \text{ as } i\to\infty$$

whenever Q is a compact subset of X and such that

$$d_g[\boldsymbol{v}(A), \boldsymbol{v}(A')] \leqq d_f(A, A') \text{ for } A\in\mathfrak{M}_f,\ A'\in\mathfrak{M}_f.$$

We shall complete the proof by showing that the functions $\boldsymbol{u}$ and $\boldsymbol{v}$

are inverse to each other.

Suppose $A \in \mathfrak{M}_f$ and $\epsilon > 0$. Letting

$$a \in A,\ b \in v(A),\ Q = \{\overset{-1}{T_i}(a) \mid i = 1, 2, 3, \cdots\} \cup \{b\},$$

we choose a positive integer j such that

$$d_f[(m_f \circ T_i)(x), (m_f \circ T_i)(x')] \leqq d_g[m_g(x), m_g(x')] + \epsilon$$

whenever $x \in Q$, $x' \in Q$, and $i > j$. We infer that

$$\begin{aligned} d_f[A, (u \circ v)(A)] &= d_f[m_f(a), (u \circ m_g)(b)] \\ &= d_f[m_f(a), \lim_{i \to \infty} (m_f \circ T_i)(b)] \\ &= \lim_{i \to \infty} d_f[m_f(a), (m_f \circ T_i)(b)] \\ &= \lim_{i \to \infty} d_f[(m_f \circ T_i)[\overset{-1}{T_i}(a)], (m_f \circ T_i)(b)] \\ &\leqq \limsup_{i \to \infty} d_g[(m_g \circ \overset{-1}{T_i})(a), m_g(b)] + \epsilon \\ &= d_g[(v \circ m_f)(a), v(A)] + \epsilon = d_g[v(A), v(A)] + \epsilon = \epsilon. \end{aligned}$$

It follows that

$$(u \circ v)(A) = A \text{ for } A \in \mathfrak{M}_f.$$

Similarly we find that

$$(v \circ u)(A) = A \text{ for } A \in \mathfrak{M}_g.$$

The proof is complete.

The preceding proposition has the following corollaries:

If V is an open subset of $\mathfrak{M}_g$ and W is a compact subset of $\{x \mid m_g(x) \in V\}$, then there is a positive integer j such that

$$T_i(W) \subset \{x \mid m_f(x) \in u(V)\} \text{ for } i > j.$$

If V is an open subset of $\mathfrak{M}_g$, then

$$\lambda_k(g, \{x \mid m_g(x) \in V\}) = \lambda_k(f, \{x \mid m_f(x) \in u(V)\}).$$

If V is an open subset of $\mathfrak{M}_g$, $R \in G_n$, and $z \in E_k$, then

$$\begin{aligned} &S(\overset{k}{p}_n \circ R \circ g, \{x \mid m_g(x) \in V\}, z) \\ &\qquad = S(\overset{k}{p}_n \circ R \circ f, \{x \mid m_f(x) \in u(V)\}, z), \\ &M(\overset{k}{p}_n \circ R \circ g, \{x \mid m_g(x) \in V\}, z) \\ &\qquad = M(\overset{k}{p}_n \circ R \circ f, \{x \mid m_f(x) \in u(V)\}, z). \end{aligned}$$

If $A \in \mathfrak{M}_g$ *and* $r > 0$, *then*

$$u(m_g[\Delta_g(A, r)]) = m_f(\Delta_f[u(A), r]).$$

If $A \in \mathfrak{M}_g$, *then*

$$L_k^*(g, A) = L_k^*[f, u(A)], \qquad L_{*k}(g, A) = L_{*k}[f, u(A)],$$
$$\mathcal{E}_k(g, A) = \mathcal{E}_k[f, u(A)].$$

If $V \subset \mathfrak{M}_g$, *then* $H_g^k(V) = H_f^k[u(V)]$.
If $V \subset \mathfrak{M}_g$ *and* $y \in E_n$, *then*

$$N(l_g, V, y) = N(l_f, u(V), y).$$

It follows that the Hausdorff N areas of l_f and l_g are equal. However the Hausdorff N areas of f and g need not be equal, and the integralgeometric N areas of f and g need not be equal. The reason why $N(f, X, y)$ may differ from $N(g, X, y)$ is that $N(h, X, y)$ is not lower semi-continuous with respect to h on $C_n(X)$. The invariants S, M, and λ_k of Fréchet equivalence are lower semi-continuous functions of compact origin.

We observe also that if $\dot{X}$ is the set of all those points of X which do not have a k-cell neighborhood in X and if

$$\dot{\mathfrak{M}}_f = \mathfrak{M}_f \cap \{A \mid \Delta_f(A, r) \cap \dot{X} \neq 0 \text{ for all } r > 0\},$$

then $u(\dot{\mathfrak{M}}_g) = \dot{\mathfrak{M}}_f$.

A *cyclic mapping* is a function $f \in C_n(X)$ whose middle space $\mathfrak{M}_f$ has no cut point. A *nondegenerate mapping* is a cyclic mapping f such that $\dot{\mathfrak{M}}_f$ has more than one element.

For example a cyclic mapping of a two-cell $X \subset E_2$ is one whose continua of constancy do not separate X, while a nondegenerate mapping is one whose continua of constancy separate neither X nor E_2.

The usefulness of the concept of Fréchet equivalence for the theory of two-dimensional Lebesgue area is largely due to the following proposition of C. B. Morrey ([MCB], [CE 4]):

If X is a two-cell, $f \in C_n(X)$, $L_2(f) < \infty$, and f is a nondegenerate mapping, then f has a Fréchet equivalent g whose partial derivatives $D_1g(x)$ and $D_2g(x)$ exist for $\mathcal{L}_2$ almost all x in $X - \dot{X}$ and whose Lebesgue area $L_2(g)$ equals the Dirichlet integral

$$\frac{1}{2}\int_{X-\dot{X}} |D_1g(x)|^2 + |D_2g(x)|^2 d\mathcal{L}_2x.$$

This proposition makes the techniques of classical analysis available for the discussion of all Fréchet invariant properties of nondegenerate mappings of two-cells with finite Lebesgue area. Furthermore there are two devices by which results concerning certain geometric properties can be extended from nondegenerate to arbitrary mappings of two-cells.

If X is a two-cell, $f\in C_n(X)$, and $r_1, r_2, r_3, \cdots$ are the monotone retractions of $\mathfrak{M}_f$ onto its proper cyclic elements, then

$$(l_f\circ r_1\circ m_f),\ (l_f\circ r_2\circ m_f),\ (l_f\circ r_3\circ m_f),\ \cdots$$

are cyclic mappings in $C_n(X)$. They are called the *cyclic components of* f. Certain properties possessed by every cyclic component of f are inherited by f. Such properties of mappings are said to be *cyclically extensible* (see [R 5]).

If X is a two-cell, $f\in C_n(X)$, and f is a cyclic but not a nondegenerate mapping, then f maps $\dot{X}$ onto a single point $y\in E_n$. For each $\epsilon>0$ there exists a two-cell T such that

$$\{x \mid |f(x)-y|\geqq\epsilon\}\subset T\subset X-\dot{X}.$$

The mapping $(f|T)\in C_n(T)$ is called an *internal approximation of* f. Every cyclic component of $(f|T)$ is a nondegenerate mapping in $C_n(T)$. Certain properties possessed by every internal approximation of f are inherited by f. Such properties of mappings are said to be *internally extensible.*

Now if a property of mappings of two-cells is both cyclically and internally extensible and if it is possessed by every nondegenerate mapping of finite Lebesgue area, then this property is successively inherited by all internal approximations of those cyclic mappings of finite Lebesgue area which fail to be nondegenerate, by all cyclic mappings of finite Lebesgue area, and by all mappings of finite Lebesgue area.

8. Mappings of locally connected finitely connected plane continua. For any two subsets A and B of E_2 we define

$$\gamma(A, B)=\sup_{x\in B}\ \inf_{y\in A}\ |x-y|$$

and the Hausdorff distance

$$\rho(A, B)=\sup\{\gamma(A, B), \gamma(B, A)\}.$$

Letting

$$\Gamma=\{A \mid A \text{ is a nonvacuous compact subset of } E_2\},$$
$$\Omega=\Gamma\cap\{A \mid A \text{ is connected}\},$$

we note that Γ is a boundedly compact metric space with respect to the Hausdorff distance, and that the set Ω of all plane continua is a closed subset of Γ.

With each $A \subset E_2$ we associate

$$A^\sharp = \text{the union of all bounded components of } E_2 - A,$$
$$\sigma(A) = \inf \{\gamma(A, W) \mid W \text{ is a component of } E_2 - A\}.$$

The following theorem characterizes the finitely connected compact subsets of the plane by means of the function σ.

THEOREM 8.1. *If* $A \in \Gamma$, *then*

$$\sigma(A) > 0$$

if and only if the number of components of $E_2 - A$ *is finite.*

PROOF. Since $\gamma(A, W) > 0$ for every component W of $E_2 - A$, the second condition implies the first.

Now suppose $\sigma(A) > 0$. Then each component of $E_2 - A$ contains an open circular disc of radius $\sigma(A)$, whence

$$\pi \cdot [\sigma(A)]^2 \cdot (\text{number of bounded components of } E_2 - A) \leq \mathcal{L}_2(A^\sharp) < \infty,$$

because $A^\sharp$ is bounded. Hence the number of bounded components of $E_2 - A$ is finite. Furthermore $E_2 - A$ has precisely one unbounded component.

LEMMA 8.2. *If* A *and* B *are such subsets of* E_2 *that*

$$\gamma(A, B) < \sigma(A) \quad \textit{and} \quad \text{Boundary } B \subset A,$$

then

$$B \subset A.$$

PROOF. Otherwise $E_2 - A$ has a component W which meets B. It follows that

$$W \cap \text{Closure }(W \cap B) \cap \text{Closure }(W - B)$$
$$\subset W \cap \text{Boundary } B \subset W \cap A = 0,$$
$$W - B = 0, \quad W \subset B, \quad \sigma(A) \leq \gamma(A, W) \leq \gamma(A, B) < \sigma(A).$$

LEMMA 8.3. *If*

$$A \in \Omega, \quad B \in \Omega, \quad A \cap B = 0, \quad A^\sharp \cap B^\sharp \neq 0,$$

then

$$\textit{either} \quad A \subset B^\sharp \quad \textit{or} \quad B \subset A^\sharp.$$

PROOF. If $A \not\subset B^f$, then $A \cap B^f = 0$ because the connected set A, which does not meet B, is contained in a single component of $E_2 - B$. It follows that

$$A \cap (B \cup B^f) = 0.$$

If also $B \not\subset A^f$, then

$$A^f \cap (B \cup B^f) = 0$$

because the connected set $B \cup B^f$, which does not meet A, is contained in a single component of $E_2 - A$ but not in A^f. It follows that $A^f \cap B^f = 0$, contrary to hypothesis.

LEMMA 8.4. *If*

$$A \in \Gamma, \qquad B \in \Omega, \qquad A \subset B^f,$$

then

$$A \cup A^f \subset B^f.$$

PROOF. The unbounded connected set $E_2 - B^f$, which does not meet A, is contained in the unbounded component of $E_2 - A$. Thus

$$E_2 - B^f \subset E_2 - (A \cup A^f), \qquad A \cup A^f \subset B^f.$$

LEMMA 8.5. *If* $A \in \Gamma$, $A^f = 0$, *and* $\epsilon > 0$, *then there is a* $\delta > 0$ *such that*

$$\gamma(A, B^f) \leqq \epsilon \quad \textit{whenever} \quad \gamma(A, B) < \delta.$$

PROOF. We choose $r > 0$ so that

$$A \subset \{x \mid |x| < r\}$$

and let

$$f(x) = \inf_{y \in A} |x - y| \qquad \text{for } x \in E_2.$$

Each point of the compact set

$$V = \{x \mid |x| \leqq r \text{ and } f(x) \geqq \epsilon\}$$

is in some unbounded connected open set W whose closure does not meet A. Hence V is contained in the union of finitely many such sets $W_1, W_2, \cdots, W_m$. Defining

$$Z = \{x \mid |x| \geqq r\} \cup \bigcup_{i=1}^{m} W_i,$$

$$\delta = \inf_{x \in Z} f(x),$$

we see that $\delta > 0$ and that Z is an unbounded connected set for which

$$\{x \mid f(x) \geqq \epsilon\} \subset Z.$$

If $\gamma(A, B) < \delta$, then Z does not meet B and is contained in the unbounded component of $E_2 - B$. Thus

$$Z \subset E_2 - B^\delta, \qquad B^\delta \subset E_2 - Z \subset \{x \mid f(x) < \epsilon\}, \qquad \gamma(A, B^\delta) \leqq \epsilon.$$

LEMMA 8.6. *If f maps the topological space X into the circle Y and if X has connected subsets A, B, C such that*

$$f(A \cap B) \not\subset f(C), \qquad f(B \cap C) \not\subset f(A), \qquad f(C \cap A) \not\subset f(B),$$

then f is an essential map of X into Y.

PROOF. Otherwise $f = h \circ g$ where g is a map of X into E_1 and h is a covering map of E_1 onto Y.

Choosing a, b, c so that

$$\begin{aligned} &a \in B \cap C, \qquad && f(a) \notin f(A), \\ &b \in C \cap A, && f(b) \notin f(B), \\ &c \in A \cap B, && f(c) \notin f(C), \end{aligned}$$

we note that $g(a)$, $g(b)$, $g(c)$ are distinct and that one of these three real numbers lies between the two others.

If $g(a)$ is between the numbers $g(b)$ and $g(c)$, both of which are in the interval $g(A)$, then $g(a) \in g(A)$ and $f(a) \in f(A)$, contrary to our choice of a.

The other two alternatives can be similarly eliminated.

LEMMA 8.7. *If g maps the topological space X into*

$$Z = E_2 - \{(\xi, \eta)\}$$

and if X has connected subsets E, W, N, S such that

$$\begin{gathered} E \cap N, \ N \cap W, \ W \cap S, \ S \cap E \text{ are nonvacuous}, \\ g(E) \subset \{(u, v) \mid u > \xi\}, \qquad g(W) \subset \{(u, v) \mid u < \xi\}, \\ g(N) \subset \{(u, v) \mid v > \eta\}, \qquad g(S) \subset \{(u, v) \mid v < \eta\}, \end{gathered}$$

then g is an essential map of X into Z.

PROOF. The preceding lemma is applicable with

$$f(x) = \frac{g(x) - (\xi, \eta)}{|g(x) - (\xi, \eta)|} \text{ for } x \in X,$$
$$A = E, \qquad B = N, \qquad C = W \cup S.$$

LEMMA 8.8. *If* $A\in\Gamma$, $g\in C_2(A\cup A^f)$, *and if* $g|A$ *is an essential map of* A *into* $E_2-\{z\}$, *then* z *is a stable value of* g.

PROOF. Assuming that

$$0 < r < \inf_{x\in A} |g(x) - z|$$

we consider the cohomology diagram:

$$\begin{tikzcd}
H^2[E_2, E_2 - K(z, r)] \arrow[d, "g^*"] & H^1[E_2 - K(z, r)] \arrow[l, "\delta"'] \arrow[d, "(g|A)^*"] & \\
H^2(A \cup A^f, A) & H^1(A) \arrow[l, "\delta'"'] & H^1(A \cup A^f) \arrow[l, "i^*"']
\end{tikzcd}$$

Since $A\cup A^f$ does not separate E_2, the group $H^1(A\cup A^f)$ and the homomorphism i^* are trivial. By exactness δ' is an isomorphism. From this and the fact that the homomorphism $(g|A)^*$ is nontrivial it follows that the homomorphism

$$g^* \circ \delta = \delta' \circ (g|A)^*$$

is nontrivial. Hence g^* is a nontrivial homomorphism.

LEMMA 8.9. *If* $A\in\Gamma$, $B\in\Gamma$, $C\in\Gamma$,

$$A \cup A^f \subset B^f, \qquad C \subset (B \cup B^f) - A^f,$$
$$g \in C_2[(B \cup B^f) - A^f],$$
$g|A$ *is an inessential map of* A *into* $E_2 - \{z\}$,
$g|C$ *is an essential map of* C *into* $E_2 - \{z\}$,

then z *is a stable value of* g.

PROOF. There is a mapping

$$f \in C_2(B \cup B^f)$$

such that

$$f|[(B \cup B^f) - A^f] = g, \qquad f(A \cup A^f) \subset E_2 - \{z\}.$$

Noting that

$$f|C = g|C, \qquad C \cup C^f \subset B \cup B^f,$$

we infer that z is a stable value of $f|(C\cup C^f)$ and of f. Since

$$\{x | f(x) = z\} \subset (B \cup B^f) - (A \cup A^f)$$

and $(B\cup B^\dagger)-(A\cup A^\dagger)$ is open in $B\cup B^\dagger$, it follows that z is a stable value of

$$f\,|\,[(B\cup B^\dagger)-(A\cup A^\dagger)] = g\,|\,[(B\cup B^\dagger)-(A\cup A^\dagger)]$$

and of g.

THEOREM 8.10. *Suppose*:

(1) *X is a locally connected, locally compact subset of E_2 for which* $\sigma(X)>0$.

(2) *$G\subset X$ and G is open relative to X.*

(3) *$u\in C_1(G)$, $v\in C_1(G)$, $g\in C_2(G)$, and*

$$g(x) = (u(x), v(x)) \text{ for } x\in G.$$

(4) *For each $z=(z_1, z_2)\in E_2$, $\nu(z)$ is the supremum of the set of all positive integers m with the following property*:

G contains m disjoint continua $A_1, A_2, \cdots, A_m$ such that

$$z_1 = u(x) \text{ for } x\in \bigcup_{i=1}^{m} A_i,$$

$$z_2 \in \text{Interior } v(A_i) \text{ for } i = 1, 2, \cdots, m.$$

Then:

(5) *$\nu(z)\leqq S(g, G, z)$ for $\mathcal{L}_2$ almost all z in E_2.*

PROOF. Let m be a positive integer.

We choose points $p_1, p_2, p_3, \cdots$ which form a countable dense subset of E_2 and define

$$P_j = \Omega\cap \{A \mid A\subset G \text{ and } p_j\in A^\dagger\} \qquad \text{for } j = 1, 2, 3, \cdots.$$

We also define

$$P_0 = \Omega\cap\{A \mid A\subset G \text{ and } A^\dagger = 0\}.$$

Corresponding to each m-termed sequence

$$q = (q_1, q_2, \cdots, q_m)$$

of non-negative integers, and for any rational numbers $r<s$, we let

$$F(q, r, s)$$

be the set of all m-termed sequences

$$A = (A_1, A_2, \cdots, A_m)$$

with the following properties:

$$A_i\in P_{q_i} \text{ for } i = 1, 2, \cdots, m;$$

$A_1, A_2, \cdots, A_m$ are disjoint;

u is constant on $\bigcup_{i=1}^{m} A_i$;

$\inf_{x\in A_i} v(x) < r < s < \sup_{x\in A_i} v(x)$ for $i = 1, 2, \cdots, m$.

Setting

$$U(A) = u(x) \text{ whenever } A \in F(q, r, s) \text{ and } x \in \bigcup_{i=1}^{m} A_i,$$

we observe that the set

$$\{z \mid \nu(z) \geq m\}$$

is the union of the countably many cartesian products

$$U[F(q, r, s)] \times \{\eta \mid r < \eta < s\}.$$

In order to prove that

$$S(g, G, z) \geqq m \text{ for } \mathcal{L}_2 \text{ almost all } z \text{ in } \{z \mid \nu(z) \geqq m\}$$

it is therefore sufficient to verify the following statement:

For each of the above triples (q, r, s) *there is a set* μ *such that*

$$\mu \subset U[F(q, r, s)], \qquad U[F(q, r, s)] - \mu \textit{ is countable,}$$
$$S[g, G, (\xi, \eta)] \geqq m \textit{ whenever } \xi \in \mu \textit{ and } r < \eta < s.$$

For the purpose of proving this we henceforth fix q, r, s and abbreviate "$F(q, r, s)$" by "F."

Associating with each $A \in F$ the $2m$-termed sequence

$$(A_1, A_1 \cup A_1^{\sharp}, A_2, A_2 \cup A_2^{\sharp}, \cdots, A_m, A_m \cup A_m^{\sharp})$$

we establish a one-to-one correspondence between F and a subset of the $2m$-fold cartesian product of Γ. Since Γ is separably metrized by ρ, the set F becomes a separable metric space with the distance

$$\sum_{i=1}^{m} \rho(A_i, B_i) + \rho(A_i \cup A_i^{\sharp}, B_i \cup B_i^{\sharp})$$

between $A \in F$ and $B \in F$.

We let R be the set of those points of F at which the function U has either a relative maximum or a relative minimum, infer that

$$U(R) \text{ is countable,}$$

and define

$$\mu = U(F) - U(R).$$

Now suppose

$$\xi \in \mu, \qquad r < \eta < s, \qquad \xi = U(A), \qquad A \in F - R.$$

We successively choose

$$t, G_1, G_2, \cdots, G_m, t', V_1, V_2, \cdots, V_m, W_1, W_2, \cdots, W_m, t'', B, C$$

in such a way that

$0<t<\sigma(X)$,
$G_i = X \cap \{x \mid \inf_{y \in A_i} |x-y| < t\}$,
$G_1, G_2, \cdots, G_m$ are disjoint subsets of G,
$0<t'<t$,
$\gamma(A_i, Z^\dagger)<t$ whenever $q_i=0$ and $\gamma(A_i, Z)<t'$,
V_i and W_i are connected and open relative to X,
Closure $(V_i \cup W_i)$ is a compact subset of G_i,
$\gamma(A_i, V_i \cup W_i)<t'$,
$V_i \cap A_i$ and $W_i \cap A_i$ are nonvacuous,
$v(x)<r$ for $x \in V_i$, $v(x)>s$ for $x \in W_i$,
$0<3t''<t'$,
$V_i \cap Z$ and $W_i \cap Z$ are nonvacuous whenever $Z \subset X$ and $\gamma(Z, A_i) < t''$,
$B \in F$, $C \in F$,
$\sum_{i=1}^{m} \rho(A_i, B_i) + \rho(A_i \cup A_i^\dagger, B_i \cup B_i^\dagger) < t''$,
$\sum_{i=1}^{m} \rho(A_i, C_i) + \rho(A_i \cup A_i^\dagger, C_i \cup C_i^\dagger) < t''$,
$U(B) < \xi < U(C)$.

It follows for each integer i between 1 and m that $V_i \cap B_i$, $W_i \cap B_i$, $V_i \cap C_i$, $W_i \cap C_i$ are nonvacuous and that one of the following three alternatives holds:

(i) $q_i=0$,
(ii) $q_i>0$ and $B_i \cup B_i^\dagger \subset C_i^\dagger$,
(iii) $q_i>0$ and $C_i \cup C_i^\dagger \subset B_i^\dagger$.

In case (i) holds we define

$$Z_i = B_i \cup C_i \cup \text{Closure } (V_i \cup W_i), \qquad X_i = Z_i \cup Z_i^\dagger$$

and conclude that

$\gamma(A_i, Z_i)<t'$,
$\gamma(X, X_i) \leqq \gamma(A_i, X_i) < t < \sigma(X)$,
Boundary $X_i \subset Z_i \subset X$,
$X_i \subset X$, $X_i \subset G_i$,
$g \mid Z_i$ is an essential map of Z_i into $E_2 - \{(\xi, \eta)\}$,
(ξ, η) is a stable value of $g \mid X_i$,

$S[g, G_i, (\xi, \eta)] \geqq 1.$

In case (ii) holds we define

$$X_i = (C_i \cup C_i^\dagger) - B_i^\dagger,$$

choose continua V_i' and W_i' such that

$$V_i' \subset X_i \cap V_i,\ W_i' \subset X_i \cap W_i,$$

$$V_i' \cap B_i,\ W_i' \cap B_i,\ V_i' \cap C_i,\ W_i' \cap C_i \text{ are nonvacuous,}$$

let

$$Z_i = B_i \cup C_i \cup V_i' \cup W_i',$$

and conclude that

$$\begin{aligned}\gamma(X, X_i) &\leqq \gamma(A_i, X_i) \leqq \gamma(A_i, B_i) + \gamma(B_i, X_i)\\ &= \gamma(A_i, B_i) + \gamma(B_i \cup B_i^\dagger, X_i)\\ &\leqq \gamma(A_i, B_i) + \gamma(B_i \cup B_i^\dagger, C_i \cup C_i^\dagger) < 3t'' < t' < t < \sigma(X),\end{aligned}$$

Boundary $X_i \subset B_i \cup C_i \subset X$,
$X_i \subset X$, $X_i \subset G_i$, $Z_i \subset X_i$,
$g | Z_i$ is an essential map of Z_i into $E_2 - \{(\xi, \eta)\}$,
$g | B_i$ is an inessential map of B_i into $E_2 - \{(\xi, \eta)\}$,
(ξ, η) is a stable value of $g | X_i$,
$S[g, G_i, (\xi, \eta)] \geqq 1.$

Since the third alternative is quite similar to the second, we obtain the result that

$$S[g, G_i, (\xi, \eta)] \geqq 1 \text{ for } i = 1, 2, \cdots, m$$

whence it follows that

$$S[g, G, (\xi, \eta)] \geqq \sum_{i=1}^{m} S[g, G_i, (\xi, \eta)] \geqq m.$$

The proof is complete.

Actually it is true that

$$\nu(z) = S(g, G, z) \quad \textit{for } \mathcal{L}_2 \textit{ almost all } z \textit{ in } E_2$$

because $\nu(z) \geqq S(g, G, z)$ for all z in E_2, but we shall neither prove nor use this fact here.

THEOREM 8.11. *Suppose*:

(1) *X is a locally connected, locally compact subset of E_2 for which* $\sigma(X) > 0$.

(2) *$G \subset X$ and G is open relative to X.*

(3) *$u \in C_1(G)$, $v \in C_1(G)$, $w \in C_1(G)$, $f \in C_3(G)$, $g \in C_2(G)$, $h \in C_2(G)$, and*

$$f(x) = (u(x),\ w(x),\ v(x)),\ g(x) = (u(x),\ v(x)),$$
$$h(x) = (u(x),\ w(x)) \text{ for } x \in G.$$

(4) *$\int_{E_2} S(g, G, z)\,d\mathcal{L}_2 z < \infty$.*

Then:

(5) *$\mathcal{L}_2[l_h(\Gamma \cap \mathfrak{M}_h - \mathfrak{M}_f)] = 0$.*

Proof. The elements of the set

$$\Gamma \cap \mathfrak{M}_h - \mathfrak{M}_f$$

are continua on which the functions h, u, and w are constant, but the functions f and v are not constant. The denial of (5) implies the existence of rational numbers $r < s$ and a Borel set P of $\mathfrak{M}_h$ such that

$$P \subset \mathfrak{M}_h \cap \Gamma, \qquad \mathcal{L}_2[l_h(P)] > 0,$$
$$\inf_{x \in A} v(x) < r < s < \sup_{x \in A} v(x) \text{ for } A \in P.$$

Defining the function ν as in the preceding theorem we find that

$$\nu(\xi, \eta) \geqq (\text{number of elements of } \{\zeta \mid (\xi, \zeta) \in l_h(P)\}) \qquad \text{for } r < \eta < s,$$

$$\infty > \int_{E_2} S(g, G, z)\,d\mathcal{L}_2 z \geqq \int_{E_2} \nu(z)\,d\mathcal{L}_2 z \geqq \int_{-\infty}^{\infty}\int_r^s \nu(\xi, \eta)\,d\mathcal{L}_1\eta\, d\mathcal{L}_1\xi$$

$$\geqq (s - r) \cdot \int_{-\infty}^{\infty} (\text{number of elements of } \{\zeta \mid (\xi, \zeta) \in l_h(P)\})\,d\mathcal{L}_1\xi,$$

$$\{\zeta \mid (\xi, \zeta) \in l_h(P)\} \text{ is finite for } \mathcal{L}_1 \text{ almost all } \xi \text{ in } E_1,$$

$$0 = \int_{-\infty}^{\infty} \mathcal{L}_1(\{\zeta \mid (\xi, \zeta) \in l_h(P)\})\,d\mathcal{L}_1\xi = \mathcal{L}_2[l_h(P)] > 0.$$

For the special case in which $X = G$ is a 2-cell the preceding Theorem 8.11 was proved by L. Cesari in [CE 2]. He used it in showing that $L_3(f)$ does not exceed the sum of the Lebesgue areas of the three orthogonal projections of f onto the coordinate planes.

Theorem 8.12. *Suppose:*

(1) *X is a locally connected, locally compact subset of E_2 for which $\sigma(X) > 0$.*

(2) *$G \subset X$ and G is open in X.*

(3) *$f \in C_n(X)$, $n \geqq 3$, and $\lambda_3(f, G) < \infty$.*

(4) *$R \in G_n$ and $h = p_n^2 \circ R \circ f$.*
Then:
(5) $\mathcal{L}_2[l_h(\Gamma \cap \mathfrak{M}_h - \mathfrak{M}_f)] = 0$.

Proof. Defining the functions

$$u, w, v_1, \cdots, v_{n-2} \in C_1(G);$$
$$g_1, \cdots, g_{n-2} \in C_2(G); \qquad f_1, \cdots, f_{n-2} \in C_3(G)$$

so that

$$(R \circ f)(x) = (u(x), w(x), v_1(x), \cdots, v_{n-2}(x)),$$
$$g_j(x) = (u(x), v_j(x)), \qquad f_j(x) = (u(x), w(x), v_j(x))$$

for $x \in G$ and $j = 1, \cdots, n-2$, we conclude that

$$h(x) = (u(x), w(x)) \quad \text{for } x \in G,$$

$$\int_{E_2} S(g_j, G, z)\, d\mathcal{L}_2 z \leqq \lambda_2(g_j, G) \leqq \lambda_2(f, G) < \infty \text{ for } j = 1, \cdots, n-2,$$

$$\Gamma \cap \mathfrak{M}_h - \mathfrak{M}_f \subset \bigcup_{j=1}^{n-2} \Gamma \cap \mathfrak{M}_h - \mathfrak{M}_{f_j},$$

$$\mathcal{L}_2[l_h(\Gamma \cap \mathfrak{M}_h - \mathfrak{M}_f)] \leqq \sum_{j=1}^{n-2} \mathcal{L}_2[l_h(\Gamma \cap \mathfrak{M}_h - \mathfrak{M}_{f_j})] = 0.$$

Theorem 8.13. *Suppose*:
(1) *X is a locally connected, locally compact subset of E_2 for which* $\sigma(X) > 0$.
(2) *$G \subset X$ and G is open in X.*
(3) *$f \in C_3(G)$ and $\lambda_2(f, G) < \infty$.*
(4) $P = \mathfrak{M}_f \cap \{A \mid A^\sharp \neq 0$ *and* $\limsup_{r \to 0+} D[f, r, \Delta_f(A, r)] > 0\}$.
Then:
(5) $\mathcal{L}_2[l_f(P)] = 0$.

Proof. It is sufficient to consider a point $q \in E_2$ and a set

$$Q \subset P \cap \{A \mid q \in A^\sharp \text{ and } S[f, G, l_f(A)] < \infty\}$$

such that l_f is univalent on Q, and to prove that Q is countable.

The subset Q of Ω is a separable metric space with the distance

$$\rho(A, B) + \rho(A \cup A^\sharp, B \cup B^\sharp)$$

between $A \in Q$ and $B \in Q$. Furthermore Q is simply ordered by the relation

$$(Q \times Q) \cap \{(A, B) \mid A \subset B^\sharp\}.$$

We recall that any function on a separable metric space into an ordered set has only countably many relative maximum or minimum values. Applying this principle to the identity map of Q we find that only countably many points of Q are isolated on the right or left. Hence Q has a countable subset Q' with the following property:

If $A \in Q - Q'$, then there exist sequences

$$B_1, B_2, B_3, \cdots; \qquad C_1, C_2, C_3, \cdots$$

of elements of Q such that

$$\begin{gathered} B_j \subset A^{f}, \quad A \subset C_j^{f} \text{ for } j = 1, 2, 3, \cdots, \\ \lim_{j \to \infty} \rho(A, B_j) + \rho(A \cup A^{f}, B_j \cup B_j^{f}) = 0, \\ \lim_{j \to \infty} \rho(A, C_j) + \rho(A \cup A^{f}, C_j \cup C_j^{f}) = 0. \end{gathered}$$

We shall prove that this property of A violates our assumptions about Q, thus showing that $Q = Q'$ and Q is countable.

The sets

$$X_j = (C_j \cup C_j^{f}) - B_j^{f}$$

are continua for which

$$\begin{gathered} A \subset \text{Interior } X_j, \text{ Boundary } X_j \subset B_j \cup C_j \subset G \subset X, \\ \gamma(A, X_j) \leqq \gamma(A, B_j) + \gamma(B_j \cup B_j^{f}, C_j \cup C_j^{f}), \\ \lim_{j \to 0} \gamma(A, X_j) = 0 < \sigma(X), \\ X_j \subset G \text{ for large } j, \\ \lim_{j \to \infty} \sup_{x \in X_j} d_f[A, m_f(x)] = 0. \end{gathered}$$

There are positive numbers $r_1, r_2, r_3, \cdots$ such that

$$\lim_{j \to \infty} r_j = 0,$$

$$D[f, r_j, \Delta_f(A, r_j)] > 0 \qquad \text{for } j = 1, 2, 3, \cdots.$$

Passing to subsequences we may assume that

$$\begin{gathered} G \supset X_j \supset \Delta_f(A, r_j) \supset X_{j+1}, \\ f(\text{Boundary } X_j) \subset \{l_f(B_j); l_f(C_j)\} \subset E_2 - K[l_f(A), r_j] \end{gathered}$$

for every positive integer j.

Now consider the cohomology diagram:

$$\begin{tikzcd}
H^2(E_2, E_2 - K[l_f(A), r_j]) \arrow[d, "f^*"] & H^1(E_2 - K[l_f(A), r_j]) \arrow[l, "\delta"'] \arrow[d, "(f \mid \text{Boundary } X_j)^*"] \\
H^2(X_j, \text{Boundary } X_j) & H^1(\text{Boundary } X_j) \arrow[l, "\delta'"']
\end{tikzcd}$$

Since $f(\text{Boundary } X_j)$ is a finite set, the homomorphism $(f \mid \text{Boundary } X_j)^*$ is trivial. It follows that the homomorphism

$$f^* \circ \delta = \delta' \circ (f \mid \text{Boundary } X_j)^*$$

is trivial. Furthermore δ is an isomorphism onto the domain of f^*. Hence f^* is a trivial homomorphism.

We conclude that X_j contains an essential canonical region Z_j of $(f, G, l_f(A), r_j)$ other than $\Delta_f(A, r_j)$. Then the sets $Z_1, Z_2, Z_3, \cdots$ are disjoint and

$$S[f, G, l_f(A)] = \infty.$$

THEOREM 8.14. *Suppose*:

(1) *X is a locally connected, locally compact subset of E_2 for which* $\sigma(X) > 0$.

(2) $f \in C_n(X)$ *and* $n \geqq 3$.

(3) $V_1, V_2, V_3, \cdots$ *and W are open subsets of* $\mathfrak{M}_f$.

(4) $U_i = \{x \mid m_f(x) \in V_i\}$ *and* $\lambda_2(f, U_i) < \infty$ *for* $i = 1, 2, 3, \cdots$.

(5) $\bigcup_{i=1}^{\infty}(V_i \cap \Gamma \cap \{A \mid A^f = 0$ *and* $A \subset \text{Interior } U_i\}) \subset W$.

(6) $R \in G_n$ *and* $h = p_n^2 \circ R \circ f \mid \bigcup_{i=1}^{\infty} U_i$.

Then:

(7) $\mathcal{L}_2[l_h(\Gamma \cap \mathfrak{M}_h - \mathfrak{M}_f)] = 0$.

(8) $S(h, \bigcup_{i=1}^{\infty} U_i, z) \leqq S(h, \{x \mid f(x) \in W\}, z)$ *for $\mathcal{L}_2$ almost all z in E_2.*

(9) $S(h, \bigcup_{i=1}^{\infty} U_i, z) \leqq \sum_{i=1}^{\infty} S(h, U_i, z)$ *for $\mathcal{L}_2$ almost all z in E_2.*

PROOF OF (7). We define

$$Q = \bigcup_{i=1}^{\infty} l_{(h|U_i)}[\Gamma \cap \mathfrak{M}_{(h|U_i)} - \mathfrak{M}_{(f|U_i)}],$$

note that

$$\mathcal{L}_2(Q) = 0,$$

and shall complete the proof by verifying that

$$l_h[\Gamma \cap \mathfrak{M}_h - \mathfrak{M}_f] \subset Q.$$

We suppose

$$A \in \Gamma \cap \mathfrak{M}_h - \mathfrak{M}_f,$$

note that $m_f(A)$ is a compact subset of

$$\bigcup_{i=1}^{\infty} V_i,$$

and choose closed sets $F_i \subset V_i$ such that

$$m_f(A) \subset \bigcup_{i=1}^{\infty} F_i.$$

Then

$$m_f(A) = \bigcup_{i=1}^{\infty} [F_i \cap m_f(A)],$$

$$m_f(A) \text{ is a nondegenerate continuum,}$$

$$m_f(A) \text{ is not zero-dimensional,}$$

and there exists a positive integer i such that

$$F_i \cap m_f(A) \text{ is not zero-dimensional,}$$

$$F_i \cap m_f(A) \text{ is not totally disconnected.}$$

Choosing a nondegenerate component B of $F_i \cap m_f(A)$ and a set C such that

$$C \in \mathfrak{M}_{(k|U_i)}, \quad \{x \mid m_f(x) \in B\} \subset C \subset A,$$

we conclude that

$$C \in \Gamma \cap \mathfrak{M}_{(k|U_i)} - \mathfrak{M}_{(f|U_i)},$$

$$l_k(A) = l_{(k|U_i)}(C) \in Q.$$

Proof of (8) and (9). We define

$$P_i = \mathfrak{M}_k \cap \{A \mid A \subset U_i,\ A^f \neq 0, \text{ and } \limsup_{t\to 0+} D[k, t, \Delta_k(A, t)] > 0\}$$

for $i = 1, 2, 3, \cdots$ and infer that

$$\mathcal{L}_2\left(l_k\left[(\Gamma \cap \mathfrak{M}_k - \mathfrak{M}_f) \cup \bigcup_{i=1}^{\infty} P_i\right]\right) = 0.$$

Now suppose

$$z \in E_2 - l_k\left[(\Gamma \cap \mathfrak{M}_k - \mathfrak{M}_f) \cup \bigcup_{i=1}^{\infty} P_i\right]$$

and Z is an essential canonical region of $(k, \bigcup_{i=1}^{\infty} U_i, z, r)$.

According to Theorem 6.10 there is a set A such that

$$A \in \Gamma \cap \mathfrak{M}_k, \qquad l_k(A) = z, \qquad \Delta_k(A, r) = Z,$$
$$\limsup_{t \to 0+} D[k, t, \Delta_k(A, t)] > 0.$$

It follows that

$$A \in \mathfrak{M}_f, \quad A \subset \text{Interior } U_i \text{ for some positive integer } i,$$
$$A^{\dagger} = 0, \qquad A \subset \{x \mid m_f(x) \in W\}$$

and we can choose t so that

$$0 < t < r, \qquad D[k, t, \Delta_k(A, t)] > 0,$$
$$\Delta_k(A, t) \subset U_i \cap \{x \mid m_f(x) \in W\}.$$

Hence Z contains the essential canonical region $\Delta_k(A, t)$ of $(k \mid U_i, U_i, z, t)$ and of $(k \mid \{x \mid m_f(x) \in W\}, \{x \mid m_f(x) \in W\}, z, t)$.

THEOREM 8.15. *Suppose:*

(1) *X is a 2-cell.*

(2) *$f \in C_n(X)$ and $L_2(f) < \infty$.*

Then:

(3) $L_2(f) \leqq H_f^2[(\mathfrak{M}_f - \mathfrak{M}_f') \cap \{A \mid \mathcal{E}_2(f, A) \geqq 1\}]$.

(4) $L_2(f) = H_f^2[(\mathfrak{M}_f - \mathfrak{M}_f') \cap \{A \mid \mathcal{E}_2(f, A) = 1\}]$.

(5) $L_2^*(f, A) = L_{*2}(f, A) = \mathcal{E}_2(f, A) = 0$ *or* 1 *for H_f^2 almost all A in $\mathfrak{M}_f$.*

(6) *The stable area*

$$\int_{E_2} S(p_n^2 \circ R \circ f, X, z)\, d\mathcal{L}_2 z$$

of the projection $p_n^2 \circ R \circ f$ of f is continuous with respect to R on G_n.

(7) *If $R \in G_n$, then*

$$S(p_n^2 \circ R \circ f, X, z) = N(p_n^2 \circ R \circ l_f, \{A \mid \mathcal{E}_2(f, A) = 1\}, z)$$

for $\mathcal{L}_2$ almost all z in E_2.

(8) *If $R \in G_n$ and W is an open subset of $\mathfrak{M}_f$, then*

$$S(p_n^2 \circ R \circ f, \{x \mid m_f(x) \in W\}, z)$$
$$= N(p_n^2 \circ R \circ l_f, W \cap \{A \mid \mathcal{E}_2(f, A) = 1\}, z)$$

for $\mathcal{L}_2$ almost all z in E_2.

PROOF OF (3) AND (6). The properties (3) and (6) of f are Fréchet invariant as well as both cyclically and internally extensive. Hence we need only consider the special case in which f is a nondegenerate mapping whose partial derivatives $D_1f(x)$ and $D_2f(x)$ exist for $\mathcal{L}_2$ almost all x in $X-\dot{X}$ and whose Lebesgue area $L_2(f)$ equals the Dirichlet integral

$$\frac{1}{2}\int_{X-\dot{X}} |D_1f(x)|^2 + |D_2f(x)|^2 d\mathcal{L}_2 x.$$

These assumptions imply that

$$|D_1f(x)| = |D_2f(x)| = (Jf(x))^{1/2},$$
$$D_1f(x) \bullet D_2f(x) = 0$$

for $\mathcal{L}_2$ almost all x in $X-\dot{X}$. Furthermore

$$\int_{E_2} S(p_n^2 \circ R \circ f, X, z) d\mathcal{L}_2 z = \int_{X-\dot{X}} J(p_n^2 \circ R \circ f)(x) d\mathcal{L}_2(x)$$

for $R\in G_n$, and for each $x\in X-\dot{X}$ such that $D_1f(x)$ and $D_2f(x)$ exist the Jacobian

$$J(p_n^2 \circ R \circ f)(x)$$

is continuous with respect to R and does not exceed $Jf(x)$.

Now (6) follows immediately.

In order to prove (3) we define

$$P = (X - \dot{X}) \cap \{x \mid Jf(x) > 0\}$$

and infer that the following ten conditions are satisfied for $\mathcal{L}_2$ almost all points x of P:

(i) $m_f(x) = \{x\} \in \mathfrak{M}_f - \dot{\mathfrak{M}}_f$.

(ii) $\lim_{r\to 0+} \mathcal{L}_2[K(x, r) - \Delta_f(x, r(Jf(x))^{1/2})]/(\pi r^2) = 0$.

(iii) There is a continuously differentiable mapping $g \in C_n(E_2)$ and a compact set G such that $\lim_{r\to 0+} \mathcal{L}_2[K(x, r) - G]/(\pi r^2) = 0$ and

$$x \in G \subset \{z \mid g(z) = f(z),\ D_1g(z) = D_1f(z),\ D_2g(z) = D_2f(z)\}.$$

(iv) There is an $R \in G_n$ such that

$$Jf(x) = Jg(x) = Jh(x), \text{ where } h = p_n^2 \circ R \circ f.$$

(v) $\lim_{r\to 0+} \mathcal{L}_2([p_n^2 \circ R \circ g][K(x, r)])/(\pi r^2) = Jf(x)$.

(vi) $\lim_{r\to 0+} \mathcal{L}_2(h[H \cap \Delta_f(x, r(Jf(x))^{1/2})])/(\pi r^2) \geqq Jf(x)$, where $H = G \cap \{z \mid Jh(z) > 0\}$.

(vii) $\lim_{r\to 0+} \mathcal{L}_2(h[H \cap \Delta_f(x, r)])/(\pi r^2) \geqq 1$.

(viii) $\mathcal{L}_2(h[H \cap \Delta_f(x, r)] - \{z \mid M[h, \Delta_f(x, r), z] \geqq 1\}) = 0$ for $r > 0$.

(ix) $p_n^2 \circ R$ is a 1-fold essential projection for f at $\{x\}$.
(x) $\mathcal{E}_2(f, \{x\}) \geqq 1$.
Certain techniques developed in [R 2], [F 1], and [W] are helpful in checking the statements (i), (ii), (iii), and (viii).

We let

$$Q = P \cap \{x \mid \{x\} \in \mathfrak{M}_f - \mathfrak{M}_f' \text{ and } \mathcal{E}_2(f, \{x\}) \geqq 1\}$$

and conclude with the aid of (i), (x), and [F 1, 3] that

$$\begin{aligned}
H_f^2[(\mathfrak{M}_f - \mathfrak{M}_f') \cap \{A \mid \mathcal{E}_2(f, A) \geqq 1\}]
&\geqq H_f^2[m_f(Q)] \geqq \int_{E_n} N(l_f, m_f(Q), y) d\mathcal{H}_n^2 y \\
&= \int_{E_n} N(f, Q, y) d\mathcal{H}_n^2 y = \int_Q Jf(x) d\mathcal{L}_2 x \\
&= \int_{X - \dot{X}} Jf(x) d\mathcal{L}_2 x = L_2(f).
\end{aligned}$$

PROOF OF (4) AND (5). We know from Theorem 6.7 that

$$L_2^*(f, A) \geqq L_{*2}(f, A) \geqq \mathcal{E}_2(f, A) \text{ for } A \in \mathfrak{M}_f$$

and infer with the help of Theorem 6.2 and (3) that

$$\begin{aligned}
\infty > L_2(f) &\geqq \int_{\mathfrak{M}_f} L_2^*(f, A) dH_f^2 A \geqq \int_{\mathfrak{M}_f} L_{*2}(f, A) dH_f^2 A \\
&\geqq \int_{\mathfrak{M}_f} \mathcal{E}_2(f, A) dH_f^2 A \geqq \int_{\mathfrak{M}_f - \mathfrak{M}_f'} E_2(f, A) dH_f^2 A \\
&= \sum_{i=1}^{\infty} H_f^2[(\mathfrak{M}_f - \mathfrak{M}_f') \cap \{A \mid \mathcal{E}_2(f, A) \geqq i\}] \\
&\geqq L_2(f) + H_f^2[(\mathfrak{M}_f - \mathfrak{M}_f') \cap \{A \mid \mathcal{E}_2(f, A) \geqq 2\}].
\end{aligned}$$

PROOF OF (7). From (4), (5), and Theorem 6.6 we know that

$$N(p_n^2 \circ R \circ l_f, \ \{A \mid \mathcal{E}_2(f, A) = 1\}, z) \geqq S(p_n^2 \circ R \circ f, X, z)$$

for $\mathcal{L}_2$ almost all z in E_2 whenever $R \in G_n$.

We define the functions u and v on G_n such that

$$u(R) = \int_{E_2} N(p_n^2 \circ R \circ l_f, \{A \mid \mathcal{E}_2(f, A) = 1\}, z) d\mathcal{L}_2 z,$$

$$v(R) = \int_{E_2} S(p_n^2 \circ R \circ f, X, z)\, d\mathcal{L}_2 z$$

for $R \in G_n$, and shall complete the proof by showing that u and v are equal.

The function v is continuous on G_n by virtue of (6).

From (4), (5), and Theorem 6.9 we infer that each of the sets

$$Y_i = \{y \mid N(l_f, \{A \mid \mathcal{E}_2(f, A) = 1\}, y) \geq i\}$$

is a Hausdorff 2 rectifiable subset of E_n. Since

$$u(R) = \sum_{i=1}^{\infty} \int_{E_2} N(p_n^2 \circ R, Y_i, z)\, d\mathcal{L}_2 z \text{ for } R \in G_n,$$

we conclude that the function u is continuous on G_n.

Thus u and v are continuous functions on G_n such that

$$u(R) \geq v(R) \text{ for } R \in G_n$$

and

$$\begin{aligned}\int_{G_n} u(R)\, d\phi_n R &= \beta(n, 2) \cdot \sum_{i=1}^{\infty} \mathcal{I}_n^2(Y_i) = \beta(n, 2) \cdot \sum_{i=1}^{\infty} \mathcal{H}_n^2(Y_i) \\ &= \beta(n, 2) \cdot \int_{E_n} N(l_f, \{A \mid \mathcal{E}_2(f, A) = 1\}, y)\, d\mathcal{H}_n^2 y \\ &= \beta(n, 2) \cdot H_f^2(\{A \mid \mathcal{E}_2(f, A) = 1\}) = \beta(n, 2) \cdot L_2(f) \\ &= \int_{G_n} v(R)\, d\phi_n R.\end{aligned}$$

It follows that $u = v$.

PROOF OF (8). Let $V_1 \supset V_2 \supset V_3 \supset \cdots$ be such open subsets of $\mathfrak{M}_f$ that

$$\mathfrak{M}_f - W = \bigcap_{i=1}^{\infty} V_i$$

and let

$$\begin{aligned} V_i' &= \{x \mid m_f(x) \in V_i\} \quad \text{for } i = 1, 2, 3, \cdots, \\ W' &= \{x \mid m_f(x) \in W\}, \qquad Q = \{A \mid \mathcal{E}_2(f, A) = 1\}, \\ g &= p_n^2 \circ R \circ l_f, \qquad h = p_n^2 \circ R \circ f. \end{aligned}$$

From (3), (4), and Theorem 6.6 as well as (7) and Theorem 8.14

we infer for $\mathcal{L}_2$ almost all z in E_2 that

$$N(g, Q, z) < \infty,$$
$$S(h, W', z) \leqq N(g, Q \cap W, z),$$
$$S(h, V_i', z) \leqq N(g, Q \cap V_i, z),$$
$$N(g, Q, z) = S(h, X, z) \leqq S(h, W', z) + S(h, V_i', z)$$

for every positive integer i. For each such point z there exists a positive integer i such that

$$N(g, Q \cap W, z) + N(g, Q \cap V_i, z) = N(g, Q, z),$$

whence it follows that

$$S(h, W', z) = N(g, Q \cap W, z).$$

THEOREM 8.16. *Suppose:*

(1) *X is a locally compact, locally connected subset of E_2 for which* $\sigma(X) > 0$.

(2) $f \in C_n(X)$ *and* $n \geqq 3$.

(3) *U is the union of all those open subsets V of $\mathfrak{M}_f$ for which*

$$\lambda_2(f, \{x \mid m_f(x) \in V\}) < \infty.$$

(4) $Q = \mathfrak{M}_f \cap \{A \mid L_2^*(f, A) = \mathcal{E}_2(f, A) = 1\}$.
Then:

(5) *If W is an open subset of U and $R \in G_n$, then*

$$S(p_n^2 \circ R \circ f, \{x \mid m_f(x) \in W\}, z) \leqq N(p_n^2 \circ R \circ l_f, W \cap Q, z)$$

for $\mathcal{L}_2$ almost all z in E_2.

(6) *If W is an open subset of U, then the 2-dimensional integral-geometric stable area of the mapping*

$$f \mid \{x \mid m_f(x) \in W\}$$

does not exceed

$$H_f^2(W \cap Q).$$

(7) *If W is an open subset of $\mathfrak{M}_f$ and $R \in G_n$, then*

$$S(p_n^2 \circ R \circ f, \{x \mid m_f(x) \in W\}, z) \leqq \sum_{A \in W,\, (p_n^2 \circ R \circ l_f)(A) = z} L_{*2}(f, A)$$

for $\mathcal{L}_2$ almost all z in E_2.

(8) *If W is an open subset of $\mathfrak{M}_f$, then the 2-dimensional integral-geometric stable area of the mapping*

$$f \mid \{x \mid m_f(x) \in W\}$$

does not exceed

$$\int_{E_n} \sum_{A \in W,\, l_f(A) = y} L_{*2}(f, A)\, d\mathcal{I}^2_n y.$$

Proof of (5). We choose $\epsilon > 0$ and define

$$g = p^2_n \circ R \circ l_f, \qquad h = p^2_n \circ R \circ f.$$

Furthermore we let F be the set of all ordered pairs

$$(V, T)$$

such that V is an open subset of W, T is a two-cell contained in X, and

$$\{x \mid m_f(x) \in V\} \subset \text{Interior } T, \qquad L_2(f \mid T) < \infty.$$

If $(V, T) \in F$, then V is an open subset of $\mathcal{M}_{(f \mid T)}$,

$$\{x \mid m_f(x) \in V\} = \{x \mid m_{(f \mid T)}(x) \in V\},$$

$$L^*_2(f, A) = L^*_2(f \mid T, A) \quad \text{for } A \in V,$$

$$\mathcal{E}_2(f, A) = \mathcal{E}_2(f \mid T, A) \quad \text{for } A \in V,$$

$$S(h, \{x \mid m_f(x) \in V\}, z) = N(g, V \cap Q, z) \text{ for } \mathcal{L}_2 \text{ almost all } z \text{ in } E_2,$$

$$H^2_f(V \cap Q) \leqq \lambda_2(f, \{x \mid m_f(x) \in V\}) < \infty.$$

Defining

$$W' = \bigcup_{(V,T) \in F} V$$

we note that

$$(W \cap \Gamma \cap \{A \mid A^f = 0 \text{ and } A \subset \text{Interior } X)\} \subset W',$$

whence, by Theorem 8.14,

$$S(h, \{x \mid m_f(x) \in W\}, z) = S(h, \{x \mid m_f(x) \in W'\}, z)$$

for $\mathcal{L}_2$ almost all z in E_2, because there are open subsets $W_1, W_2, W_3, \cdots$ of W such that

$$W = \bigcup_{i=1}^{\infty} W_i,$$

$$\lambda_2(f, \{x \mid m_f(x) \in W_i\}) < \infty \text{ for } i = 1, 2, 3, \cdots.$$

Finally we choose pairs (V_1, T_1), (V_2, T_2), (V_3, T_3), $\cdots$ in F such that

$$W' = \bigcup_{i=1}^{\infty} V_i,$$

$$H_f^2\left(Q \cap V_{i+1} \cap \bigcup_{j=1}^{i} V_j\right) \leq \epsilon 2^{-i}.$$

It follows from the Theorems 8.14 and 8.15 that

$$\begin{aligned} S(k, \{x \mid m_f(x) \in W'\}, z) &\leq \sum_{i=1}^{\infty} S(k, \{x \mid m_f(x) \in V_i\}, z) \\ &= \sum_{i=1}^{\infty} N(g, V_i \cap Q, z) \end{aligned}$$

for $\mathcal{L}_2$ almost all z in E_2.

However

$$\begin{aligned} \mathcal{L}_2\left(g\left[\bigcup_{i \neq j} (Q \cap V_i \cap V_j)\right]\right) &\leq H_f^2\left[\bigcup_{i \neq j} (Q \cap V_i \cap V_j)\right] \\ &\leq \sum_{i=1}^{\infty} H_f^2\left(Q \cap V_{i+1} \cap \bigcup_{j=1}^{i} V_j\right) \\ &\leq \sum_{i=1}^{\infty} \epsilon 2^{-i} = \epsilon \end{aligned}$$

and

$$\sum_{i=1}^{\infty} N(g, V_i \cap Q, z) = N(g, Q \cap W', z) \leq N(g, Q \cap W, z)$$

whenever

$$z \in E_2 - g\left[\bigcup_{i \neq j} (Q \cap V_i \cap V_j)\right].$$

We conclude that

$$\mathcal{L}_2(\{z \mid S(k, \{x \mid m_f(x) \in W\}, z) > N(g, Q \cap W, z)\}) \leq \epsilon.$$

Proof of (6). We define

$$Y_i = E_n \cap \{y \mid N(l_f, W \cap Q, y) \geq i\} \qquad \text{for } i = 1, 2, 3, \cdots$$

and compute:

$$\begin{aligned}
H_f^2(W\cap Q) &\geqq \int_{E_n} N(l_f, W\cap Q, y)\,d\mathcal{H}_n^2 y \\
&= \sum_{i=1}^{\infty} \mathcal{H}_n^2(Y_i) \geqq \sum_{i=1}^{\infty} \mathcal{J}_n^2(Y_i) \\
&= \sum_{i=1}^{\infty} \beta(n, 2)^{-1} \int_{G_n} \int_{E_2} N(p_n^2 \circ R, Y_i, z)\,d\mathcal{L}_2 z\,d\phi_n R \\
&= \beta(n, 2)^{-1} \int_{G_n} \int_{E_2} \sum_{i=1}^{\infty} N(p_n^2 \circ R, Y_i, z)\,d\mathcal{L}_2 z\,d\phi_n R \\
&= \beta(n, 2)^{-1} \int_{G_n} \int_{E_2} N(p_n^2 \circ R \circ l_f, W \cap Q, z)\,d\mathcal{L}_2 z\,d\phi_n R \\
&\geqq \beta(n, 2)^{-1} \int_{G_n} \int_{E_2} S(p_n^2 \circ R \circ f, \{x \mid m_f(x) \in W\}, z)\,d\mathcal{L}_2 z\,d\phi_n R.
\end{aligned}$$

PROOF OF (7). For $z \in E_2$ we define

$$g(z) = \sum_{A \in W, (p_n^2 \circ R \circ l_f)(A) = z} L_*(f, A)$$

and observe that

$$L_*(f, A) = \infty \text{ for } A \in W - U,$$

$$g(z) = \infty \text{ for } z \in (p_n^2 \circ R \circ l_f)(W - U).$$

On the other hand if

$$z \in E_2 - (p_n^2 \circ R \circ l_f)(W - U),$$

$$S(p_n^2 \circ R \circ f, \{x \mid m_f(x) \in W\}, z) \\ \qquad = S(p_n^2 \circ R \circ f, \{x \mid m_f(x) \in U \cap W\}, z),$$

and by (5) and Theorem 6.7 the second member of this equation is for $\mathcal{L}_2$ almost all such z less than or equal to

$$N(p_n^2 \circ R \circ l_f, U \cap W \cap Q, z) \leqq g(z).$$

PROOF OF (8). From (7) we infer that

$$\begin{aligned}
&\beta(n, 2)^{-1} \int_{G_n} \int_{E_2} S(p_n^2 \circ R \circ f, \{x \mid m_f(x) \in W\}, z)\,d\mathcal{L}_2 z\,d\phi_n R \\
&\qquad \leqq \beta(n, 2)^{-1} \int_{G_n} \int_{E_2} \sum_{A \in W, (p_n^2 \circ R \circ l_f)(A) = z} L_{*2}(f, A)\,d\mathcal{L}_2 z\,d\phi_n R
\end{aligned}$$

$$= \beta(n, 2)^{-1} \int_{G_n} \int_{E_2} \sum_{(p_n^2 \circ R)(y) = s} \; \sum_{A \in W,\, l_f(A) = y} L_{*2}(f, A)\, d\mathcal{L}_2 s\, d\phi_n R$$

$$= \int_{E_n} \sum_{A \in W,\, l_f(A) = y} L_{*2}(f, A)\, d\mathcal{I}_n^2 y.$$

REMARK. *In case* $n = 3$ *we can add the following three statements to the conclusions of Theorem* 8.16:

(9) $L_2^*(f, A) = L_{*2}(f, A) = \mathcal{E}_2(f, A) = 0$ *or* 1 *for* H_f^2 *almost all* A *in* U.

(10) $L_2^*(f, A) = L_{*2}(f, A) = 0$ *or* 1 *or* ∞ *for* H_f^2 *almost all* A *in* $\mathfrak{M}_f$.

(11) *If* W *is an open subset of* $\mathfrak{M}_f$, *then*

$$\sum_{A \in W,\, l_f(A) = y} L_2^*(f, A) = \sum_{A \in W,\, l_f(A) = y} L_{*2}(f, A)$$

is a non-negative integer or ∞ *for* $\mathcal{H}_3^2$ *almost all* y *in* E_3.

We observe that (9) implies (10) because

$$L_{*2}(f, A) = \infty \text{ for } A \in \mathfrak{M}_f - U,$$

and that (10) implies (11) because l_f maps sets of H_f^2 measure zero onto sets of $\mathcal{H}_3^2$ measure zero.

In order to prove (9) it is sufficient to consider an open subset V of $\mathfrak{M}_f$ for which

$$\lambda_2(f, \{x \mid m_f(x) \in V\}) < \infty$$

and to show that

$$L_2^*(f, A) = 0 \text{ for } H_f^2 \text{ almost all } A \text{ in } V - Q.$$

Suppose $\epsilon > 0$. Since

$$H_f^2(V \cap Q) \leqq \lambda_2(f, \{x \mid m_f(x) \in V\}) < \infty,$$

we can select an open subset W of V such that

$$V - Q \subset W, \qquad H_f^2(W \cap Q) < \epsilon.$$

Letting

$$W' = \{x \mid m_f(x) \in W\},$$

we infer from (6) that

$$\lambda_2(p_3^2 \circ R \circ f, W') < \epsilon \text{ for } R \in G_3.$$

Choosing $R^1, R^2, R^3 \in G_3$ so that

$$R^1(x) = x, \qquad R^2(x) = (x_2, x_3, x_1), \qquad R^3(x) = (x_3, x_1, x_2)$$

whenever $x = (x_1, x_2, x_3) \in E_3$, we find that

$$\lambda_2(f, W') \leq \sum_{i=1}^{3} \lambda_2(p_3^2 \circ R^i \circ f, W') < 3\epsilon.$$

This conclusion can be justified by extending the fundamental inequality of Cesari [CE 2] from mappings of two-cells to mappings of arbitrary finitely triangulable subsets of E_2 into E_3. This extension presents no great difficulty.

It follows from Theorem 6.2 that

$$\int_{V-Q} L_2^*(f, A)\,dH_f^2A \leq \int_{W} L_2^*(f, A)\,dH_f^2A \leq \lambda_2(f, W') < 3\epsilon.$$

The arbitrary nature of ϵ implies that

$$\int_{V-Q} L_2^*(f, A)\,dH_f^2A = 0,$$

$$L_2^*(f, A) = 0 \text{ for } H_f^2 \text{ almost all } A \text{ in } V - Q.$$

THEOREM 8.17. *If X is a 2-cell and $f \in C_3(X)$, then*

$$\begin{aligned} L_2(f) &= \int_{\mathfrak{M}_f} L_2^*(f, A)\,dH_f^2A = \int_{\mathfrak{M}_f} L_{*2}(f, A)\,dH_f^2A \\ &= \int_{E_n} \sum_{l_f(A)=y} L_2^*(f, A)\,d\mathcal{H}_n^2y = \int_{E_n} \sum_{l_f(A)=y} L_{*2}(f, A)\,d\mathcal{H}_n^2y \\ &= \int_{E_n} \sum_{l_f(A)=y} L_2^*(f, A)\,d\mathcal{J}_n^2y = \int_{E_n} \sum_{l_f(A)=y} L_{*2}(f, A)\,d\mathcal{J}_n^2y. \end{aligned}$$

PROOF. By virtue of Theorem 6.2 each of the above seven numbers is less than or equal to $L_2(f)$ and greater than or equal to the last integral. Furthermore Theorem 8.16 implies that the last integral is greater than or equal to the 2-dimensional integralgeometric stable area of f, which equals $L_2(f)$.

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Brown University

THE FEBRUARY MEETING IN NEW YORK

The 478th meeting of the American Mathematical Society was held at Columbia University, New York City, on Saturday, February 23, 1952. The meeting was attended by about 200 persons, including the following 175 members of the Society.

Milton Abramowitz, R. D. Anderson, R. L. Anderson, Joseph Andrushkiw, N. C. Ankeny, R. G. Archibald, N. J. Artin, Sholom Arzt, Enrique Bayó, E. G. Begle, D. L. Bernstein, D. W. Blackett, J. H. Blau, Volodymyr Bohun-Chudyniv, Samuel Bourne, J. W. Bower, H. W. Brinkmann, Paul Brock, F. E. Browder, A. B. Brown, L. J. Burton, D. W. Bushaw, J. H. Bushey, Jewell H. Bushey, Eugenio Calabi, W. R. Callahan, Evelyn Carroll-Rusk, P. W. Carruth, G. Y. Cherlin, P. J. Cocuzza, E. A. Coddington, L. W. Cohen, R. M. Cohn, W. H. H. Cowles, M. L. Curtis, R. B. Davis, C. R. DePrima, Jesse Douglas, Beno Eckmann, Samuel Eilenberg, C. C. Elgot, Joanne Elliott, Bernard Epstein, Donald Epstein, M. P. Epstein, Trevor Evans, W. H. Fagerstrom, W. E. Ferguson, Arrigo Finzi, W. T. Fishback, Edward Fleisher, A. D. Fleshler, H. A. Forrester, L. G. Fourès, Gerald Freilich, Bernard Friedman, Murray Gerstenhaber, Abolghassem Ghaffari, B. P. Gill, Sidney Glusman, H. E. Goheen, J. K. Goldhaber, Daniel Gorenstein, J. W. Green, L. W. Green, H. M. Griffin, Emil Grosswald, Laura Guggenbuhl, Felix Haas, G. A. Hedlund, Henry Helson, Aaron Herschfeld, W. M. Hirsch, C. C. Hsiung, S. T. Hu, Ralph Hull, H. G. Jacob, W. S. Jardetzky, R. V. Kadison, Shizuo Kakutani, Hyman Kamel, M. E. Kellar, D. E. Kibbey, J. F. Kiefer, H. S. Kieval, V. L. Klee, George Klein, Morris Kline, Daniel Kocan, E. R. Kolchin, B. O. Koopman, Alice Krikorian, J. C. Laffan, Serge Lang, C. E. Langenhop, J. A. Larrivee, Solomon Lefschetz, Marguerite Lehr, Benjamin Lepson, Howard Levi, M. A. Lipschutz, W. G. Lister, E. R. Lorch, Eugene Lukacs, Brockway McMillan, Wilhelm Magnus, E. W. Marchand, A. J. Maria, W. T. Martin, J. P. Mayberry, A. E. Meder, Jr., K. S. Miller, W. H. Mills, Don Mittleman, C. T. Molloy, Deane Montgomery, W. L. Murdock, C. A. Nelson, W. J. Nemerever, Morris Newman, J. A. Nohel, I. L. Novak, Adolf Nussbaum, A. F. O'Neill, H. A. Osborn, Robert Osserman, J. C. Oxtoby, L. E. Payne, H. O. Pollak, E. L. Post, G. N. Raney, H. E. Rauch, H. W. Raudenbush, Helene Reschovsky, Moses Richardson, J. E. Robinson, Robin Robinson, I. H. Rose, David Rosen, Maxwell Rosenlicht, J. E. Rosenthal, Walter Rudin, H. E. Salzer, Arthur Sard, Leo Sario, Samuel Schecter, Abraham Schwartz, I. E. Segal, H. N. Shapiro, Maurice Sion, O. K. Smith, P. A. Smith, H. H. Snyder, J. J. Sopka, I. A. Stegun, S. K. B. Stein, J. J. Stoker, J. T. Tate, R. L. Taylor, D. E. van Tijn, J. L. Tits, B. W. Volkmann, Alan Wayne, G. C. Webber, Alexander Weinstein, Louis Weisner, David Wellinger, A. B. Willcox, Otto Wolf, N. Z. Wolfsohn, Arthur Wouk, D. M. Young, L. A. Zadeh, J. A. Zilber, Leo Zippin.

Professor Alexander Weinstein of the University of Maryland delivered an invited address entitled *Generalized axially symmetric potential theory* at 2:00 P.M. Professor B. O. Koopman presided.

The morning session for contributed papers was presided over by Professor Ralph Hull. Professors Wilhelm Magnus and E. R. Lorch presided at the afternoon sessions.

Abstracts of the contributed papers are listed below, those with a "t" after their numbers having been read by title. Paper number 300 was presented by Dr. Young. Dr. Warga was introduced by Dr. Paul Brock.

ALGEBRA AND THEORY OF NUMBERS

268t. L. M. Blumenthal: *Two existence theorems for systems of linear inequalities.*

Let A denote an $m\times n$ matrix of rank r, A^T its transpose, and x a column matrix of n indeterminates. It is shown that the system of m linear inequalities $Ax\geqq 0$ (with at least one left-hand member positive) has a solution if and only if rows and corresponding columns of the matrix AA^T may be so shifted that (i) the upper left-hand principal minor M of order r is nonsingular and (ii) each rth order minor of AA^T formed from M by replacing its last row with that part of the jth row of AA^T contained in the first r columns ($j=r+1, r+2, \cdots, m$) is non-negative. It follows that the system of inequalities (in the wide sense) $Ax\geqq 0$ has a nontrivial solution if and only if whenever $r=n$, conditions (i), (ii) are satisfied. The theorems are consequences of a lemma in which the sides of the convexly metrized n-sphere determined by a great $(n-1)$-sphere are metrically characterized. (Received December 17, 1951.)

269. Volodymyr Bohun-Chudyniv: *Square matrices differing in some properties from Euler's squares.*

From the schemes solving Euler's problem proposed by: (1) L. Euler (Novi Comm. Acad. Petrop. vol. 15 (1770) p. 75; Comm. Arith. vol. 1, pp. 427–443); (2) C. Avery (Math. Misc., New York, 1839, p. 101); (3) G. K. Perkins (Ibid., pp. 102–105); (4) V. A. Lebesque (Nouv. Ann. Math. vol. 15 (1856) pp. 403–407); (5) L. Bastien (Spinx-Oedipe vol. 7 (1912) p. 12); (6) N. Fuss (Mem. Acad. Sci., St. Petersburg vol. 4 (1813) pp. 240–247); just two types of square matrices are obtained. (1) Matrices which satisfy only 20 conditions of Euler, (2) those which satisfy all 22 conditions. The present paper gives other types of matrices which can be obtained from the author's scheme (1) solving Euler's problem (V. Bohun-Chudyniv, *Solution of Euler's problem*, Ukrainian Free Acad. of Sci., Regensburg, 1947), and not previously determined. These types are as follows: I. Matrices which satisfy besides Euler's 20 conditions also the six: (a) $A_{1,1}^2+A_{1,4}^2+A_{4,1}^2+A_{4,4}^2=A_{1,2}^2+A_{1,3}^2+A_{4,2}^2+A_{4,3}^2=A_{2,2}^2+A_{2,3}^2+A_{3,2}^2+A_{3,3}^2=A_{2,1}^2+A_{2,4}^2+A_{3,1}^2+A_{3,4}^2=S$. (b) $A_{1,1}A_{1,3}+A_{1,2}A_{1,4}+A_{4,1}A_{4,3}+A_{4,2}A_{4,4}=A_{2,1}A_{2,3}+A_{2,2}A_{2,4}+A_{3,1}A_{3,3}+A_{3,2}A_{3,4}=0$. Conditions (a) were noted also by Perkins (ref. 3 above). These matrices are obtained from the author's scheme (1) if its arguments satisfy the following relationships: $u_2=u_0t$, $u_3=u_1t$, $v_1=(2t/t^2-1)v_2$. Arbitrary values can be substituted for v_0, v_2, u_0, u_1, t. II. Matrices which satisfy 16, 14, 12, 8, 6, 4, and 2 conditions of Euler's problem and six conditions (a) and (b). These matrices are obtained from scheme (1) using relationships (2) on p. 4 of the author's paper mentioned above. (Received January 14, 1952.)

270t. Leonard Carlitz: *A divisibility property of the Bernoulli polynomials.*

Let u be integral (mod p) and put $\sigma_m(u)=p^{-r}(B_m(u)+1/p-1)$. It is proved that if $p\geqq 3$, $(p-1)p^r|m$, then $\sigma_m(u)$ is integral (mod p); indeed the residue (mod p) is specified. The corresponding result involving Bernoulli polynomials of higher order is also obtained. (Received December 24, 1951.)

271*t*. Leonard Carlitz: *Bernoulli numbers and polynomials of higher order.*

In the notation of Nörlund let $(x/(e^x-1))^k e^{xu} = \sum_{m=0}^{\infty} B_m(u)x^m/m!$. Among the results of this paper the following may be mentioned. We assume $k<p-1$, $m \not\equiv 0, \cdots, k-1 \pmod{p-1}$, $m \geqq k \geqq 1$, and u integral (mod p). 1. $B_m^{(k)}(u)$ is integral (mod p). 2. If $p^r \mid (m)_k$, where $(m)_k = m(m-1) \cdots (m-k+1)$, then the numerator of $B_m^{(k)}$ is divisible by p^r. 3. If $(p-1)p^{e-1} \mid b$, $m \geqq rb+k$, then $\sum_{s=0}^{r} (-1)^{r-s} C_{r,s} T_{m+sb}^{(k)} \equiv 0 \pmod{p^{re}}$, where $T_m^{(k)} = B_m^{(k)}(u)/(m)_k$; also $\sum_{s=0}^{r}(-1)^{r-s}(C_{r,s}B_{m+sb}^{(k)}(u) \equiv 0 \pmod{p^{(r-k)e}}$. (Received December 24, 1951.)

272*t*. Leonard Carlitz: *Distribution of primitive roots in a finite field.*

Davenport (Quart. J. Math. Oxford Ser. vol. 8 (1937) pp. 308–312) proved that for large p one can always find a primitive root of the form $\theta+a$, where θ generates the field. He also showed that for given p, there exist fields such that no number $a\theta+b$ can be a primitive root. In the present note these results are extended in several directions. 1. Results are obtained concerning the number of numbers $\beta = \theta^r + a_1\theta^{r-1} + \cdots + a_r$ that belong to the exponent e, where $e \mid p^n-1$. 2. For fixed p and r, there exist fields such that no number β is a primitive root. (Received December 24, 1951.)

273*t*. Leonard Carlitz: *Primitive roots in a finite field.*

Ore (Trans. Amer. Math. Soc. vol. 36 (1934) pp. 243–274) has introduced the notion of a number of $GF(p^{nm})$ belonging to the exponent $a(x)$, where $a(x)$ *corresponds* to the ordinary polynomial $A(x)$, where $A(x) \mid x^m-1$. In particular β is a primitive root in this sense if it belongs to $x^{p^{m}}-x$. It is natural to ask whether there exist numbers that are simultaneously primitive roots both in the ordinary and in Ore's sense. The principal result of this paper is the following. Let $e \mid p^{nm}-1$, $A(x) \mid x^m-1$. Then the number of numbers of $GF(p^{nm})$ belonging to e and $a(x) = \phi(e)\Phi(A)/p^{nm} + O(p^{nm(1/2+\epsilon)})$ $(p^{nm} \to \infty)$. (Received December 24, 1951.)

274*t*. Leonard Carlitz: *Sums of primitive roots in a finite field.*

Let $e_i \mid p^n-1$, $k_i \mid p^n-1$. Let β_i denote a number of $GF(p^n)$ that belongs to the exponent e_i, $i=1, \cdots, r$, and let ξ_i be an arbitrary number of $GF(p^n)$. We consider the following problems. 1. The number of solutions $\beta_1, \cdots, \beta_r$ of $\alpha = \alpha_1\beta_1 + \cdots + \alpha_r\beta_r$, where $\alpha_i \neq 0$. 2. The number of solutions $\beta_1, \cdots, \beta_r, \zeta_1, \cdots, \zeta_s$ of $\alpha = \alpha_1\beta_1 + \cdots + \alpha_r\beta_r + \delta_1\xi_1^{k_1} + \cdots + \delta_s\xi_s^{k_s}$. Asymptotic results are obtained for both problems. 3. The number of solutions $\beta, \xi_1, \cdots, \xi_{2m}$ of $\alpha = \beta_1 + \delta_1\xi_1^2 + \cdots + \delta_{2m}\xi_{2m}^2$. Here a simple exact formula is obtained. (Received January 7, 1952.)

275*t*. F. Marion Clarke: *Note on the effect of commutativity in a finite loop.*

It is noted that, as a direct consequence of the definition, loops of order less than or equal to 4 are commutative and associative; that a noncommutative loop of order 5 is not necessarily associative; but that commutativity implies associativity. It is shown by counter examples that in loops of higher order, prime or composite, commutativity alone does not produce associativity. If, however, every element of a commutative loop L_n of *any* order n has order n, and if L_n satisfies the weak associative law: $x\cdot[b\cdot(c\cdot b)]=[(x\cdot b)\cdot c]\cdot b$; then L_n satisfies the general associative law and L_n is a group. (Received January 10, 1952.)

276*t*. Anne C. Davis: *On order types whose squares are equal.*

The question of the existence of order types α and β such that $\alpha^2=\beta^2$ and $\alpha\neq\beta$ has been open. The answer turns out to be positive. Denumerable order types α and β satisfying the above conditions can be constructed as follows. Separate the set R of rationals into two disjoint sets S and T, both dense in R. Replace every element of S by an ordered set of type ω, leaving every element of T unchanged. Let α be the order type of the ordered set thus obtained and let $\beta=\alpha+\omega$. It is easily seen that $\alpha\neq\beta$, and, by an essential use of a theorem of Skolem (Skrifter utgit av Videnskapsselskapet i Kristiania, I. klasse 1920, no. 4, p. 32, Theorem 2) one can show that $\alpha^2=\beta^2=\alpha$. (Received February 4, 1952.)

277*t*. Chandler Davis: *Structure of certain Boolean algebras with operators.*

An S5 modalgebra is a Boolean algebra $\mathfrak{B}$ with several S5 closure operators (McKinsey-Tarski) acting on it. Study of such systems is motivated by applications to logic, and because they provide generalizations of statements involving equivalence relations. (In case $\mathfrak{B}$ is complete atomic and ϕ an equivalence relation among its points, assign to each $a\in\mathfrak{B}$ the join a^ϕ of all ϕ-equivalence classes including any point $\subseteq a$; this is an S5 closure operator.) Any projective algebra (Everett-Ulam) is evidently an S5 modalgebra under the operators $a\rightarrow a^x$ and $a\rightarrow a^y$ used in Chin and Tarski's axiom set (Bull. Amer. Math. Soc. Abstract 54-1-90); an S5 modalgebra can be obtained similarly from any projective algebra *relativized* to an arbitrary fixed element. Theorem: Every S5 modalgebra with two operators is obtained in this way. Generalization to k operators is immediate. This theorem may be thought of as asserting "co-ordinatizability" of these systems. (Received January 3, 1952.)

278. Beno Eckmann: *Cohomology of groups and transfer.*

For an arbitrary group A and an A-module J (an Abelian group, written additively, with A as group of left operators) let $H^p(A, J)$, $p=1, 2, \cdots$, be the (equivariant) cohomology groups of A with coefficient group J. For any subgroup B of A there is a coefficient group J' such that $H^p(B, J)$ is naturally isomorphic to $H^p(A, J')$; namely, $J'=A$-module of all functions ψ from A to J such that $\psi(ba)=b\cdot\psi(a)$ for all $b\in B$, $a\in A$. An A-homomorphism f of J' into J induces therefore homomorphisms S_f^p of $H^p(B, J)$ into $H^p(A, J)$, for all p.—Special case: If the index n of B in A is finite, let $a_1, \cdots, a_n$ be any representatives of the left cosets and define f by $f(\psi)=\sum a_i^{-1}\psi(a_i)$. The induced homomorphisms $S_f^p=T^p$ of the cohomology groups generalize the "transfer of A into B": If A operates trivially in J, the homomorphism T^1 of $H^1(B, J)=\mathrm{Hom}\,(B, J)$ into $\mathrm{Hom}\,(A, J)$ is dual to the transfer. Properties analogous to those of the transfer hold for all the T^p. E.g., let I^p denote the homomorphism of $H^p(A, J)$ into $H^p(B, J)$ induced by the injection of B into A; then $T^p(I^p h)=nh$ for all $h\in H^p(A, J)$. (Received January 24, 1952.)

279. Trevor Evans: *Embeddability and the word problem.*

A method exists for solving the word problem in an equationally defined class of abstract algebras if, and only if, a method exists for deciding whether an incomplete algebra can be embedded. This includes the previous result of the author (*The word problem for abstract algebras*, J. London Math. Soc. vol. 26 (1951)) that the word problem can be solved for a class of algebras having the property that any incomplete algebra can be embedded. Using the methods of that paper it can be assumed that, in solving the word problem, the two words to be compared are generators and that the generators and relations form an incomplete algebra. If the two generators are not equivalent, by equating some of the generators, an embeddable incomplete algebra can be found containing the two generators as distinct elements. If the two generators are equivalent no such incomplete algebra exists. On the other hand, if the word problem can be solved, the embeddability problem can also be solved since an incomplete algebra can be embedded only if its elements are not equivalent in the algebra it freely generates. (Received January 8, 1952.)

280*t*. W. H. Gottschalk: *The extremum law.*

The extremum law is defined to be the class E of all statements S such that S is equivalent to the axiom of choice and such that the conclusion of S asserts the existence of an extremal (maximal or minimal) element. A member of E is called a form of the extremum law. Various forms of the extremum law are known under the names of Zorn's lemma, Hausdorff maximality principle, etc. The purpose of this note is to point out a particularly simple form of the extremum law, namely: If R is a binary relation, then there exists a maximal set A such that $A \times A \subset R$. (Received December 26, 1951.)

281*t*. I. N. Herstein: *A generalization of a theorem of Jacobson.* II.

In his paper *A generalization of a theorem of Jacobson* (Amer. J. Math. vol. 73 (1951) pp. 756–762) the author proved the following theorem: Let R be a ring with center Z such that $x^n - x \in Z$ for all $x \in R$, n a fixed integer larger than 1; then R is commutative. In this paper this result is extended to the case where $n = n(x)$ is no longer fixed but an integer depending on x, where, however, we assume that $n(x) < N$ for all $x \in R$. The methods of the previous paper, together with several applications of the "Dirichlet box principle" yield the result. (Received January 11, 1952.)

282*t*. I. N. Herstein: *Finite multiplicative subgroups in division rings.*

This paper is concerned with extending the result that any finite multiplicative subgroup of the multiplicative group of a commutative field is cyclic. The following results are obtained: (1) Let K be a division ring of characteristic $p \neq 0$. Then any finite multiplicative subgroup in K is cyclic. (2) Let K be any division ring. Then any multiplicative, odd-ordered subgroup in K is metacyclic. (3) Let Q be the quaternions over the reals. Then multiplicative subgroups of odd order in K must be cyclic. It is conjectured that any multiplicative subgroup of odd order in a division ring must be cyclic. (Received January 11, 1952.)

283. Ralph Hull: *Cyclotomy and cyclic fields.*

For $e = 3, 4, 5, 6, 8, 12$, all cyclic fields of degree e are determined by exhibiting the equations satisfied by the respective periods of roots of unity which generate the fields. The coefficients of the equations are expressed in terms of the conductor c of the field and its factors, and representations of c or one of its factors by certain quadratic forms. For example, when $e=8$, the quadratic forms are x^2+4y^2 and a^2+2b^2. (Received January 11, 1952.)

284. Serge Lang: *On quasi algebraic closure.*

A field F is defined to be C_i if every form with coefficients in F in n variables, of degree d, and $n>d^i$, has a nontrivial zero in F. For $i=1$ this is the quasi algebraic closure of Artin. We prove that a function field in s variables over a C_r constant field is C_{r+s}, and also some conjectures of Artin concerning C_1 and C_2 fields. In particular, the power series field over a finite field is C_2 and the following fields are C_1: A field F complete under a discrete valuation with algebraically closed residue class field, and the maximal unramified extension of a field with a discrete valuation and perfect residue class field. In fact any subfield F_0 of F which is dense and algebraically closed in F is C_1. These results can be applied to local class field theory to prove Chevalley's theorem that over a p-adic field a 2-cocycle of period r is split by every field of degree r. (Received January 10, 1952.)

285*t*. D. J. Lewis: *Norm groups of certain simple algebras.*

The principal result obtained is that the reduced norm group of a central simple algebra of order nine over an algebraic number field is the multiplicative group of that field. The proof utilizes the fact that a central simple algebra of order nine is either a total matrix algebra or a division algebra. In either case the reduced norm function is a homogeneous polynomial of degree three in nine indeterminates. Use is then made of the author's result on cubic homogeneous polynomials over p-adic fields and the Maass, Hasse, Schilling norm theorem. A somewhat similar result is obtained for central simple algebras of order four. (Received January 10, 1952.)

286. Maxwell Rosenlicht: *Differentials of second kind for algebraic function fields of one variable.*

Let K be a field of algebraic functions of one variable over the constant field k, which, for simplicity, we assume algebraically closed. The differential ω of K is said to be of the second kind if for each place P of K/k there exists a function $f_P \in K$ such that $\omega - df_P$ is finite at P. The differentials of second kind form a vector space over k that contains the space of exact differentials. The quotient space is known to have dimension $2g$, where g is the genus of K, in case the characteristic of K is zero; if the characteristic of K is $p\neq 0$, the dimension is g. If a differential ω can be approximated arbitrarily closely at a certain fixed place P by exact differentials, then, in characteristic zero, $\omega=0$; if the characteristic is $p\neq 0$, ω is exact. (Received January 10, 1952.)

287*t*. A. R. Schweitzer: *An identity in Grassmann's extensive algebra.*

Let $E_1, E_2, \cdots, E_n$ $(n=2, 3, \cdots)$ be mutually perpendicular unit vectors in Grassmann n-space. If $X_n=\sum_{i=1}^n x_iE_i$ and $Q(X_nE_i)=\sum_{j=1}^n x_jQ(E_jE_i)$ $(i=1, 2, \cdots, n)$ where $Q(X_nE_i)$ and $Q(E_jE_i)$ denote operators on E_i which transform E_i into X_n and E_j respectively, then the following expression is an identity provided $Q(E_iE_i)=1$ and $Q(E_iE_j)=-Q(E_jE_i)$ $(i\neq j)$: $\sum_{i=1}^n x_i^2=\sum_{i=1}^n x_iQ(X_nE_i)$. This identity permits the ready calculation of $Q(Y_nX_n)$ where the latter symbol is an operator on X_n which transforms X_n into the vector $Y_n=\sum_{i=1}^n y_iE_i$ as is easily verified by multiplying the above identity on both sides by $Q(Y_nX_n)$ and noting that

$Q(Y_nX_n)\times Q(X_nE_n)=Q(Y_nE_n)$. Reference is made to the author's article in Math. Ann. vol. 69 (1910). (Received January 9, 1952.)

288. Bodo Volkmann: *On classes of sets of integers.*

For any infinite set $\mathfrak{A}=\{a_1, a_2, \cdots\}$ of positive integers the dyadic value ("Dualwert") is defined to be $\Gamma(\mathfrak{A})=\sum_{a\in\mathfrak{A}} 2^{-a}$. The author (in a paper to be published in the J. Reine Angew. Math.) has investigated classes $N=\{\mathfrak{A}, \mathfrak{B}, \cdots\}$ of such sets and applied Hausdorff's measure theory to the corresponding point sets $_\Gamma N=\{\Gamma(\mathfrak{A}), \Gamma(\mathfrak{B}), \cdots\}$. Based on Knichal's and Besicovitch's methods, some general theorems on Hausdorff dimensions are obtained by means of which dim $_\Gamma N$ is determined in the cases when N is (a) the class of infinite subsets of a fixed set $\mathfrak{A}$; (b) the class of all sets $\mathfrak{A}$ for which a given number $\zeta(0\leq\zeta\leq 1)$ lies between their lower and upper asymptotic densities. The corresponding results are: (a) dim $_\Gamma N=$ lower asymptotic density of $\mathfrak{A}$ (containing the proof of a conjecture due to E. F. and R. C. Buck, Amer. J. Math. vol. 69, pp. 413–420); (b) dim $_\Gamma N=\log\gamma_s/\log 2$, γ_s being the positive root of the equation $\xi^s-\xi^{s-1}-\cdots-\xi^0=0$; (c) dim $_\Gamma N(\zeta\log\zeta+(1-\zeta)\log(1-\zeta))/\log(1/2)$. (Received January 9, 1952.)

ANALYSIS

289. Joseph Andrushkiw: *On the zeros of integral functions.*

Let the zeros of the polynomial $f(z)$ be pure imaginary. Applying a theorem of Hurwitz (Math. Ann. vol. 46, *Ueber die Bedingungen, unter welchen eine Gleichung nur Wurzeln mit negativen reellen Teilen besitzt*) to the polynomial $F(z)=f(z+x)$, $x\neq 0$, one obtains sufficient condition that a polynomial with real coefficients have all its zeros pure imaginary: All principal minors of even order of a determinant of $2n$th order, whose elements are the coefficients of the polynomial, are positive. Similar sufficient conditions can be deduced for a polynomial with all positive or all negative zeros. By proper transformation of the above determinants A_r they are shown to be symmetric functions of the zeros and the sufficient conditions appear to be also necessary. Let n_1 and n_2 be the numbers of sign variations in the sequences $1, A_k/A_{k+1}$ and $\bar{A}_k/\bar{A}_{k+1}$, $1\leq k\leq 2n-1$, where $\bar{A}_r=(-1)^{C_{r+1,2}}A_r\neq 0$, respectively. Then the number of negative zeros of a polynomial is $n-n_1$, and the number of positive zeros $n-n_2$; $n_1+n_2=n$ is the necessary and sufficient condition that all zeros be real and distinct. The same methods applied to the results of Grommer (J. Reine Angew. Math. vol. 144, *Ganze transzendente Funktionen mit lauter reellen Wurzeln*) extend the criteria to some types of integral transcendental functions. (Received December 31, 1951.)

290*t*. S. D. Bernardi: *A critical evaluation of the classical and variational methods in the theory of schlicht functions.*

The paper consists primarily of a critical review of the classical results in the theory of functions $f(z)=z+a_2z^2+a_3z^3+\cdots$ regular and schlicht in the unit circle $|z|<1$. It is suggested that the "area-principle" of Gronwall be generalized by the introduction of proper weight functions. It is also suggested that an improvement of the inequality $\lim_{n\to\infty}|a_n|/n\leq e/2$ might possibly result from the generalized form of the area-principle due to Prawitz by the application of an induction process. An example is given showing the independence of the inequality of Prawitz from the parametric representations of the coefficients a_3 and a_4 given by A. C. Schaeffer and D. C. Spencer. It is also shown that if a_n are all real and $f(z)$ is convex in the direction of the imaginary axis, then $|a_n| \leq 1$. The proof of this last result follows readily since, under the above conditions, $t(z) = zf'(z)$ is typically-real for $|z| < 1$ and hence $p(z) = (1-z^2)t(z)/z = 1 + \sum_1^\infty b_n z^n$, $b_n = (n+1)a_{n+1} - (n-1)a_{n-1}$, is a function of positive real part in $|z| < 1$ so that $|b_n| \leq 2$, from which the result follows. (Received January 9, 1952.)

291*t*. R. P. Boas: *Integrability along a line for a class of entire functions.*

Duffin and Schaeffer have shown in a paper presented to the International Congress of Mathematicians in 1950 that if $f(z)$ is an entire function of exponential type $c < \pi$, then $\int_{-\infty}^{\infty} |f(x)|^2 dx \leq K \sum_{-\infty}^{\infty} |f(\lambda_n)|^2$ provided $|\lambda_n - n| \leq L$ and $|\lambda_{n+m} - \lambda_n| \geq \delta > 0$, $m \neq 0$. This is generalized to $\int_{-\infty}^{\infty} \phi(H|f(x)|)dx \leq K \sum_{-\infty}^{\infty} \phi(|f(\lambda_n)|)$ for a class of functions which includes $\phi(t) = t^p$, $p > 0$, and all increasing positive convex functions. The proof depends on an interpolation series. (Received December 14, 1951.)

292. F. E. Browder: *The Dirichlet problem for linear elliptic differential equations of even order with variable coefficients. The Fredholm alternative.*

The Dirichlet problem for a linear elliptic differential operator K of order $2m$ on a bounded domain D of E^n is the following: Given $h \in C^1(D) \cap L^2(D)$, $g \in C^m(D)$ such that the mth derivatives of g are square-summable, to find $u \in C^{2m}(D)$ for which $Ku = h$ and such that $u - g$ and its derivatives of order not greater than $m-1$ vanish on the boundary of D in a suitable sense. Suppose that the coefficients of the jth derivatives in K belong to $C^{2m+j}(D)$ and the mth derivatives of these coefficients are continuous on the closure of D. It is shown that under these conditions if the solution of the Dirichlet problem is unique, then it always exists and conversely. In the general case $Ku = h$ has a solution for zero boundary values for all h such that $\int_D hw = 0$ for all w in the finite-dimensional set of null solutions of the adjoint equation $\overline{K}w = 0$. The dimension of the set of null solutions for K is equal to that for $\overline{K}$. (Received January 9, 1952.)

293*t*. F. E. Browder: *The eigenvalues and eigenfunctions of the Dirichlet problem for self-adjoint linear elliptic equations.*

If L is a linear elliptic differential operator on a bounded domain D in E^n, A a linear differential operator of order less than $2m$, the order of L, then $w \in C^{2m}(D)$ is an eigenfunction of L with respect to A with eigenvalue k if $Lw = kw$ and w together with all its derivatives of order less than m vanishes on the boundary of D in a suitable sense. Under differentiability conditions on the coefficients of L and A, a complete theory of such eigenfunctions and eigenvalues is established. In particular it is shown that if L is positive self-adjoint and A is self-adjoint, then the eigenfunctions of L with respect to A are complete in the space of functions vanishing on neighborhoods of the boundary of D. (Received January 9, 1952.)

294*t*. R. M. Conkling and D. O. Ellis: *On metric groupoids.*

A single operation algebra (groupoid) which forms a metric space is called a *metric groupoid* if the operation is simultaneously continuous in the metric topology. It is called a *uniformly metric groupoid* (u.m.g.) if this simultaneous continuity is simultaneously uniform. Certain algebraic properties of subsets of metric groupoids are shown to extend to their topological closures. Among these are commutativity, associativity, identity element, ideal property, and subgroupoid property. It is shown that the Cantor-Hausdorff completion of a u.m.g. may be made into a u.m.g. in which the original u.m.g. is isometrically isomorphically imbedded. In particular, it is shown that a uniformly metric group is a topological group and that any uniformly metric (Abelian) group may be isometrically isomorphically imbedded in a metrically complete uniformly metric (Abelian) group. (Received December 17, 1951.)

295. R. B. Davis: *A boundary value problem for third-order linear partial differential equations of composite type.*

It is proved that, by suitable changes of variables, all third-order partial differential equations of composite type (i.e., having exactly one real characteristic), in which the third-order derivatives occur linearly and with continuously differentiable coefficients, can be reduced to a stated canonical form. Using this canonical form, the following boundary value problem for a certain linear composite equation $L(u)=f$ with coefficients belonging to C^1 is solved in the sense described: for a region for which the classical Green's function exists and satisfies certain inequalities, boundary values belonging to C^3 are prescribed on the entire boundary for the derivative of u along the characteristics, and are prescribed along a suitable curve for u itself. It is then proved that either the resulting boundary value problem has a unique solution, or else the homogeneous problem obtained by prescribing zero boundary values for $L(u)=0$ has a nontrivial solution. Two cases are established in which the second alternative is impossible. The methods employed are strongly analogous to those used in the study of second-order elliptic equations. (Received January 8, 1952.)

296. J. W. Green: *Approximately convex and approximately subharmonic functions.*

Hyers and Ulam have defined (Bull. Amer. Math. Soc. Abstract 57-4-380) an ϵ-convex function to be a function f in a convex domain of Euclidean n-space satisfying the inequality $f(hx+(1-h)y)\leqq\epsilon+hf(x)+(1-h)f(y)$ for $0\leqq h\leqq 1$. They proved the theorem: If f is continuous and ϵ-convex, there exists a convex function differing from f by not more than $k_n\epsilon$, where $2k_n=1+(n-1)(n+2)/2(n+1)$. In this paper, this theorem is proved in a different manner with an improved value of $2k_n \sim \log_2 (n+1)$, and only slightly poorer results obtained for discontinuous f. Sharpness of the inequality for small n is shown, and some properties of the extreme functions discussed. An ϵ-subharmonic function is defined to be an upper semicontinuous function f such that if h is a harmonic function dominating f on the boundary of a domain, then $h+\epsilon$ dominates f inside the domain. It is proved that if f is ϵ-subharmonic, there exists a subharmonic function u, namely the greatest subharmonic minorant of f, such that $u\leqq f\leqq u+\epsilon$. (Received December 14, 1951.)

297. Henry Helson: *Isomorphisms of abelian group algebras.*

Let G and H be locally compact abelian groups with group algebras $L(G)$ and $L(H)$ respectively. Let T be a linear operator mapping $L(G)$ onto $L(H)$ isomorphically. Then G and H are isomorphic groups, under the following hypotheses: the dual group of G (or of H) is connected, and T has bound less than two. The proof depends on a method found by A. Beurling and applied by him to a related problem on the real line. The conclusion of the theorem holds without restriction on the groups if the bound of T is one. (This result has been extended by J. G. Wendel to nonabelian group algebras.) (Received January 7, 1952.)

298t. M. L. Stein: *Sufficient conditions for the convergence of Newton's method in complex Banach spaces.*

Let $T(y)$ be an operator defined on one Banach space into another and let $\delta T(y; h)$ be its first variation. The Banach space analogue of Newton's method is given by the iteration formula $(*)$ $y_{i+1} = y_i - \delta T^{-1}(y_i; T(y_i))$, $i = 0, 1, 2, \cdots$, where δT^{-1} denotes the inverse with respect to h of δT. Kantorovič [Uspehi Matematičeskih Nauk N.S. vol. 3 (1948) pp. 1237–1240] has established conditions under which a sequence $\{y_i\}$ determined by $(*)$ will converge to a solution of $T(y) = 0$ in a real Banach space. The present paper establishes the following convergence theorem for complex Banach spaces: Let Y and Z be complex Banach spaces and let $T(y)$ be defined on $S: \|y\| < \rho$, $\rho > 0$, into Z. Let the following conditions hold: (i) $T(y)$ is G-differentiable, (ii) there exists y_0 in S, a positive number a, and a sufficiently small positive constant M such that $\|T(y)\| \leq M$ if $\|y - y_0\| \leq a$, (iii) $\delta T(y_0; h)$ is a 1–1 mapping of a subspace of Y onto a subspace of Z. Then the sequence $\{y_i\}$ given by $(*)$ is well defined and there exists $d > 0$ such that $\{y_i\}$ converges quadratically to a unique element $\bar{y}$ of the d-neighborhood of y_0. Furthermore $\bar{y}$ is the unique solution of $T(y) = 0$ in this neighborhood. (Received December 17, 1951.)

299t. J. L. Walsh: *Degree of approximation to functions on a Jordan curve.*

Let C be an analytic Jordan curve contained in an annular region D. Let the sequence of functions $F_n(z)$ analytic in D satisfy the inequality $|F_n(z)| \leq AR^n$ in D and for some $f(z)$ the inequality $|f(z) - F_n(z)| \leq A_0/n^{p+\alpha}$ on C. Then on C the function $f(z)$ is of class $L(p, \alpha)$ if $0 < \alpha < 1$ and of class Z_p if $\alpha = 1$. For the respective components $F_{1n}(z)$ and $F_{2n}(z)$ of $F_n(z)$ and $f_1(z)$ and $f_2(z)$ of $f(z)$ one has $|f_j(z) - F_{jn}(z)| \leq A_1 \log n/n^{p+\alpha}$ on C, $j = 1, 2$. In particular this conclusion holds if $z = 0$ lies interior to C and if $F_n(z) = F_{1n}(z) + F_{2n}(z)$, $F_{1n}(z) = \sum_0^n a_{nk}z^k$, $F_{2n}(z) = \sum_{-n}^{-1} a_{nk}z^k$. (Received January 29, 1952.)

300. J. L. Walsh and D. M. Young: *A new bound for the moduli of continuity of harmonic functions.*

The use of the analogue for harmonic functions of Milloux's theorem for analytic functions shows that if R is an arbitrary bounded simply connected region, if the modulus of continuity of $u(x, y)$ on the boundary of R is $\omega(\delta)$, if M is the oscillation of $u(x, y)$ on the boundary, and if $u(x, y)$ is harmonic in R and continuous in the closure $\overline{R}$ of R, then the modulus of continuity $\omega^*(\delta)$ of $u(x, y)$ in $\overline{R}$ satisfies the inequality $\omega^*(\delta) \leq (D(\delta/D)^{1/2}) + (8M/\pi)(\delta/D)^{1/2}$, where D is any positive constant. A sharper inequality which is valid under stronger assumptions on R has been presented by the authors (Bull. Amer. Math. Soc. Abstract 58-2-206). (Received January 11, 1952.)

301. Jack Warga: *On a class of iterative procedures for solving normal systems of differential equations.* Preliminary report.

Given a system of differential equations $x' = f(x, t)$, $x(t_0) = x_0$, where x and $f(x, t)$

are n-dimensional vectors, a sequence of vector functions $y_j(t)$ is defined by $y_j' = G_j(y_j, t)$, $y_j(t_0) = x_0$. The vector functions $G_j(x, t)$ are bounded and continuous in some domain D of the x, t space, satisfying a Lipschitz condition in x and $f(y_j(t), t) - G_{j+1}(y_j(t), t) \to 0$ uniformly in t. A proof is given of the convergence of the sequence $y_j(t)$ to the solution of the given system and a study is made of convergence properties in particular cases. Various iterative procedures described in the literature are shown to be of the type under discussion. New procedures are suggested which are applicable to numerical work and practical calculations. (Received January 8, 1952.)

302*t*. Bertram Yood: *Difference algebras of linear transformations on a Banach space.*

Let $E(X)$ be the algebra of all bounded linear transformations on an infinite-dimensional Banach space X into X. Let $K(X)$ be the subset of all completely continuous transformations in $E(X)$. Let π be the natural homomorphism of $E(X)$ onto the difference algebra $E(X)-K(X)$. Let G be the set of regular elements of $E(X)-K(X)$ and let R be its radical. It is shown that if $T\in\pi^{-1}(G)$, $U\in\pi^{-1}(R)$, then T, $T+U$, T^*, and T^*+U^* have finite nullity and nul $(T^*)-$nul $(T)=$nul $(T^*+U^*)-$nul $(T+U)$. For $T=I$, $U\in K(X)$ this reduces to a well known result of Schauder, Studia Math. vol. 2 (1930). The relations between the sets of (left, right) regular elements in $E(X)$ and in $E(X)-K(X)$ are investigated in some detail. (Received January 9, 1952.)

APPLIED MATHEMATICS

303. Abolghassem Ghaffari: *The behavior of hodograph functions for slow motion.*

In a recent paper (Bull. Amer. Math. Soc. Abstract 58-2-112) the author showed that the elementary solutions of the hodograph equations depend on the functions (1) $\psi_m(\zeta)=\zeta^{m/2}F(a_m, b_m, c_m; \zeta)$ for $m>0$, $\psi_{-n}(\zeta)=[\zeta^{-n/2}-K\zeta^{n/2}\zeta_s^{-n}][1+O(\zeta)]$ for $m=-n<0$, and $\phi_m(\zeta)=P/m\partial\psi_m/\partial\zeta$ for all values of m, where $\zeta=w^2/w_m^2$ (w being the compressible fluid speed with w_m its maximum, ζ_s being the value of ζ at the sonic speed, K being independent of ζ, F being the hypergeometric function formed with a_m, b_m, and c_m, and P being $2\zeta(1-\zeta)^{-\beta}$ where $\beta=(\gamma-1)^{-1}$ with γ the adiabatic index). The functions (1) which shall be referred to as ***hodograph functions*** can be written in the form (2) $\psi_m(\zeta)=\zeta^{m/2}+O(\zeta^{m/2+1})$, $\phi_m(\zeta)=\zeta^{m/2}+O(\zeta^{m/2+1})$, for all values of m. Since ϕ_m and ψ_m behave like $\zeta^{m/2}$ for small ζ, as shown in (2), it follows that when $m>0$, the hodograph functions are both positive monotone increasing functions of ζ and their first derivatives are positive near $\zeta=0$. Hence they are all positive throughout the subsonic range. It can be shown that when $m<0$, ψ_m is positive, but this is not true of its derivative or of ϕ_m. (Received January 7, 1952.)

304*t*. R. L. Sternberg: *A criterion for stability of nonlinear pulsed systems.*

Consider a pulsed system for which the output sequence $\{y_n\}$ is related to the input sequence $\{x_n\}$ by a relation of the form $y_n=\sum_{k=1}^{\infty} w_k(x_{n-k})$ ($n=0, \pm 1, \pm 2, \cdots$) where the $w_k(x)$ are arbitrary real single finite-valued functions of a real variable x. Call the system stable if $\{x_n\}$ bounded implies that $\{y_n\}$ is defined and bounded. Then by generalizing a method given by W. Hurewicz the following theorem is obtained. If the system is stable, then for each m one has $(*)$ $\left|\sum_{k=1}^{\infty} w_k[m \operatorname{sgn} w_k(m)]\right| \neq \infty$, i.e., for each m the series in $(*)$ converges or oscillates; moreover, in case for each $a>0$ there exists an m_a such that on $[-a, a]$ one has $0\leqq w_k(x)\leqq w_k(m_a)$ $(k=1, 2, \cdots)$, then the system is stable if and only if for each m the series in $(*)$ converges. This work was supported in part by a contract with the Rome Air Development Center of the U.S. A. F. (Received January 11, 1952.)

GEOMETRY

305t. Edward Kasner and Don Mittleman: *Strongly orthogonal algebraic curves.*

If p is a point of intersection of two algebraic curves $C_1:\phi=0$ and $C_2:\psi=0$ at which all of the branches of C_1 are orthogonal to all the branches of C_2, then C_1 and C_2 are called totally orthogonal at p. Two algebraic curves which are totally orthogonal at each finite point of intersection and for which, at each point of intersection which lies on the infinite line, the order of $\omega=\phi_x\psi_x+\phi_y\psi_y$ is at least r_i+s_i-1 are said to be strongly orthogonal. The fundamental theorem of M. Nöther is shown to be applicable and ω is a linear combination of ϕ and ψ. Necessary and sufficient conditions that two algebraic curves be strongly orthogonal are given (a) when one of the curves is a straight line, the other being arbitrary, and (b) when each of the two curves is a nondegenerate conic. (Received January 8, 1952.)

306. V. L. Klee: *The critical set of a convex body.*

For a convex body C in E^n, $x\in \text{Int } C$, and $y\in FC$, let $\rho(y, x)=yx/yp$, where $[y, p]$ is the chord containing $[y, x]$. Let $\rho(x)=\sup_{y\in FC}\rho(y, x)$, $S(x)=\{y \mid y\in FC$ and $\rho(y, x)=\rho(x)\}$, $r=\inf_{x\in \text{Int } C}\rho(x)$, and $C^*=\{x \mid \rho(x)=r\}$. C^* and r are the critical set and critical ratio investigated in E^2 by B. H. Neumann [J. London Math. Soc. vol. 14 (1939) pp. 262–272] and in E^n by Hammer and Sobczyk [Bull. Amer. Math. Soc. Abstract 57-2-112]. This paper provides a more detailed discussion in E^n. Typical results (with $C\subset E^n$, $n\geqq 2$) are (1) $\dim C^*\leqq n-2$; (2) for each $x\in C^*$, $S(x)$ contains at least three points; (3) $r(1-r)^{-1}+\dim C^*\leqq n$, and if greater than $n-1$, then for no $x\in C^*$ is $S(x)$ contained in an open half-space parallel to C^* but not containing C^*. Critical sets of certain special bodies are investigated in detail. One of the principal tools is an extension of Helly's theorem on the intersection of convex sets. (Received January 16, 1952.)

307t. T. K. Pan: *Pseudo normal curvature and related curves.*

Let $\mathbf{v}$ be the vector of a vector field in a surface of ordinary space. The derived vector of $\mathbf{v}$ along a curve of the surface can be decomposed into a component tangential and a component normal to the surface. The former component called the angular spread has been studied by W. C. Graustein and R. M. Peters. It is the latter component, which is evidently a generalization of the normal curvature vector, with which the paper is concerned. Pseudo normal curvature, pseudo asymptotic line, pseudo line of curvature, indicatrix and curve of a vector field in a surface are defined, new characteristics of principal curvature and line of curvature are derived, and most of the theorems connected with normal curvatures, such as Meusnier's theorem, Euler's theorem, etc., are generalized first for a surface in an ordinary space and then extended to V_n in V_{n+1} and to V_n in V_m. (Received January 8, 1952.)

308. Helene Reschovsky: *Sets associated with convex bodies.*

Let $D(k, n)$, $1/2\leqq k<1$, $n=0, 1, 2, \cdots$, be set of points within a plane convex body such that there exist exactly n chords of C which are divided by these points in the ratio k, the ratio of the larger segment to the whole chord. B. H. Neumann (J. London Math. Soc. vol. 14) has proved that there exists in the interior of a plane convex body a unique point such that the maximum ratio in which a point divides all chords passing through it assumes a minimum value r^*, $1/2 \leq r^* \leq 2/3$. In this note it is proved that (1) $D(k, 0)$ is two-dimensional when $k > r^*$ and empty when $k \leq r^*$, (2) $D(k, n)$, $n \neq 0$, is two-dimensional for an even number n and one-dimensional for an odd number n, when $k > 1/2$, (3) $D(1/2, n)$, $n \neq 0$, is two-dimensional for an odd number n and one-dimensional for an even n. (Received January 4, 1952.)

309*t*. A. R. Schweitzer: *On the derivation of the regressive product in Grassmann's geometrical calculus.* III.

Values are derived for the regressive products $P_2(\alpha_1\alpha_2\alpha_3\alpha_4) = \alpha_1\alpha_2 \cdot \alpha_1\alpha_3\alpha_4$ and $P_2'(\alpha_1\alpha_2\alpha_3\alpha_4) = \alpha_1\alpha_3\alpha_4 \cdot \alpha_1\alpha_2$ (Grassmann, *Gesammelte Werke*, vol. 1, part 1, pp. 231, 244). From $P_1(\alpha_1\alpha_2\alpha_3\alpha_4) = c_1q \cdot \alpha_1\alpha_2$ and $P_2(\alpha_1\alpha_2\alpha_3\alpha_4) = c_2q^2 \cdot \alpha_1$ since the product $P_2(\alpha_1\alpha_2\alpha_3\alpha_4)$ is associative, one obtains $c_2q^2 \cdot \alpha_1 = c_1q \cdot \alpha_1\alpha_2 \cdot \alpha_1\alpha_3\alpha_4$ and (by a change of notation) $c_2q^2 \cdot \alpha_1 = \alpha_1\alpha_3\alpha_4 \cdot c_1q \cdot \alpha_1\alpha_2$. It is assumed that the latter statements are respectively equivalent to $c_2q \cdot \alpha_1 = c_1\alpha_1\alpha_2 \cdot \alpha_1\alpha_3\alpha_4$ and $c_2q \cdot \alpha_1 = \alpha_1\alpha_3\alpha_4 \cdot c_1 \cdot \alpha_1\alpha_2$ and that $c_1 = c_2 = 1$. Definitions: $[\alpha_1\alpha_2\alpha_3\alpha_4] \cdot \alpha_1 = \alpha_1\alpha_2 \cdot \alpha_1\alpha_3\alpha_4$ means: There exist e_1, e_2, e_3, e_4 such that $q \cdot \alpha_4 = \alpha_1\alpha_2 \cdot \alpha_1\alpha_3\alpha_4$ and analogously for $[\alpha_1\alpha_2\alpha_3\alpha_4] \cdot \alpha_1 = \alpha_1\alpha_3\alpha_4 \cdot \alpha_1\alpha_2$. Therefore $\alpha_1\alpha_2 \cdot \alpha_1\alpha_3\alpha_4 = \alpha_1\alpha_3\alpha_4 \cdot \alpha_1\alpha_2 = [\alpha_1\alpha_2\alpha_3\alpha_4] \cdot \alpha_1$. (Received January 9, 1952.)

310*t*. A. R. Schweitzer: On the derivation of the regressive product in Grassmann's geometrical calculus. IV.

The complete statements governing the regressive products $P_1(\alpha_1\alpha_2\alpha_3\alpha_4) = \alpha_1\alpha_2\alpha_3 \cdot \alpha_1\alpha_2\alpha_4$, $P_2(\alpha_1\alpha_2\alpha_3\alpha_4) = \alpha_1\alpha_2 \cdot \alpha_1\alpha_3\alpha_4$, $P_2'(\alpha_1\alpha_2\alpha_3\alpha_4) = \alpha_1\alpha_3\alpha_4 \cdot \alpha_1\alpha_2$ are represented by the abstract statement $A \cdot CD = AD \cdot C$ where for P_1, $A = \alpha_1\alpha_2\alpha_3$, $C = \alpha_1\alpha_2$, $D = \alpha_4$; for P_2, $A = \alpha_1\alpha_2$, $C = \alpha_1$, $D = \alpha_3\alpha_4$; and for P_2', $A = \alpha_1\alpha_3\alpha_4$, $C = \alpha_1$, $D = \alpha_2$. Reference is made to Grassmann, *Gesammelte Werke*, vol. 1, part 1, pp. 213–219, 295–296. (Received January 9, 1952.)

LOGIC AND FOUNDATIONS

311*t*. Hyman Kamel: *Relational algebra.*

A new system of axioms is given for a relational algebra $\mathfrak{A} = (\mathfrak{C}, ', \cap, \cup, {}^\cup, \cdot, \Lambda, V, I)$, namely: (1) $(\mathfrak{C}, ', \cap, \cup, \Lambda, V)$ is a boolean algebra, (2) $(RS)T = R(ST)$, (3) $RI = R$, (4) $R^{\cup\cup} = R$, (5) $(RS)^\cup = S^\cup R^\cup$, (6) $R\Lambda = \Lambda = \Lambda R$, (7) $RS \cap T \subset (R \cap TS^\cup)(S \cap R^\cup T)$. The axiom system is proved equivalent to Tarski's (Journal of Symbolic Logic (1941) pp. 76–77). A proof is given that $\mathfrak{A}$ is simple ($RV = V$ or $VR' = R$) if and only if $R \neq \Lambda$ implies $VRV = V$. The bulk of the identities in Schroder's book are deduced. The notion of a uniform relational algebra (u.r.a.) is introduced by postulating a subclass $\mathfrak{U} = [\alpha, \beta, \gamma, \cdots]$ of $\mathfrak{A}$ such that: (1) $\mathfrak{U}$ is a filter on $\mathfrak{A}$, (2) $I \subset \alpha$, (3) $\alpha^\cup \in \mathfrak{U}$, (4) For every α there exists a β such that $\beta\beta \subset \alpha$. It is proved that every u.r.a. is a closure algebra with closure defined by $\overline{R} = \bigcap_\alpha \alpha R \alpha$. A beginning is made in developing u.r.a. as an algebraization of uniform space theory. (Received January 17, 1952.)

STATISTICS AND PROBABILITY

312*t*. H. W. Becker: *Permutation isomorphs.* I.

The number of n letter words in which the alphabetically kth letter may not occur prior to the kth position (factorial words) is $f_n = n!$. An n in the mth position of an n letter permutation corresponds to a terminal m in an n letter factorial word. This isomorphism between f_n and the $P_n = n!$ permutations makes $abc \cdots n = f_n \cap P_n$ self-correspondent. Double entry tables $f_{n,m} \to P_{n,m}$ then furnish many new classifications of permutations, based on the notions of convergent, and omega. The $(n-1)$th, $(n-2)$th, $\cdots$ convergents of a permutation are the permutations remaining on removal of the letters n, $n-1$, $\cdots$. The letter k is the omega of the kth convergent. Thus ${}_{\rm I}f_{n,m} = m \cdot {}_{\rm I}f_{n,m} + (n-m+1) \cdot {}_{\rm I}f_{n-1,m-1}$, the subset characterized by n letters m different$\to {}_{\rm I}P_{n,m}$, the n letter permutations whose omegas occupy m different positions in their convergents (enumerating also those with $m-1$ inversions, or m ascending runs, J. Riordan, Proceedings of the American Mathematical Society vol. 2 (1951) p. 429). Likewise ${}_{\rm III}f_{n,m} = (n-1) \cdot {}_{\rm III}f_{n-1,m} + {}_{\rm III}f_{n-1,m-1}$, characterized by m a's$\to {}_{\rm III}P_{n,m}$, characterized by m convergents with initial omegas (or else, by m cycles, J. Touchard, Acta Math. vol. 70 (1939) p. 243). Of the two interpretations of $n!$: P_n is the most important to physical science; f_n, to aesthetics. They have further isomorphisms, in terms of square and triangular chessboards. (Received January 8, 1952.)

313*t*. H. W. Becker: *Permutation isomorphs*. II.

Pitch is said to be an element of form in Chinese poetry. If we non-repetitively permute the pitch of each member of a rhyme group in all possible ways, we get L_n pitch-rhyme schemes, where (Riordan) $L_{n+1} = (2n+1)L_n - n(n-1)L_{n-1} = S(L+S)^n$ is the Laguerre number whose generating function is $e^{tL} = e^{t/(1-t)}$. If the first member of each rhyme group is exempted from permutation, we get $R_n = n!$ pitch-rhyme schemes, whose generating function is $e^{tR} = e^{-\log(1-t)}$, which also generates the cycles of substitutions $S_n = n!$ (Touchard, ibid.). Some classification isomorphisms $S_{n,m} \to R_{n,m}$ are the well known tables of m cycles$\to m$ different letters, and m unit cycles $\to m$ singletons; and the new tables, of n in mth cycle$\to$terminal m, m the bottom letter of the first cycle$\to$last "a" in mth position, last underanged letter$\to$last singleton, first m letters all in different cycles$\to$initial ascending run of length m, $n-m-1$ cocyclic pairs of consecutive letters$\to$m non-couplets. These may be specialized to $S^{(k)}_{n,m} \to R^{(k)}_{n,m} = (n-1)S^{(k)}_{n-1,m} + (n-1)_{k-1} \cdot S^{(k)}_{n-1,m-1}$, characterized by at least k letters in each cycle, or rhyme group, whose generating function is $\exp tS^{(k)} = \exp\left(\sum_k^\infty t^i/i\right)$. $S^{(2)}_n = (S-1)^n =$ subfactorial n. (Received January 8, 1952).

314. Miriam A. Lipschutz: *Some results on strong laws*.

Chung and Erdös (*On the application of the Borel-Cantelli lemma*, Trans. Amer. Math. Soc. vol. 72 (1952) pp. 179–186) have proved the following theorem. Let $\{E_n\}$ be an infinite sequence of events with $\sum P(E_n) = \infty$. Under a set of auxiliary conditions involving the joint probabilities of pairs of events E_j, we can conclude that $P(E_n \text{ i.o.}) = 1$ providing the following holds: For every positive integer k there exists a constant c and an integral valued function $H(n)$ such that for every $k \geq H(n)$ and $n \geq k$ we have (M): $P(E_k | E'_k \cdots E'_n) > cP(E_k)$. On the basis of a general result on the validity of (M) for (a) certain types of events connected with sums of independent r.v., (b) recurrent events, the author has applied this theorem to obtain strong laws for the distribution of the number of positive sums of independent r.v., covering all cases where the arcsine law has been shown to hold. Similarly she has obtained strong upper bounds for the number of realizations of $\mathcal{E}$ in n trials for recurrent events $\mathcal{E}$ with infinite variance when the distribution of the recurrence time is of the form $P(X>x)=h(x)/x^{\alpha}$, with $\lim_{x\to\infty} h(cx)/h(x)=1$ and $\alpha=1/2$ (Feller, *Fluctuation theory of recurrent events*, Trans. Amer. Math. Soc. vol. 67, p. 106). This includes the result of Chung-Hunt on the number of zeros in $\sum\pm 1$ (Ann. of Math. vol. 50, p. 389). (Received January 9, 1952.)

TOPOLOGY

315. R. D. Anderson: *A monotone interior dimension-raising mapping defined over S^3.*

The author shows that there exists a continuous collection G of mutually exclusive compact continua filling up S^3 such that G with respect to its elements as points is homeomorphic to a space in which the Hilbert cube is imbedded. This result settles affirmatively the question as to whether there exists an interior map of a manifold on a space of higher dimension (S. Eilenberg, *On the problems of topology*, Ann. of Math. vol. 50 (1949) pp. 247–260). As a corollary of the methods used it is also established that if S is any compact triangulable n-manifold ($n>1$) and ϵ is any positive number, there exists a monotone interior mapping f of S onto S such that the inverse image under f of any point S is within ϵ of each point of S. (Received January 10, 1952.)

316*t*. D. O. Ellis: *Orbital topologies.*

Let S be a set and γ a mapping of S into itself. Define: x is an accumulation point of X in S_γ if and only if x is in the γ-orbits of infinitely many distinct points of X. S_γ is a T_1 space but not, in general, a T_2 space. The major results are: (1) *If γ is finite-to-one, then γ is continuous in S_γ*; (2) *If δ is finite-to-one mapping of S into itself and δ commutes with γ, then δ is continuous in S_γ*; (3) *If γ is biuniform and onto, γ is a homeomorphism of S_γ*; and, as a novelty, (4) *If k is any infinite cardinal, there is a dense-in-itself T_1 space of cardinal 2^k whose homeomorphism group contains an Abelian subgroup of cardinal 2^k of involutory homeomorphisms.* (Received December 17, 1951.)

317*t*. S. T. Hu: *A cohomology theory with higher coboundary operators.* III. *The homotopy axiom and the groups for spheres.*

In our first note of this series [Indagationes Mathematicae vol. 11 (1949) pp. 418–424], a generalization of the classical cohomology theory is given by introducing coboundary operators of higher order. In our second note [Indagationes Mathematicae vol. 12 (1950) pp. 1–7], the algebraic axioms, the exactness axiom, and the excision axiom of Eilenberg and Steenrod are verified; and the m-dimensional (p, q)-cohomology groups of a single point are computed. At that time, the author was not able to see if the homotopy axiom is satisfied. In a recent note of J. W. Keesee [Ann. of Math. vol. 54 (1951) pp. 247–249], it is shown that the continuity axiom and the algebraic axioms imply the homotopy axiom. This opens an easy way to check the homotopy axiom for our generalized theory. In the present note, the author indicates the verification of the homotopy axiom for our generalized theory by means of Keesee's theorem. By means of these proved axioms, the m-dimensional (p, q)-cohomology groups of an n-sphere over a coefficient group G are computed in the last section of the present note. (Received January 8, 1952.)

318*t*. S. T. Hu: *On local structure of finite-dimensional groups.*

Let G denote a connected locally compact separable metric topological group of finite dimension n. In a work of Deane Montgomery [Ann. of Math. vol. 52 (1950) pp. 591–605], it is proved that G contains a compact zero-dimensional set Z and an invariant connected, locally connected, locally compact n-dimensional local group C such that $U=ZC$ is an open set containing the identity element e and that the correspondence $(z, c)\rightarrow zc$ defines a homeomorphism of the topological product space $Z\times C$ onto U. In general, it is not known whether or not the set Z can be selected to be a subgroup of G. In the present paper, it is proved by means of local group extensions that the answer is affirmative at least for the particular case that the center of G is locally connected. As a corollary, we have the assertion that if G has no arbitrarily small subgroups, then G must be locally connected. (Received January 8, 1952)

L. W. COHEN
Associate Secretary

BOOK REVIEWS

Activity analysis of production and allocation. Ed. by T. C. Koopmans. (Cowles Commission Monograph, no. 13.) New York, Wiley, 1951. $4.50.

This volume consists of twenty-five papers presented at a conference in Chicago in June 1949. The conference and the research leading to the papers presented were supported by The Rand Corporation. The theme of the conference was the application of mathematics to economics, and the papers are grouped by the editor under four headings: theory of programming and allocation, applications of allocation models, mathematical properties of convex sets, and problems of computation. Of these the third is partly expository, the fourth deals with problems of computing optimal points and situations some of which are discussed in the first part, and the second deals with some special applications to economic situations. (For recent results on the computing problem the reader is also referred to Brown and v. Neumann, pp. 73–80 of the volume *Contributions to the theory of games*, edited by H. W. Kuhn and A. W. Tucker.)

Several of the contributions to the first part deal with linear technological systems. A linear technological system is one such that the quantity of output of each commodity produced is multiplied by k if each input (of commodities and labor) is multiplied by k. The totality of points which correspond to possible outputs constitute a convex set in a Euclidean space. There is an obvious partial ordering of the points according to their "efficiency." Koopmans and Georgescu-Roegen discuss these efficient points, which obviously have to be on the surface of the convex body and must have suitably inclined support planes. These ideas have been current in mathematical statistics since Wald proved that the totality of Bayes solutions form a complete class. Basing himself on a paper by Wald and Wolfowitz (*Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability*, University of California Press, pp. 149–158), Gerard Debreu has since discussed these ideas in an economic context in a paper in Econometrica (July, 1951, pp. 273–292).

Three contributions (by Samuelson, Koopmans, and Arrow) are devoted to proving a theorem enunciated and proved by Samuelson. Let X_i, $i=1, \cdots, n$, be the total production of the ith output, which is divided into a final output C_i and inputs X_{ji}, $j=1, \cdots, n$, which are used to produce the jth input. Thus

$$X_i = C_i + \sum_{j=1}^{n} X_{ji} \qquad (i = 1, \cdots, n).$$

Labor, the $(n+1)$st good, is thought of as the sole nonproduced good, and its given total X_{n+1} is allocated among the different industries so that

$$X_{n+1} = \sum_{j=1}^{n} X_{j,(n+1)}.$$

Let each good be subject to a production function F_i which is homogeneous of the first order. Equilibrium requires that any C, say C_1, be at a maximum subject to fixed values of X_{n+1} and the other C's. This means that

$$C_1 = F_1(X_{11}, X_{12}, \cdots, X_{1,(n+1)}) - \sum_{j=1}^{n} X_{j1}$$

is to be a maximum subject to

$$C_i = F_i(X_{i1}, X_{i2}, \cdots, X_{i,(n+1)}) - \sum_{j=1}^{n} X_{ji} \qquad (i = 2, \cdots, n),$$

$$X_{n+1} = \sum_{j=1}^{n} X_{j,(n+1)}.$$

Samuelson's theorem asserts that the maximizing values $\{X_{ij}\}$ are such that the $\{X_{ij}/X_i\}$ are independent of the fixed values $C_2, \cdots, C_n, X_{n+1}$.

Linear programming of an economy is a matter of allocating the available resources so as to maximize the utility of the economy. Mathematically the problem is one of maximizing a linear function of several variables constrained by linear inequalities. Dantzig proves that this problem is equivalent to the problem of solving a zero-sum two-person game. More general results are given by Gale, Kuhn, and Tucker, who prove general duality and existence theorems. (For nonlinear programming see Kuhn and Tucker, *Proceedings of the Second Berkeley Symposium on Probability and Statistics*, pp. 481–492.)

An introduction to the volume by Koopmans gives descriptions of the papers and their interrelations.

J. WOLFOWITZ

Classical mechanics. By H. Goldstein. Cambridge, Addison-Wesley, 1951. 12+399 pp.

This book gives an advanced course in classical mechanics, with emphasis on those methods and principles which have proved to be most useful in various branches of modern physics.

The exposition is based on the assumption that the reader is already familiar not only with elementary principles of mechanics, but also with a range of mathematics somewhat beyond the usual undergraduate program, in particular with vector analysis which is used freely throughout the book.

The scope of the book may be most readily presented by giving a brief statement concerning the successive chapters:

The first chapter gives, in 29 pages, a survey of elementary principles of the mechanics of a system of particles, including the Lagrange equations, velocity-dependent potentials, and the dissipation function. This succinct treatment is made possible by use of vector analysis and by omission of applications.

The second chapter (28 pages) starts with Hamilton's principle, derives the Lagrange equations anew, and discusses advantages of a variational principle formulation of the basis of mechanics. The digression on the calculus of variations is, of course, very brief.

The third chapter (33 pages) is devoted to applications to central forces. It includes Kepler's laws, and a discussion of orbits in cases of attraction where the potential is an integral power of the distance. The problem of scattering of particles in a central force field is also analyzed.

In chapter four (50 pages) we find a treatment of the kinematics of rigid body motion. There is a digression on orthogonal transformations and matrix theory, providing mathematical tools to be used systematically in this and following chapters. The Eulerian angles, the Cayley-Klein parameters, the eigenvalue problem, infinitesimal rotations, and the Coriolis force are centers of interest.

Chapter five (42 pages) is devoted to the equations of motion of a rigid body. Tensors and dyadics are introduced, and used effectively. Problems which are discussed include the force-free motion of a rigid body, the heavy symmetrical top with a fixed point, and precession of charged bodies in a magnetic field.

Chapter six (30 pages) introduces special relativity into classical mechanics, with a minimum of historical material. The Lorentz transformation is neatly derived, and the idea of a four-dimensional world advanced. The force and energy equations in relativistic mechanics are found and Lagrangian formulations discussed.

In chapter seven (22 pages) we find a study of the Hamiltonian equations of motion and of the principle of least action. Most of the discussion is devoted to non-relativistic mechanics, but the problem of relativistic mechanics is briefly presented.

Chapter eight (36 pages) is entitled canonical transformations. It includes Poincaré's integral invariants, the Lagrange and Poisson brackets as canonical invariants, and infinitesimal contact transformations. As an application of Poisson brackets a fundamental theorem of statistical mechanics, Liouville's theorem, is proved.

Chapter nine (46 pages) is devoted to the Hamilton-Jacobi theory. A study of action-angle variables leads up to brief remarks on Bohr's quantum theory. A final section shows how the Hamilton-Jacobi formulation is particularly suited to generalize from classical to wave mechanics.

Chapter ten (29 pages) treats of small oscillations of a mechanical system about a point of stable equilibrium. With the customary approximations, the Lagrangian equations appear as a system of linear differential equations with constant coefficients, the constants being restricted by the conditions that the kinetic energy is a positive definite form and that the potential energy has a minimum at the equilibrium point. The analysis of this system is carried out elegantly by free use of matrix theory. A very interesting application is made to free vibrations of a linear triatomic molecule.

Chapter eleven (26 pages) concludes the book with an introduction to the Lagrangian and Hamiltonian formulations for continuous systems and fields. The transition from the discrete variable to the continuous variable is explained with care. Sound vibrations in gases is presented as an example of the Lagrangian formulation. The final section contains a description of fields by variational principles, providing a brief introduction to recent methods in theoretical physics.

At the end of each chapter references are given to well known books, with useful suggestions in each case as to desirable supplementary reading. At the end of the book are listed seventy-one treatises in mechanics and closely related branches of physics and mathematics; these would form a valuable reference library.

The exercises given at the end of the chapters are for the most part extensions of the theory in the text, but some serve to illustrate the use of the theory in solving interesting physical or astronomical problems. They will be difficult for most students, but their solution will add greatly to the understanding and power of the student.

The brevity and clarity arising from the use of vector analysis and matrix theory, and the wise selection of topics and illustrations enable the author to encompass a wide range of fundamental theory in a comparatively small volume. The well-prepared reader will find the presentation lucid and interesting; and if he goes further with his studies he will find this book a most useful volume to have at his elbow.

The few errors noted by the reviewer were of no great importance. Perhaps some mathematicians will be disturbed by an occasional lack of completeness or of precision, but such defects seem trivial by comparison with the high merits of the book as a whole.

The reviewer considers the book to be a very valuable addition to mathematical literature, bridging the gap as it does between the mechanics of the nineteenth century and more recent developments.

E. J. MOULTON

Mechanics. By S. Banach. Trans. by E. J. Scott. (Monografie Matematyczne, vol. 24.) Warszawa-Wrocław, 1951. 4+546 pp. $6.00.

This work was first published in Polish in 1938.

The book is notable for its clarity, and for the completeness of its exposition of the range of material covered. Assuming that the mathematical preparation of the student includes nothing beyond the elements of the calculus, the author undertakes to give as easy a presentation of classical mechanics as possible. To him this means the giving of a logically arranged set of definitions, assumptions, and theorems, with detailed proofs and with numerous illustrative examples. He has carried out his task exceptionally well—at least from the viewpoint of a mathematician.

As a text book the volume would be improved by the inclusion of problems to be solved by the student (none are given), but a teacher may select such exercises from the many which are available in standard works.

The illustrative examples are interesting, and cover a wide range. Thus we find (a) the reactions when a three-legged stool rests on a floor, (b) the fuel load required for an interplanetary rocket, (c) the determination of the mass of a planet, and so on. Chapter VI, on statics of a rigid body, is particularly designed for students of technology, and is so written as to be independent of much of the material in the preceding chapters.

The first chapter is devoted to that portion of the algebra of vectors which is most important in the study of mechanics. This tool is then freely and effectively used throughout the remainder of the book.

The mathematical quality of the exposition may be suggested by the following introductory paragraph: "Time. In kinematics, in addition to known geometric concepts, there arises the concept of time.

For purposes of theoretical kinematics it is sufficient to assume that to each moment there is assigned a certain number t, and that there are assigned smaller numbers before t than for moments after t. Conversely, to each ordering of numbers t there should correspond a certain moment: to a larger number a later moment."

The order of presentation and the extent of the theory covered is indicated by the titles of the successive chapters: I. *Theory of vectors.* II. *Kinematics of a point.* Subheadings are: motion relative to a frame of reference; change of frame of reference. III. *Dynamics of a material point.* Subheadings: dynamics of an unconstrained point; dynamics of a constrained point; dynamics of relative motion. IV. *Geometry of masses.* Subheadings: systems of points; solids, surfaces and material lines. V. *Systems of material points.* VI. *Statics of a rigid body.* Subheadings: unconstrained body; constrained body; systems of bodies. VII. *Kinematics of a rigid body.* VIII. *Dynamics of a rigid body.* IX. *Principle of virtual work.* X. *Dynamics of holonomic systems.* XI. *Variational principles of mechanics.*

The commendable detail with which the author discusses his material expands the text to some 546 pages; but by a judicious use of boldfaced titles, italicized statements, and displayed equations he has made it easy to pick out essential results. The careful insertion of references to previous formulas or theorems by page and number will save the reader's time and energy, and add materially to the value of the volume as a reference book.

While the discussion is detailed it is not carried to the point of pedantry. For example, the author devotes nearly three pages to the definitions of density of a body, surface, or line in terms of its mass (as a limit of a ratio), and to the formulas for the mass when the density is given (as integrals). He notes that the integral formulas check when the density is constant, and may be used to find the mass of an arbitrary part of the whole body, surface, or line. He does not, however, invoke the aid of some variant of Duhamel's theorem in arriving at the integral formulas, nor does he elaborate on the character of the curves and surfaces involved; in fact, the definition of density at a point of a body does not give a density for points on the surface of the body which would yield a continuous density function. To look after all such matters adequately is clearly impossible in an elementary course in mechanics.

The book is characterized by a high standard of precision, but is marred by occasional slips, some of which should, perhaps, be charged against the translator. Besides such editorial items as the omission of commas, the division of the word "makes" into two syllables, the occasional dropping of the dot in the scalar product of two vectors, and so on, there may be found logical errors which may be corrected by changing a "the" to an "a" or to "certain," or "both" to "the two" (pp. 53, 55), and errors such as "a virtual displacement is said to be every displacement of the point A . . . " (pp. 470, 471). Such slips are, however, not numerous enough to be very serious. An error of another kind arises from the inclusion of a factor 1/6 in a formula for the volume of a parallelepiped (p. 13).

The most serious blunder which the reviewer noted is the statement (p. 144) that if the density of the earth is distributed symmetrically with respect to the center of mass then it can be proved that the force of attraction is directed constantly toward the earth's center of mass. It is a vitally important fact in celestial mechanics that this is not so; the conclusion is incorrect even for a homogeneous oblate spheroid. It is an interesting problem to determine for what surfaces and what laws of density the attraction of a body for a particle exterior to it is constantly directed toward the center of mass of the body. The appropriate vector equation, equivalent to three scalar equations, is an integral equation for the density function, has an unknown function in an equation of the surface of the body, and a third unknown proportionality function. Solutions to this problem are known, but they do not agree with the one in the book.

We must express our appreciation to the author, to the translator, and to the publisher for adding this fine book to the collection of works on classical mechanics.

E. J. MOULTON

The theory of the Riemann zeta-function. By E. C. Titchmarsh. Oxford University Press, 1951. 6+346 pp. $8.00.

The zeta-function was introduced almost 100 years ago by Riemann in his famous memoir on the number of primes less than a given number. While since then enough has been discovered about the zeta-function to justify its use in analytic number theory, the question raised by Riemann about the location of its zeros remains unanswered. The milder hypothesis of Lindelöf that $\zeta(1/2+it)=O(t^{\epsilon})$ for every $\epsilon>0$ also remains unsettled. The zeta-function continues to be a major challenge to mathematicians.

The author in his well known Cambridge Tract of 1930 gave a remarkably comprehensive and concise account of the zeta-function. Now he has given an expanded account in order to include recent results of which the most notable are due to A. Selberg. The sole prerequisite for reading this treatise is a knowledge of the fundamentals of function theory. The author is an excellent expositor of the kind of analysis that up to now has been the major tool in research on the zeta-function.

A sketch of the contents of the book follows. The first chapter gives the Euler product and Dirichlet series representation of $\zeta(s)$. Representation of $\log \zeta(s)$, $1/\zeta(s)$, $\zeta^k(s)$, etc., are also given as well as series involving Ramanujan sums as coefficients.

The second chapter deals with the analytic character of $\zeta(s)$. It contains seven methods for proving the functional equation.

The third chapter contains the Hadamard-de la Vallée Poussin theorem that $\zeta(1+it)\neq 0$ and the prime number theorem, and considers $\zeta(s)$ and related functions for s slightly to the left of the line $R(s)=1$.

The fourth chapter takes up approximate formulae for $\zeta(s)$ including the very useful approximate functional equation.

The fifth chapter is on the order of $\zeta(s)$ in the critical strip and uses the methods of Weyl as developed by Hardy and Littlewood and of van der Corput for the estimation of exponential sums which occur in the approximate formulae for $\zeta(s)$. In the sixth chapter exponential sums are treated by Vinogradoff's method.

The seventh chapter treats the mean values of powers of $|\zeta(s)|$ along lines $R(s)=\text{constant}$ in the critical strip. The approximate formulae for $\zeta(s)$ are used as is the convexity of the mean values of analytic functions.

The symbol Ω is used to denote negation of o so that $F(t)=\Omega(\phi(t))$ means that there exists an $A>0$ such that $|F(t)|>A\phi(t)$ for some arbitrarily large values of t. In chapter eight Ω theorems for $\zeta(\sigma+it)$ and $1/\zeta(\sigma+it)$ are obtained by diophantine methods and also by dealing directly with integrals of high powers of the functions being studied.

The distribution of zeros in the critical strip is considered in chapter nine and recent results of A. Selberg are included. The zeros on the critical line are studied in chapter ten. Many proofs are given of Hardy's theorem that there are an infinite number of zeros on the critical line. The number of zeros of $\zeta(s)$ of the form $1/2+it$, $0<t\leqq T$, is denoted by $N_0(T)$. The result of A. Selberg that $N_0(T)>AT\log T$ for some $A>0$ and all large T is given. (It will be recalled that if the Riemann hypothesis is true $N_0(T)\sim(1/2\pi)T\log T$.)

In chapter eleven the general distribution of values of $\zeta(s)$ is studied. Near the line $R(s)=1$, Picard's theorem is used. In the critical strip diophantine approximation is used.

Let the number of divisors of n be denoted by $d(n)$ and $\sum_{n\leqq x} d(n)$

$=x\log x+(2\gamma-1)x+\Delta(x)$. It was shown by Dirichlet that $\Delta(x)=O(x^{1/2})$ and better results have been obtained since. The function $\Delta(x)$ is related by the Mellin transform to $\zeta^2(w)/w$ in the critical strip. The divisor problem, that is, the order of $\Delta(x)$, and generalizations are considered in chapter twelve.

In chapter thirteen the Lindelöf hypothesis is assumed to be true and consequences of it are investigated and in chapter fourteen consequences of assuming the truth of the Riemann hypothesis are investigated.

The final chapter is on calculations relating to the zeros of $\zeta(s)$. After indicating how the early zeros are shown to lie on the critical line the author observes that if the Riemann hypothesis is false this could be shown by using modern calculating devices.

N. LEVINSON

Advances in applied mechanics. Vol. II. Ed. by R. von Mises and T. von Kármán. New York, Academic Press, 1951. 10+233 pp. $6.50.

Kármán and Lin's paper, *On the statistical theory of isotropic turbulence*, represents the views of two distinguished specialists in the current semi-empirical turbulence theory. It purposes to clarify the notion of spectral similarity in flows where there is turbulent diffusion of energy. The authors divide the spectrum into three frequency ranges, in each of which only two parameters are considered significant; similarity laws follow by the usual dimensional argument. Since this paper is of a nonmathematical character, the reviewer does not detail its contents here.

Kuerti's survey, *The laminar boundary layer in compressible flow*, is successful in its expressed aim: "Although the mathematics used in the boundary layer literature presents no particular conceptual difficulties, it is complicated on account of the number of parameters involved and because of the different approaches tried by different authors. The study of the original literature thus requires a patient reader. Under these circumstances it seemed desirable to put together what may be called a guide to rather than a review of the existing literature on the subject." Attention is restricted to steady plane flow of a perfect gas with constant specific heats, the viscosity and thermal conduction being assumed functions of temperature only, and the Prandtl number constant. The bibliography of thirty items seems to be complete through 1948.

The author does not mention any work of his own, and there appears to be no original contribution made by the article. One notes, for example, that the derivation in Part II is based upon the usual order of magnitude assertions, even more disagreeable and even less convincing in the case of a compressible fluid than for an incompressible one. It would seem to be possible to extend the method of Lin [A.D. Michal, *Matrix and tensor calculus*, New York, see Chap. 18], which, while purely formal, at least makes clear what is assumed and what one might try to prove. The major part of the article classifies and presents "the transformations that have evolved in the search for practical solutions of the boundary layer problem," accompanied by numerical graphs. The mathematically inclined reader might gather some suggestions of possible approaches to a future strict treatment, but apparently work on this subject so far has been of a rather heuristic type. While the author's careful ordering and scrupulous presentation provide a reliable encyclopedic article, they are insufficient to arouse the reader's interest in this dry topic.

Clark and Reissner's paper, *Bending of curved tubes*, makes a solid contribution to engineering elasticity theory. The problem stated in the title, whose history is summarized in the introduction, is treated for the first time as a problem in the theory of thin toroidal shells of revolution. The appropriate differential equations were derived earlier by E. Reissner [Proc. Nat. Acad. Sci. U.S.A. vol. 35 (1949) pp. 204–208]. The meridian curve of the torus is expressed in the form $r=r(\xi)$, $z=z(\xi)$, where r and z are Cartesian coordinates in a meridian plane. The problem differs from those discussed in books on shell theory in that an azimuthal displacement $v=kr\theta$ is assumed, although the stress system is taken as axially symmetric.

The major part of the paper deals with the circular torus $r=a+b\sin\xi$, $z=-b\cos\xi$. Calling β the change of the latitude angle and retaining only the terms of lowest order in b/a, the authors reduce the equations of the problem to the form

$$\beta''+\mu\,(\sin\xi)\psi=0,$$
$$\psi''-\mu\,(\sin\xi)\beta=\mu k\cos\xi,$$

where ψ is a dimensionless stress function and $\mu=(12(1-\nu^2))^{1/2}b^2/(ah)$, h being the shell thickness. They set up a formal Fourier series solution, in which the coefficients are to be obtained by solving an infinite system of linear equations. They show that previous formal results concerning the problem correspond to taking the first few terms in one of two different ways. The series solution appears to be rapidly convergent for small μ. To supplement it the authors obtain an asymptotic solution for large μ.

If the applied moment m be written as $\rho EIk/a$, the authors find for the "rigidity factor" ρ the expressions

$$\rho = \begin{cases} \dfrac{16 + 5\mu^2/36}{16 + 41\mu^2/36 + \mu^4/576} & \text{for small } \mu, \\ \dfrac{2}{\mu} & \text{for large } \mu. \end{cases}$$

These formulae together give excellent agreement with independently computed numerical values based on a six term Fourier polynomial. The authors' results include simple approximate formulae for the stress and displacement distributions. In the last part of the paper they obtain asymptotic solutions for the elliptic torus.

The most interesting and stimulating article is Neményi's, *Recent developments in inverse and semi-inverse methods in the mechanics of continua*. He calls inverse "an investigation of a partial differential equation of physics if in it the boundary conditions (or certain other supplementary conditions) are not prescribed at the outset. Instead, the solution is defined by the differential equation and certain additional analytical, geometrical, kinematical, or physical properties of the field. In the semi-inverse method some of the boundary conditions are prescribed at the outset, whereas others are left open and obtained indirectly." This article does not attempt to present the subject exhaustively, but aims rather "to elucidate the nature, value, and the potentialities of this approach." Incidentally, however, it explains the essence of nearly all the main contributions.

The paper is divided into sections of about equal length concerning inviscid incompressible fluids, perfect gases, elastic bodies in equilibrium, and plastic bodies. Much of the work mentioned is very recent, some unpublished. The author in many cases is able to correlate, compare, and contrast the various investigations, often in different fields. At the end of the paper is a table summarizing results known up to the present on five semi-inverse problems in seven domains of continuum mechanics. Among the more interesting investigations cited are those of Taylor and Trkal on decaying motion of a viscous fluid; those of Tollmien, Prim, and Neményi-Prim on limiting lines, problems of invariance, and "generalized Beltrami flows" in gas dynamics; and those of Neményi on an "influence principle" and on stress trajectories in elasticity.

The author emphasizes the value of simple exact solutions as typical and suggestive illustrative cases, and he shows that such solutions have been or can be obtained by inverse or semi-inverse methods in many cases. His examples, too numerous to list here, substantiate also the following conclusions. "1. Inverse and semi-inverse methods [may] lead to solutions of important boundary value problems." "2. They [may] lead to the discovery of unsuspected discontinuities, limitations, or general field properties. . . ." "3. [They] may settle existence questions in a positive sense, or may decide a uniqueness problem in a negative sense." "4. [They] are essential for the comparative study of the differential equations of the various problems of mechanics."

The last and longest paper, *Theory of filtration of liquids in porous media*, by Polubarinova-Kochina and Falkovich, is a translation of an article in Prikladnaia Matematikha i Mekhanika vol. 11 (1947) pp. 629–677. There is no indication that the translation was made with the author's approval or even cognizance. The contrary is suggested by the odd remark at the head of the references: "The original reference style has been retained as presumably the most effective for the purpose of identification to Russian libraries. The abbreviations which the translator has been able to decode are explained below. . . ." One wonders which readers will be able to secure the assistance of a Russian librarian to locate the originals. The references are in fact absolutely incomprehensible in many instances (e.g. Izv. NIIG, GONTI, Nauchnîe zap. MGMI), erroneous in some others. This defect is aggravated by the extreme condensation of the article, which is hardly more than a list of results, so that it is only rarely possible to see how one step follows from the preceding. Even the figures are not always understandable. The interested reader will have to obtain the sources, and for most of these he will have to carry out a tedious detective work in bibliographical indices.

There are short sections on motion which do not obey Darcy's law, three-dimensional motions, and unsteady motions, but the bulk of the article deals with steady plane flow of an incompressible fluid obeying Darcy's law:

$$\vec{v} = -\kappa \operatorname{grad}\left[\frac{p}{\rho g} + y\right] = \operatorname{grad}\phi,$$

where $\vec{v}$ is the filtration velocity, κ the filtration coefficient, and y the vertical coordinate. The velocity potential ϕ is harmonic, so that the variety of mathematical techniques developed in plane potential theory are all applicable. The authors choose to deal only with Russian work, mainly exact solutions. There are a few references to Muskat's book [*The flow of homogeneous fluids through porous media*, New York and London, 1937], none to Neményi's [*Wasserbauliche*

Strömungslehre, 1933]. One is impressed by the variety of problems having both mathematical and practical interest. Because of the quantity of the material and the incomprehensible way in which it is presented, the reviewer makes no further attempt at summary. He notes passing reference to some investigations of an interesting mathematical character. (1) A justification of a method of series development given by Kufarev and Vinogradov [Akad. Nauk SSR Doklady vol. 57 (1947)]. (2) Qualitative theorems by Lavrentiev [*Conformal mapping*, OGIZ, 1946] concerning flow under a dam with cut-offs, situated over an impermeable rock; increasing the length of a cut-off increases pressure under the dam immediately upstream, decreases it downstream, and the most effective means of reducing the exit velocity is to lengthen the cut-off which is farthest downstream. No indication of how to obtain these results is given.

The language in the whole volume is the next poorest the reviewer has seen in a mathematical publication issued in an English-speaking country. The translator of the Russian article preserves a terse and ragged style, stringing together with the aid of parentheses (sometimes misplaced) unusual expressions which, while possibly elegant in Russian, ring oddly in English. In the article on shells infinitives are split and participles suspended by rule. The paper on inverse methods is an example of original grammar set off by stochastic punctuation. The paper on the boundary layer is written almost entirely in the passive voice. While the format of the book is pleasant, the setting is careless: some parts have dropped out of the formulae, some of the letters are broken or mis-set, some blank slugs have printed black bars here and there, there are a good many textual misprints, and the running heads for one article read "Continua mechanics." While at first sight it would seem that all formulae are intentionally left altogether unsupported by punctuation, careful search reveals a favored few which are set off by commas, periods, and even in one case a semicolon. In the copy sent for review there are holes in some of the pages. One observes also that although the volume was issued in the middle of 1951, apart from notes added in proof the bibliographies seem to indicate that all the articles were completed in 1949 or earlier.

The publishers display some audacity in offering this slim and carelessly printed little book of 233 pages, nearly a third of which is a translation, for a price equal to or greater than the cost of a volume of a first rate research journal containing three or four times as much material.

C. TRUESDELL

Problèmes concrets d'analyse fonctionnelle. By P. Lévy. Paris, Gauthier-Villars, 1951. 14+484 pp. 4000 fr.

This work is a revised edition of Lévy's *Leçons d'analyse fonctionnelle* published in 1922. The purpose of the new title is to emphasize the difference in method and point of view between this book and the very abstract and general theories to which the name "functional analysis" is now commonly attached. Apart from the fact that Lévy restricts attention to certain special function spaces, there is a more striking difference between this work and most other publications on functional analysis, in that Lévy devotes most of his space to the study of functional differential equations. These are the analogues in function space of total differential equations, first order partial differential equations, and second order partial differential equations, especially the Laplace equation. The last type of problem (considered in Part III) involves considerable difficulties, even in choosing a suitable formulation for it. It is necessary to consider problems of measure in function space, the definition of mean value of a functional, and the proper definition of the Laplace operator. Lévy's procedure is to consider these ideas in n-dimensional space, and then to proceed to the limit as n tends to infinity, so as to obtain a mean value for certain functionals defined in L_2. When the method of passage to the limit is chosen so that certain requirements are satisfied, other aspects of the geometrical situation become rather bizarre. Lévy has extensively revised Part III, but he states in his Preface that he still does not regard it as in definitive form. Choice of another method of passage from finite-dimensional space to the space C of continuous functions leads to the Wiener integral, which has other applications in functional analysis (cf. Paley and Wiener, *Fourier transforms in the complex domain*, Amer. Math. Soc. Colloquium Publications, vol. 19, 1934, Chaps. 9 and 10; also later memoirs by R. H. Cameron and W. T. Martin). It seems reasonable to forecast that investigation will reveal still other methods of defining integrals in the spaces C and L_2, which will have interesting applications.

A noteworthy change from the first edition is the addition of Part IV, by Franco Pellegrino, on analytic functionals of analytic functions. About forty percent of Part I has been omitted, including the old Chapter 3 on Lebesgue and Stieltjes integrals, Chapter 7 on orthogonal functions, and Chapter 8 on the equations of Fredholm and Volterra and integrodifferential equations. For a review of the first edition by C. A. Fischer, see Bull. Amer. Math. Soc. vol. 29 (1923) p. 229.

Part I is entitled *The Foundations of the functional calculus.* The function spaces to be considered are described in Chapter 1. Principal emphasis is placed on the space L_2. Chapter 2 proceeds to discuss linear functionals and their representations, and the first variation of functionals. Chapters 3 and 4 are concerned with functionals of the second degree, and the second variation, and with functionals of arbitrary degree. Principal attention is paid to the case when these functionals have the special forms $\int_a^b k(t)x(t)dt$, $\int_a^b\int_a^b k(s, t)x(s)x(t)dsdt$, $\int_a^b k(t)x^2(t)dt$, etc.

Part II is almost unchanged, even to the numbering of the paragraphs. It is concerned with total "differential" equations in function space, with application of these results to the theory of the Green's function, with partial functional differential equations of the first order, and with applications of this last idea to set up a Hamilton-Jacobi theory for multiple integrals in the calculus of variations. It is not clear how this Hamilton-Jacobi theory provides any effective illumination of the difficulties inherent in the theory of extrema of multiple integrals. In connection with total differential equations in function space, it may be noted that such problems have been studied in abstract Banach space by Michal and Elconin (see Acta Math. vol. 68 (1937) pp. 71–107).

Part III is entitled *The notion of mean value in function space, and the generalised Laplace equation*. The approximating n-dimensional space in L_2 (for the interval $[0, 1]$) is composed of the functions constant on the interior of each of the intervals $((i-1)/n, i/n)$. When a complete orthogonal system of functions in L_2 is chosen so as to set up a correspondence with classical Hilbert space, another natural method of choosing an approximating n-dimensional space is suggested. The mean values of a functional given by these two methods of approximation need not be the same, even when both exist, and the mean value given by the Hilbert space method may depend on the order of the axes. Lévy gives conditions on the choice of the coordinate system to ensure that these mean values shall agree. Chapter 5 of this section, on harmonic functionals, discusses Green's formula, potentials of simple and of double layers and of volume distributions, the Dirichlet problem, and the problem of Plateau, in function space.

The fourth and concluding part of the book consists of six chapters —121 pages— on the subject of analytic functionals of analytic functions, written by Franco Pellegrino, a pupil of Luigi Fantappiè, who initiated this branch of the theory of functionals about 1925.

The functions admitted as arguments of analytic functionals are those called "biregular." Each biregular function $y_0(t)$ has a domain

M_0 which is open on the complex sphere, does not include the whole sphere, and need not be connected. A biregular function is locally analytic, and vanishes at ∞ if defined there. A "continuation" of a biregular function is not necessarily an analytic continuation. The class of biregular functions is made into a topological space $S^{(1)}$ by the system of neighborhoods (A, σ) of its points (y_0, M_0). For convenience we shall denote by $N(y_0, M_0; A, \sigma)$ the neighborhood (A, σ) of (y_0, M_0) consisting of all (y, M) with $M \supset A$ and $|y(t)-y_0(t)| < \sigma$ on A, where A is closed, $A \subset M_0$, $\sigma > 0$. With these neighborhoods $S^{(1)}$ is a T_0-space. A set consisting of a single function (y_0, M_0) has as points of accumulation all the restrictions of y_0 which lie in $S^{(1)}$. A linear set in $S^{(1)}$ is a collection $E = [(y, M)]$ such that every finite subcollection of the domains M has a non-null intersection, and every finite linear combination $\sum c_j y_j$ with complex coefficients lies in E. The domain of the sum $\sum c_j y_j$ is of course the intersection of the domains M_j. It is shown that a linear open set in $S^{(1)}$ consists of all the functions $y(t)$ which are biregular on a fixed closed proper subset A of the complex sphere. It is noteworthy that the space $S^{(1)}$ itself is not linear.

An analytic curve in $S^{(1)}$ is given by a parametric representation $y(t, \alpha)$, where for each α_0 in a set Ω open on the α-sphere, the corresponding function $y(t, \alpha_0)$ is biregular on its domain $M(\alpha_0)$ (which is open on the t-sphere), and for each t_0 in a set Ω' open in the t-sphere $y(t_0, \alpha)$ is regular in α, and where the complement $I(\alpha)$ of $M(\alpha)$ is continuous with respect to α in terms of the usual distance of point sets. A functional F defined on an open set R in $S^{(1)}$ is said to be locally analytic in case $F(y_1) = F(y_2)$ whenever y_2 is a continuation of y_1, and for every analytic curve $y(t, \alpha)$ lying in R, $F[y(\cdot, \alpha)]$ is a locally analytic function of α. The author states that it is unknown whether every analytic functional is continuous. However, an analytic functional which is bounded near a point of $S^{(1)}$ is necessarily continuous there.

A linear functional $F(y)$ is one which is defined and linear on a linear open set. It is continuous if and only if it is analytic. The class of linear analytic functionals is isomorphic to $S^{(1)}$, and in fact it follows from Cauchy's integral formula that every linear analytic functional F is expressible in the form

$$F(y) = \frac{1}{2\pi i} \int_C u(t) y(t) dt$$

where the path C encloses the singularities of u and excludes those of

y, and the function u is the "antisymmetric indicatrix" $u(\alpha) = F[1/(\alpha - t)]$.

Chapter IV is a very brief chapter devoted to "mixed functionals," i.e., to transformations on the space $S^{(1)}$. The operation D of differentiation is a linear analytic transformation, and linear transformations permutable with D are called "operators of the closed cycle" (cf. Volterra, *La teoria dei funzionali applicata ai fenomeni ereditaria*, Atti Congresso Int. di Bologna, vol. IV, 1928). The antisymmetric indicatrix

$$u(\alpha, z) = F\left(\frac{1}{\alpha - t}; z\right)$$

of such an operator $F(y; z)$ is a function of the difference $(\alpha - z)$. "Normal operators" are those having the functions t^n $(n = 0, 1, 2, \cdots)$ as characteristic vectors. The antisymmetric indicatrix of a normal operator is a function of z/α. The properties of normal operators are used to derive the theorem of Hadamard which relates the singularities of the functions defined by the power series $\sum a_n t^n$, $\sum b_n t^n$, and $\sum a_n b_n t^n$. However, little else is given in the direction of explicit applications of these concepts.

In Chapter V the author discusses the application of analytic linear functionals to the solution of linear differential equations, both ordinary and partial. The justification of the symbolic calculus is taken up first. The later sections outline five methods for the explicit solution of certain classes of linear partial differential equations.

Chapter VI takes up the first and higher variations of nonlinear functionals, and defines the first derivative, for example, as the antisymmetric indicatrix of the first variation. These notions lead to an analogue of the Taylor expansion, called "the series of Fantappiè." Brief reference is made to maxima and minima of real analytic functionals.

Most of the details are omitted in the later chapters. References are included to the work of various authors, listed in the bibliography of 119 items, 49 of which are by Fantappiè. Unfortunately no page numbers are given in many of the references. Mention is made of a forthcoming work by Fantappiè and Pellegrino, which is to contain a more detailed exposition of the theory.

The theory of Fantappiè is interesting in being an explicit development of functional analysis for a case where the basic function space is not a Banach space. It may throw some light on the form which an abstract theory of analytic functionals should take.

L. M. GRAVES

Introduction to Hilbert space and the theory of spectral multiplicity. By P. R. Halmos. New York, Chelsea, 1951. 114 pp. $3.25.

There is little doubt that the author of this book enjoyed himself thoroughly during its preparation. Reading the result afforded this reviewer considerable pleasure. In one hundred and nine well-packed pages one finds an exposition which is always fresh, proofs which are sophisticated, and a choice of subject matter which is certainly timely. Some of the vineyard workers will say that P. R. Halmos has become addicted to the delights of writing expository tracts. Judging from recent results one can only wish him continued indulgence in this attractive vice.

The present work may confidently be recommended. However, beginners in the field should be cautioned before they rush off to secure a copy. Unless one is equipped and in training, one should not attempt the expedition. One must not be misled by the title. For this introduction to Hilbert space, one has to be an expert in measure theory. As a matter of fact it is best to have read the author's book on measure theory or its equivalent. One has to know enough about Banach spaces to be conversant with the Riesz representation theorem for the linear functionals on the space of continuous functions. We would be ready to wager that most young mathematicians learn that theorem subsequent to the theorem on the spectral resolution of hermitian operators and not prior to it—which is the scheme of things here. But, for those who know this material and wish an excellent introduction to multiplicity theory, this tract is just right.

The subject matter of the book is funnelled into three chapters: The geometry of Hilbert space; the structure of self-adjoint and normal operators; and multiplicity theory for a normal operator. For the last, an expert knowledge of measure theory is indispensable. Indeed, multiplicity theory is a magnificent measure-theoretic tour de force. The subject matter of the first two chapters might be said to constitute an introduction to Hilbert space, and for these, an a priori knowledge of classic measure theory is not essential. Thus one may question the author's decision to unveil his virtuosity in this direction sooner than was necessary, or perhaps desirable.

Chapter I has some features which differentiate it from previous texts in this domain. The Hilbert spaces under consideration are not assumed separable. The handling in proofs and notation from this point of view is completely successful. For another thing, considerable use is made of the parallelogram identity for vectors: $\|f+g\|^2+\|f-g\|^2=2\|f\|^2+2\|g\|^2$. The spirit of this and similar identities has certainly pervaded many smoke-filled colloquium rooms in the past decade and it is a pleasure to see it make its bow in book form. In terms of it certain proofs are very satisfactory. See, for example, the theorem concerning the existence of a vector perpendicular to a proper closed linear manifold.

As stated above, the second chapter is devoted to self-adjoint and normal operators. The bounded case only is treated. This calls for the usual preliminary development of a theory of projections. Also discussed are the elementary properties of the spectrum of an operator. In this connection the author gives a bizarre proof of the famous elementary theorem that if $\|1-A\|<1$, then A^{-1} exists. The series $(1-x)^{-1}=\sum_0^\infty x^n$ is probably the most extensively generalized in all of mathematics. Why not use it here instead of relying for a proof on Theorem 21.3, or any other for that matter? As mentioned before, the proof of the spectral theorem is based on "the external analytic crutch of measure theory." The author seems to regret (p. 71) "the lot of apparently formidable machinery" that he used; thus we shall not overburden his conscience with further allusion to the matter.

Before setting forth on the arduous journey through multiplicity theory, the reader is advised to be well-rested and of strong determination. Though the fundamental ideas and the ultimate result are quite reasonable and lucid, one's patience is often taxed by the annoying complexity inherent to the problem. Thus, such simple phenomena as those concerning orthogonality of manifolds can be compounded with such tirelessness as to wilt one's spirit. Before one reaches the point of no return in the proof, the temptation is great to let intuition become master over logic and consign mere proof to the antipurgatory. The original results in multiplicity theory were given by Hellinger in 1909. The treatment of non-separable spaces seems to be essentially more difficult. The first results and subsequent refinements are due to Wecken (1939), Nakano (1941), and Plessner and Rohlin (1946). It is these authors who have most influenced the present treatment. The problem here considered may be phrased in the form: Determine the unitary invariants of a single normal operator. That is, given two normal operators A and B, find significant, satisfactory, necessary, and sufficient conditions that for some unitary operator U, $B=U^{-1}AU$. For the finite-dimensional case, the answer is obviously that A and B should have the same spectral (or characteristic) values and that the manifold of vectors associated with each spectral value should have the same dimensionality for A as for B. The infinite-dimensional situation is enormously more complicated. The reader who is interested in a treatment which contains the present one as a special case, indeed, gives the unitary invariants of any commutative W^* algebra, should consult the recently published paper of I. E. Segal (Memoirs of the American Mathematical Society, no. 9, II, 1951).

The general result of the third chapter should be described briefly. Suppose that X is a space, S is a σ-Boolean algebra of sets over X, and that μ is a finite measure on S. Consider the Hilbert space $L_2(\mu)$ of square integrable functions on X. In this space one may construct for every set M in S the projection operator $E(M)f = \chi_M \cdot f$ where f is arbitrary in $L_2(\mu)$ and χ_M is the characteristic function of M. $E(M)$ is an example of a spectral measure. Now suppose u is a cardinal number. Form the direct sum of u copies of the space $L_2(\mu)$—this leads to phenomena of multiplicity u. Consider then the spectral measure defined by $E(M)\{f_k\} = \{\chi_M \cdot f_k\}$. Here $\{f_k\}$ is the general element in the direct sum space and k is an index ranging over a set of cardinal number u. Finally, let μ_j be orthogonal finite measures on S, for every j let u_j be a cardinal number, and construct the spectral measure $E(M)\{f_{jk}\} = \{\chi_M \cdot f_{jk}\}$. The latter is called the canonical example of a spectral measure.

Now let A be a normal operator. Then there is associated with A a complex resolution of the identity $E(\lambda)$ such that $A = \int \lambda dE(\lambda)$. If M is a Borel set in the complex plane, $E(M)$ is a spectral measure. The fundamental result of multiplicity theory is essentially that by a suitable isomorphism (unitary transformation between Hilbert spaces) this spectral measure is equivalent to a suitable canonical spectral measure. A critical and classical step in going from the spectral measure $E(M)$ associated with a normal operator A to an ordinary measure μ is the following: If x is a vector in the underlying Hilbert space $\mathfrak{H}$, then $\mu(M) = (E(M)x, x)$ is a finite measure. Here M is an arbitrary Borel set. Furthermore, if we write $U\chi_M = E(M)x$, then U may be extended to an isomorphism from the space $L_2(\mu)$ onto the closed linear manifold of $\mathfrak{H}$ generated by all elements of the form $E(M)x$. For this isomorphism U, one has $U^{-1}E(M)Uf = \chi_M \cdot f$. This fact is the building block for multiplicity theory.

The author's treatment of multiplicity theory seems to be quite satisfactory. Although the material is complicated, it is subdivided into pleasant compact packages which the reader absorbs as he proceeds on his journey. The author meanwhile is a veritable ringmaster who marshals his troupe of techniques, cracking the whip over each aspect of the theory in turn, thus keeping it at the proper pitch for its part. Several well written sections give the beginner a much desired heuristic approach to the situation.

The book closes with an informal discussion of source material and references and a bibliography containing 52 entries. Approximately one-third of these are vital to the present undertaking, the others being either marginal in value or representing a caprice of the author. No errors major or minor were detected, a fact which is only one indication of the very careful way in which the booklet was prepared. Most pages exhibit a zest for play as well as work which is refreshing. Indeed, at times one may have a vague apprehension that the author is preparing a prank or baiting a trap; however it seldom turns out to be more than a friendly tweak given with a wink. Such an intimate style, in the present desert of works written with an unexceptionable scientific detachment, is warmly welcome. It is certainly a facet to the general success enjoyed by Halmos' previous books.

E. R. Lorch

A theory of formal deducibility. By H. B. Curry. (Notre Dame Mathematical Lectures, no. 6.) University of Notre Dame, 1950. 9+126 pp.

The monograph contains a detailed account of the predicate calculus as presented by Gentzen (Math. Zeit. vol. 39 (1934) pp. 176–210, 405–431) in a sequence calculus in which the rules of inference follow in a natural way from the intended meanings of the logical connectives. However, Curry's treatment differs in several major respects from earlier ones. The predicate calculus is approached as an episystem over a basic formal system of elementary propositions. Various portions of the classical and intuitionistic systems are studied separately. There is a discussion of alternative concepts of negation. And a final chapter suggests a new approach to modal logic.

A formal system is specified by a primitive frame which defines inductively terms, elementary propositions, and theorems. (The author develops here notions presented in a paper in Bull. Amer. Math. Soc. vol. 47 (1941) pp. 221–241.) In Curry's usage, the Hilbertian formal systems have as elementary propositions, propositions of the form "A is a provable formula," the formulas being terms in Curry's terminology. In studying a formal system it is customary and convenient not to limit attention only to elementary propositions, but to consider in addition compound propositions such as "Not for all formulas A, is A provable." These compound propositions are formed from the elementary ones by use of the logical connectives. Curry speaks of this broader system as an episystem over a formal system. (An episystem is not to be confused with a metasystem over a calculus.) The episystem may itself be formalized of course. Curry considers the predicate calculus as a formal system over an unspecified elementary formal system. In giving a detailed description of the notion of formal system, the author introduces a tentative theory of grammatics, relating formal systems to the actual language of communication in which they are imbedded, in the present case mathematical English.

In studying the predicate calculus, first the systems for the finite positive connectives ("and," "or," and "implies") are discussed. Systems corresponding to both Gentzen's L- and N-calculi are presented as well as axiom systems for the propositional algebras of these segments of the classical and intuitionistic systems. Gentzen's Hauptsatz receives a careful treatment, which differs in some details from the original one.

The chapter on quantification involves extensive detail designed to meet the well known difficulties in formulating this portion of logic. A notation is introduced to specify, for each deduction, the range of variables for which it is valid. It is shown that any theorem can be proved using only those free variables which appear in it.

A further chapter discusses three basic negation concepts, invalidity, refutability, and absurdity. A proposition A is said to be invalid in a system S just in case a proof for it does not exist in S. Such a negation has the disadvantage that, unlike the other logical connectives, it is not extensible, since A may be valid in some extension of S. A proposition is said to be refutable in S in case it implies R, where R is a proposition of a class defined to be directly refutable. A formula is said to be absurd in case it implies every formula of S. Curry considers four formal systems. The minimal, the intuitionistic, a system LD, and the classical one. Negation for all of the systems is of the refutability type. That for the intuitionistic one is particularized to absurdity. For the system LD the concept is not of the absurdity type, but it does satisfy a principle of excluded middle. Classical negation embodies all these features. This list of negation concepts is, of course, not exhaustive. Negation of yet another sort is described in the reviewer's note (Journal of Symbolic Logic vol. 14 (1949) pp. 16–26; Fitch has also used this type of negation in a system described in the same volume, pp. 209–218).

In a final chapter devoted to modal logic, Curry suggests that the proposition $\nmid A$ (A is necessary), where A is a proposition of a formal system S, be interpreted to mean that A is derivable by the rules of some specified subsystem $S\nmid$ of the system S. Rules for the operation $\nmid$ consistent with the interpretation are presented. The adjunction of these rules to the classical calculus results in a system equivalent to Lewis' system S4. A related notion for possibility is also suggested. Consider a family of systems; A is said to be possible in a system S_1 of the family in case it is provable in some stronger system of the family.

In making available in monograph form the important ideas of Gentzen, a definite need in the literature is filled. The work throughout gives meticulous attention to precise formulations and to detail in proof. It is almost entirely self-contained, and in spite of the great detail of the treatment should be of interest to the general mathematical public as well as to the specialist in foundations.

D. NELSON

Lezioni di analisi. By F. Severi and G. Scorza Dragoni. Bologna, Cesare Zuffi, 1951. Vol. 3, 6+255 pp. 2700 lire.

Volume 1 was written by the senior author alone and published by Zanichelli in 1933, it was reviewed by T. H. Hildebrandt in vol. 41, January 1935, of this Bulletin. A second edition appeared in 1938 and a third edition is in course of publication. Part 1 of volume 2 with G. Scorza Dragoni as coauthor appeared in 1943, but does not seem to have been reviewed in the Bulletin.

The present volume 3 contains a discussion of ordinary differential equations (existence and uniqueness theorem for single equations and for systems and problems in the large, mainly boundary value problems) followed by a chapter on trigonometric series and a brief discussion of the differential geometry of surfaces.

The main text gives a clear, fluent and rigorous account of the basic facts based on the Riemann integral. In each chapter, except the first, this exposition is supplemented by a section in smaller print labelled Complements and Exercises. Of exercises in our sense there are comparatively few, but the complements serve to open vistas to a multitude of more advanced questions. There is a wealth of historical material here and in the main text, over two hundred authors are quoted with dates though normally without textual references. The complement to the second chapter is particularly rich; it occupies a fourth of the book and the subject matter is kaleidoscopic. There is internal unity, however, and the brief sketches of the many topics are skilfully done. This section contains some interesting material on analytic functions of several variables and various extensions of Cauchy's theorem and Cauchy's integral to such functions. The Lebesgue integral is used in the complements. The discussion in the large in Chapter 3 takes the fixed point theorems of Brouwer and

Birkhoff-Kellogg as its point of departure. Orthogonal series, mean convergence, Vitali's closure theorem, and Hilbert spaces are discussed briefly in the complement to Chapter 4. The last chapter ends with a two page historical survey of differential geometry in higher dimensions and projective differential geometry.

There are undoubtedly readers who will claim that the authors would have given more, had they given less. To me the point of view is refreshing and the multitude of facts make the book a treasure house. The cover and the press work do credit to printer and publisher alike.

EINAR HILLE

Proceedings of a Second Symposium on Large-Scale Digital Calculating Machinery. (Annals of the Computation Laboratory of Harvard University, Vol. 26.) Cambridge, Harvard University Press, 1951. 38+393 pp. $8.00.

The first of the Harvard Symposia on this topic was held early in 1947, at the dedication of the Mark II Calculator. The Second Symposium was held at the dedication of the Mark III Calculator, at present in operation at the Navy Proving Ground in Dahlgren. Sessions were devoted to engineering developments (which will not be covered in this review), to numerical methods and computational problems in various sciences. There is, in this volume, a reasonably comprehensive survey of the field, both in the United States and in Europe, as it was in 1949.

The most significant mathematical contribution in the present volume is due to C. Lanczos, who, in his picturesque way, presents a method of minimized iterations for the solution of characteristic value problems. W. E. Milne examines various finite difference approximations to the two-dimensional Laplacian operator.

The problem of semi-automatic instruction is discussed by H. D. Huskey and a beginning of a unified theory of computing machines is presented by G. W. Patterson.

Most interesting are the papers describing some of the problems which await solution. Those who are interested in handling differential equations will find enough problems in the papers by H. Feshbach on nuclear physics and by R. D. O'Neal, E. T. Welmers, and H. W. Emmons on aeronautics and aerodynamics. On the other hand, the papers by F. Mosteller, W. W. Leontief, L. R. Tucker, and H. Chernoff on various topics in the economic and social sciences provide ample material for those interested in the manipulation of matrices.

It is interesting to compare the changes in the consumer demand which have taken place in the time between the two Symposia. As an instance, Leontief considered in 1947 a static model of inter-industry relationships, in 1949 he proposed a dynamic model (and, at the present time, plans are being made for actual computation on such a model): progress is at the rate of 1 dimension in two years!

Readers of these two Symposia volumes will look forward to more. They will hope that in succeeding volumes there will be more contributions to modern numerical analysis and to the problems of organization of a high speed digital computing center, and accounts rendered of major computational problems, such as the paper on a problem in physical chemistry in the present volume by H. A. Scheraga, J. T. Edsall, J. Orten Gadd, Jr., a problem which occupied two weeks computing time on the Mark I Calculator.

JOHN TODD

BRIEF MENTION

Proceedings of the International Congress of Mathematicians, Cambridge, Massachusetts, U.S.A., August 30–September 6, 1950. Providence, American Mathematical Society, 1952. Vol. 1, 8+769 pp. Vol. 2, 2+461 pp. $15.00

These volumes contain the list of officers and members, the report of the Secretary, and the text of the addresses and communications to the Congress; the proceedings of the Conference on Algebraic Tendencies in Analysis, however, are scheduled to appear separately as a volume in the series of Mathematical Surveys.

Fourier Series. By G. H. Hardy and W. W. Rogosinski. Cambridge University Press, 1950. 12+100 pp. 10s 6d.

Except for the correction of a few minor mistakes, this is the same as the first edition of 1944, reviewed in Bull. Amer. Math. Soc. vol. 51 (1945) pp. 212–214.

NOTES

The Board of Trustees met on April 25, 1952 and voted to continue, for the three year period beginning July 1, 1952, the $5.00 page rate now used in the computation of institutional dues. The Board also voted to increase the rate for the publication charge applicable to nonmember institutions from $5.00 to $10.00 per page. The cost of publishing contributed papers has advanced very rapidly. The $10.00 rate just adopted does not fully cover the assembly cost and is less than one-half the full cost of publication. These rates were reconsidered at this time in accordance with a recommendation of the Committee to Study Problems of Publication which was approved by the Council and the Board of Trustees in December, 1951. An announcement concerning the charge for the publication of contributed papers credited to nonmember institutions was made on page 115 of the March issue of this Bulletin. This note revises the page rate stated in the announcement.

The United States Naval Postgraduate School, which was founded in 1909 at Annapolis, Maryland, and until 1947 was a department of the United States Naval Academy, has been moved to Monterey, California.

Associate Professor A. L. Blakers of Lehigh University has been awarded The Alfred Noble Robinson Award for outstanding performance in the service of the University and unusual promise of professional achievement.

Professor I. W. Burr of Purdue University received the Brumbaugh award of the American Society for Quality Control, May, 1951.

Professor Richard Courant of New York University has been elected a foreign member of the Accademia Nazionale dei Lincei.

Professor Einar Hille of Yale University has been elected to the Royal Swedish Academy of Sciences.

Assistant Professor F. E. Hohn of the University of Illinois has received a Ford Foundation grant for the advancement of education during 1951–52.

Mr. M. A. Hyman of the Naval Ordnance Laboratory is a Fulbright Scholar for the year 1951–52 and is at the Technische Hogeschool, Delft, Netherlands.

Professor Nathan Jacobson of Yale University has a Fulbright Award and Guggenheim Fellowship and is studying in Paris for the academic year 1951–52.

Mr. P. G. Kirmser of the Institute of Technology, University of

Minnesota, has a Fulbright Fellowship in Mathematics and is studying at the University of Amsterdam for the academic year 1951–52.

Dr. W. G. Lister of Yale University has been awarded a Postdoctoral Atomic Energy Commission Fellowship.

Dr. W. R. Orton of the University of Illinois has a Fulbright fellowship and is studying in Paris.

Professor George Pólya of Stanford University has been elected an honorary member of the Board of the Mathematical Society of France.

Assistant Professor I. E. Segal of the University of Chicago has a Guggenheim Fellowship and is studying at the Institute for Advanced Study for the academic year 1951–52.

Professor M. H. Stone of the University of Chicago, Visiting Lecturer of the American Mathematical Society 1951–52, has been elected a member of the Academia Brasileira de Ciencias.

Professor Oscar Zariski of Harvard University has been elected a member of the American Philosophical Society. He has also been elected a member of the Istituto Lombardo delle Scienze.

Professor E. F. Beckenbach of the University of California, Los Angeles, is on leave of absence and has been appointed a member of the Institute for Advanced Study for the year 1951–52.

Dr. Stefan Bergman of Harvard University has been appointed to a professorship at Stanford University.

Dr. Archie Blake of the Mechanical Research Corporation has accepted a position as head of the International Business Machines Computing Section of the Cornell Aeronautical Laboratory, Buffalo, New York.

Professor Richard Brauer of the University of Michigan has been appointed to a professorship at Harvard University.

Mr. H. H. Brown of the Armour Research Foundation, Illinois Institute of Technology, has accepted a position as senior research engineer with the Franklin Institute Laboratories for Research and Development, Philadelphia, Pennsylvania.

Assistant Professor R. E. Carr of Michigan State College has accepted a position as research engineer with North American Aviation, Inc., Downey, California.

Dr. P. F. Conrad of the University of Illinois has been appointed to an assistant professorship at Newcomb College, Tulane University of Louisiana.

Dr. G. F. Cramer of the Defense Department has accepted a position as engineer with R. C. A. Victor.

Assistant Professor S. P. Diliberto of the University of California, Berkeley, is on leave of absence and has been appointed to a visiting assistant professorship at Princeton University.

Professor F. C. Gentry of Arizona State College is on leave of absence and is teaching at the University of New Mexico for the academic year 1951–52.

Dr. T. R. Greene of Carnegie Institute of Technology has accepted a position as engineer with The General Electric Company, Schenectady, New York.

Mr. S. G. Kneale of Harvard University has accepted a position as mathematician with the Philco Corporation, Philadelphia, Pennsylvania.

Dr. Waclaw Kozakiewicz of the University of Montreal has been appointed to an associate professorship at McGill University.

Professor E. P. Lane of the University of Chicago has retired.

Dr. R. A. Leibler of the Navy Department has accepted a position as mathematician with the Sandia Corporation, Albuquerque, New Mexico.

Dr. B. A. Lengyel of the Office of Naval Research has accepted a position as consultant with the Howard Hughes Aircraft Company, Culver City, California.

Mr. O. T. McMillan of Michigan State College has accepted a position as research associate at the Willow Run Research Center, University of Michigan, Ypsilanti, Michigan.

Mr. H. G. Mazurkiewicz of the Willow Run Research Center, University of Michigan, has accepted a position as engineer with the Glenn L. Martin Company, Baltimore, Maryland.

Mr. R. E. Montgomery has accepted a position as a mathematical research analyst and staff member with the Sandia Corporation, Albuquerque, New Mexico.

Dr. J. D. Newburgh of the Institute for Advanced Study has been appointed to an assistant professorship at Tulane University of Louisiana.

Mr. Adolf Nussbaum of Columbia University has accepted a position as staff member with the Electronic Computer Project, Institute for Advanced Study.

Dr. I. S. Reed of Northrup Aircraft, Hawthorne, California, and the University of Southern California has accepted a position on the D. I. C. Staff with the Massachusetts Institute of Technology.

Professor C. N. Reynolds of West Virginia University has retired with the title emeritus.

Professor G. E. F. Sherwood of the University of California, Los Angeles, has retired with the title emeritus.

Associate Professor J. L. Vanderslice of the University of Maryland is on leave of absence and has accepted a position as mathematician with the Johns Hopkins Applied Physics Laboratory, Silver Spring, Maryland.

Mr. Thirukkannapuram Vijayaraghavan of the Ramanujan Institute of Mathematics has been appointed a visiting lecturer at the University of Chicago.

Mr. R. F. Wheeling of Brown University has accepted a position as technologist with the Socony-Vacuum Oil Company, Paulsboro, New Jersey.

Mr. Herman Zabronsky of the University of Pennsylvania has accepted a position as associate mathematician with Union Carbide, Oak Ridge, Tennessee.

The following promotions are announced:

B. H. Arnold, Oregon State College, to an associate professorship.

Clifford Bell, University of California, Los Angeles, to a professorship.

W. M. Boothby, Northwestern University, to an assistant professorship.

B. W. Brewer, Oregon State College, to an associate professorship.

Hermann Chernoff, University of Illinois, to an associate professorship.

L. A. Colquitt, Texas Christian University, to an associate professorship.

F. D. Faulkner, United States Naval Postgraduate School, to an associate professorship.

J. B. Giever, Boston University, to an assistant professorship.

Michael Golomb, Purdue University, to a professorship.

F. C. Leone, Case Institute of Technology, to an assistant professorship.

Jakob Levitzki, Hebrew University, to a professorship.

Harry Levy, University of Illinois, to a professorship.

A. B. Mewborn, United States Naval Postgraduate School, to a professorship.

K. S. Miller, New York University, to an assistant professorship.

Sam Perlis, Purdue University, to an associate professorship.

R. M. Redheffer, University of California, Los Angeles, to an assistant professorship.

F. Virginia Rohde, University of Florida, to an assistant professorship.

J. P. Scholz, Lebanon Valley College, to an associate professorship.

H. E. Vaughan, University of Illinois, to an associate professorship.

The following appointments to instructorships are announced: Boston University: Dr. F. E. Browder; University of Detroit: Mr. B. V. Ritchie; Drexel Institute of Technology: Mr. J. S. Biggerstaff; Illinois Institute of Technology: Mr. Albert Soglin; University of Maryland: Mr. G. L. Spencer; University of Rochester: Mr. R. W. MacDowell; Northwestern University: Mr. W. E. Jenner, Mr. T. C. Littlejohn; Syracuse University: Dr. Jerome Blackman; Wayne University: Mr. C. H. Kraft, Mr. C. K. Tsao.

Mr. E. M. DiDomenica of Drexel Institute of Technology died on October 5, 1951 at the age of forty-five years.

Assistant Professor F. G. Graff of Oberlin College died on February 22, 1952 at the age of forty years.

Mr. E. B. Fackler, Vice President and Actuary of The Western and Southern Life Insurance Company, died on January 8, 1952, at the age of seventy-two years.

Professor Emeritus W. B. Fite of Columbia University died on March 1, 1952 at the age of eighty-two years. He had been a member of the Society for thirty-seven years.

Professor V. S. Lawrence, Jr., of the Virginia Polytechnic Institute died on February 20, 1952 at the age of fifty-six years. He had been a member of the Society for twenty-one years.

Professor Emeritus N. J. Lennes of Montana State University died on November 21, 1951 at the age of seventy-seven years. He had been a member of the Society for thirty-eight years.

Dr. P. F. Neményi, of the Naval Research Laboratory, died on March 1, 1952 at the age of fifty-six years.

The following two hundred and seventeen doctorates, with mathematics, mathematical physics, or statistics as a major subject, were conferred during 1951 in universities in the United States and Canada. The university, month in which degree was conferred, minor subjects, (other than mathematics), and title of the dissertation are given in each case if available.

C. M. Ablow, Brown, June, *Perturbations of free surface flows.*

O. R. Ainsworth, California, Berkeley, September, *Theory of waves from a point source in one of two semi-infinite contiguous elastic media.*

M. I. Aissen, Stanford, September, *On the estimation and computation of torsional rigidity.*

N. C. Ankeny, Princeton, January, *Consequences of the extended Riemann hypothesis.*

Sholom Arzt, New York, May, *On a mean value theorem for certain divisor functions taken over exponential sequences.*

D. G. Austin, Northwestern, June, *Some Lipschitzian approximations to differentiable and approximately differentiable functions.*

W. G. Bade, California, Los Angeles, June, *An operational calculus for operators with spectrum confined to a strip.*

J. H. Barrett, Texas, June, *Ordinary differential equations of non-integral order.*

R. G. Bartle, Chicago, August, *Singular points of functional equations.*

M. A. A. Bassam, Texas, June, *Holmgren-Riesz transform.*

R. H. Battin, Massachusetts Institute of Technology, June, minor in mechanical engineering, *Investigation of oscillation in flows over curved surfaces.*

R. E. Bechhofer, Columbia, June, minor in statistical quality control, *The effect of preliminary tests of significance on the size and power of certain tests of univariate linear hypotheses.*

F. A. Beeler, Michigan, February, *A study of the variation in the moments of a frequency distribution due to grouping.*

L. D. Berkovitz, Chicago, August, I. *Circular-summations and localizations of double trigonometric series.* II. *On double trigonometric integrals.* III. *On double Sturm-Liouville expansions.*

S. D. Bernardi, New York, May, *On the coefficients of schlicht functions.*

D. W. Blackett, Princeton, October, *Simple and semisimple near rings.*

Jerome Blackman, Cornell, June, minor in physics, *Studies in the heat equation.*

A. A. Blank, New York, January, *The diffraction and reflection of pulses by wedges and corners.*

G. M. Bloom, Northwestern, June, *On the total variation of solutions of the bounded variation moment problem.*

Jean M. Boyer, Maryland, August, *An arc problem.*

Fred Brafman, Michigan, February, *Generating functions of Jacobi and related polynomials.*

Paul Brock, New York, May, *The nature of solutions of a Raleigh type forced vibration equation with a large coefficient of damping.*

R. L. Broussard, Louisiana State, August, *A necessary and sufficient condition that a set be homeomorphic to a plane region bounded by a finite number of nonintersecting circles.*

R. H. Brown, Columbia, June, *The solution of a certain difference equation with applications to probability.*

C. E. Burgess, Texas, June, *Concerning continua and their complementary domains in the plane.*

L. P. Burton, North Carolina, June, *Minimax solutions of ordinary differential systems.*

G. H. Butcher, Pennsylvania, February, *An extension of the sum theorem of dimension theory.*

P. L. Butzer, Toronto, October, minor in quantum mechanics, *On Bernstein polynomials.*

Eugenio Calabi, Princeton, October, *Isometric complex analytic embedding of Kähler manifolds.*

J. W. Carr, III, Massachusetts Institute of Technology, September, minor in physics, *Direct method of solution of the Schrödinger eigenvalue problem by a high-speed digital computing machine.*

H. H. Chang, Syracuse, January, *Approximately analytic functions of bounded type and boundary behavior of solutions of elliptic partial differential equations.*

J. O. Chellevold, Lehigh, June, *Singular quadratic functions for n dependent variables.*

L. P. H. Cheo, Oregon, June, *The density of the sum of sets of Gaussian integers.*

G. Y. Cherlin, Rutgers, June, minor in logic, *Bounds for the coefficients of some classes of odd multivalent functions.*

Elsie T. Church, Kentucky, June, *Inversion with respect to a cubic of the syzygetic pencil.*

A. B. Clarke, Brown, June, *On the representation of cardinal algebras by directed sums.*

W. J. Conner, Tulane, June, *On locally closed sets.*

W. S. Connor, North Carolina, August, *The structure of balanced incomplete block designs and the impossibility of certain unsymmetrical cases.*

P. F. Conrad, Illinois, June, minor in philosophy, *Imbedding theorems for Abelian groups with valuations.*

S. D. Conte, Michigan, February, *Thin plate problem involving bipolar coordinates.*

J. M. Cook, Chicago, August, *The mathematics of second quantization.*

Philip Cooperman, New York, January, *An extension of the method of Trefftz for finding local bounds on the solutions and their derivations of boundary value problems.*

C. W. Curtis, Yale, June, *Additive ideal theory in general rings.*

M. L. Curtis, Michigan, February, *Deformation-free continua in Euclidean n-space.*

Elizabeth H. Cuthill, Minnesota, December, minor in physics, *Integrals on spaces of functions which are real and continuous on finite and infinite intervals.*

E. A. Davis, California, Berkeley, June, *A dynamic economic theory.*

R. B. Davis, Massachusetts Institute of Technology, June, minor in physics, *Boundary value problem existence theorems for third order semi-elliptic partial differential equations.*

R. J. Diamond, California Institute of Technology, June, minor in physics, *A type of pseudo-norm.*

V. E. Dietrich, Purdue, August, minor in physics, *Some classifications of projective planes by incidence matrices.*

Flora Dinkines, Chicago, August, *Semi-automorphisms of groups.*

W. F. Donoghue, Jr., Wisconsin, August, minor in physics, *The bounded closure of locally convex spaces.*

Leila A. Dragonette, Pennsylvania, June, *Asymptotic formulae for the mock theta series of Ramanujan.*

Milton Drandell, California, Los Angeles, June, *Generalized convex sets in the plane.*

G. F. D. Duff, Princeton, April, *Limit-cycles and rotated vector fields.*

D. G. Duncan, Michigan, February, *Some results in Littlewood's algebra of S-functions.*

B. J. Eisenstadt, Michigan, June, *The space of inessential continuous functions into the circle.*

R. L. Ely, Carnegie Institute of Technology, June, *Forced vibrations of continuous beams under pulsating, moving loads.*

J. L. Ericksen, Indiana, September, minor in physics, *Some geometrical problems connected with ideal gas flows.*

G. W. Evans, New York, May, *The motion of the interface in heat conduction problems involving recrystallization.*

R. L. Evans, Minnesota, June, minor in physical chemistry, *Asymptotic solutions in the neighborhood of a turning point for linear ordinary differential equations containing a parameter.*

O. B. Faircloth, Texas, June, *On the number of solutions of some general types of equations in a finite field.*

A. L. Fass, Columbia, June, *Homology and cohomology of algebras.*

J. M. G. Fell, California, Berkeley, January, *On L-spaces.*

J. V. Finch, Chicago, June, *On Banach spaces with inner products.*

Robert Finn, Syracuse, June, *On some properties of the solutions of a class of non-linear partial differential equations.*

W. H. Fleming, Wisconsin, August, *Boundary and related notions for generalized parametric surfaces.*

A. E. Foster, Kentucky, August, *A generalized harmonic conjugate for commutative algebras.*

L. K. Frazer, Indiana, June, minor in physics, *One-parameter families of linear line complexes.*

M. S. Friberg, Minnesota, June, minor in physics, *A method for the effective determination of conformal maps.*

D. R. Fulkerson, Wisconsin, January, *Quasi-Hermite forms of row-finite matrices.*

J. W. Gaddum, Missouri, August, *Metric methods in integral and differential geometry.*

M. P. Gaffney, Jr., Chicago, June, *The harmonic operator for exterior differential forms.*

Dieter Gaier, Rochester, November, minor in physics, *On some Tauberian theorems.*

Leonard Geller, Massachusetts Institute of Technology, June, minor in English and history, *Absolute convergence of Lebesgue-Stieltjes trigonometric integrals.*

Harvey Gellman, Toronto, October, minor in nuclear physics, *Computation of the magnetic field produced by a moving liquid.*

Murray Gerstenhaber, Chicago, June, *Rings of derivations.*

Sister M. Seraphim Gibbons, Minnesota, June, minor in physics, *A solution by iteration of the integral equations connected with the Dirichlet and Neumann problems in three dimensions.*

W. M. Gilbert, Princeton, June, *Completely monotonic functions of several variables.*

I. L. Glicksberg, California, Los Angeles, June, *Cesaro summation in harmonic analysis on groups.*

S. I. Goldberg, Toronto, October, minor in quantum mechanics, *Cohomology Lie algebras and Lie groups.*

L. A. Goodman, Princeton, October, *The estimation of population size using sequential sampling tagging methods.*

C. H. Gordon, Massachusetts Institute of Technology, September, minor in physics, *Tunnell wall effects for non-uniform two dimensional motion of thin airfoils in incompressible flow.*

R. P. Gosselin, Chicago, August, *The theory of localization for double trigonometric series.*

T. R. Greene, Carnegie Institute of Technology, June, *Water waves in channels of infinite depth.*

E. V. Greer, Oklahoma, June, minor in physics, *A theorem on planar transformations.*

W. T. Guy, California Institute of Technology, June, minor in physics, *Contributions to tensor analysis.*

Violet B. Haas, Massachusetts Institute of Technology, June, minor in physics, *Singular perturbations of an ordinary differential equation.*

Melvin Hausner, Princeton, June, *Dirichlet's principle and generalized boundary values.*

Nickolas Heerema, Tennessee, August, *An algebra determined by a binary cubic form.*

L. J. Heider, Michigan, June, *Characterizations of continuous function Banach spaces.*

Henry Helson, Harvard, March, *Fourier transforms and spectral synthesis on locally compact Abelian groups.*

Melvin Henriksen, Wisconsin, August, *On the ideal structure of the ring of entire functions and other function rings.*

R. T. Herbst, Duke, June, minor in physics, *Reduction of passive orthonormic differential systems to passive systems of the first order.*

B. E. Howard, Illinois, June, minor in electrical engineering, *Hydrodynamic properties of an electron gas.*

Burrowes Hunt, Colorado, August, minors in physics and philosophy, *Parameters determining the numbers of solutions of certain congruences or equations in a finite field.*

E. R. Immel, California, Los Angeles, June, *Problems of estimation and of hypothesis testing connected with birth-and-death Markov processes.*

B. M. Ingersoll, Syracuse, June, *On a problem of Cauchy type for linear hyperbolic equations.*

J. S. Isenberg, Brown, June, I. *On some problems in three-dimensional boundary layer flow.* II. *Supersonic flow past a solid body—two-dimensional case.*

H. W. Ito, Minnesota, August, minor in mechanics, *On conjugate functions.*

W. W. Jacobs, George Washington, May, *Random matrices.*

G. B. Kallianpur, North Carolina, August, *Some topics in the theory of stochastic processes.*

E. L. Kaplan, Princeton, January, *Infinite permutations of stationary random sequences.*

Josef Kates, Toronto, October, minor in quantum electrodynamics, *Space change effects in cathode ray storage tubes.*

Stanley Katz, New York, February, *On the representation of power free integrals by a system of polynomials.*

J. W. Keesee, Tulane, June, *Cohomology groups of a collection.*

E. P. King, Carnegie Institute of Technology, June, *The operating characteristic of the control chart for sample means when process standards are unspecified.*

J. R. Kinney, Illinois, June, minor in physics, *Continuity properties of sample functions of Markov processes.*

George Klein, Chicago, August, *On the approximation of functions by polynomials.*

Erwin Kleinfeld, Wisconsin, June, *Alternative rings.*

J. R. Knudsen, New York, May, *A study of effects of viscosity and heat conductivity on the transmission of sound waves in a compressible fluid.*

L. D. Kovach, Purdue, August, *On elementary nilpotent algebras.*

K. H. Kramer, Carnegie Institute of Technology, June, *The distribution of range in compositions of normal universes.*

S. A. Kushneriuk, Toronto, June, minor in quantum mechanics, *Central and tensor forces in the scattering of protons by protons.*

J. M. A. L'Abbé, Princeton, June, *Systems of transfinite types involving n-conversion.*

R. J. Lambert, Iowa State, July, minor in physics, *Extension of normal theory to general matrices.*

N. D. Lane, Toronto, October, minor in quantum mechanics, *A symmetric proof of the Riemann-Roch theorem and related properties.*

S. R. Lang, Princeton, June, *On quasi algebraic closure.*

G. E. Latta, California Institute of Technology, June, minor in aeronautics, *Singular perturbation problems.*

G. F. Leger, Illinois, June, minor in physics, *On cohomology theory for Lie algebras.*

Norman Levine, Ohio State, August, *Absolutely continuous product transformations in the plane.*

W. G. Lister, Yale, June, *A structure theory of Lie triple systems.*

A. J. Lohwater, Rochester, June, minor in physics, *The boundary values of a class of analytic functions.*

A. R. Lovaglia, California, Berkeley, September, *Locally uniformly convex Banach spaces.*

J. R. Lurye, New York, May, *Electromagnetic reflection and transmission matrices of a continuously stratified anisotropic medium by variational technique.*

John McCarthy, Princeton, June, *Projection operators and partial differential equations.*

J. S. MacNerney, Texas, June, *Continued fractions in which the elements are operators in a linear space.*

Paul Mandl, Toronto, October, minor in atomic and molecular spectra, *Transition through a weak shock front.*

W. H. Marlow, Iowa, February, *On limits of phalanxes.*

George Marsaglia, Ohio State, March, *Stochastic processes and classes of random variables.*

J. E. Maxfield, Oregon, June, *Normal numbers in k dimensions.*

Margaret W. Maxfield, Oregon, June, *Fermat's theorem for matrices over a modular ring.*

Paul Meier, Princeton, October, *Weighted means and lattice designs.*

D. M. Merriell, Chicago, June, *On almost alternative flexible algebras.*

E. A. Michael, Chicago, August, *Locally m-convex algebras.*

P. T. Mielke, Purdue, June, minor in engineering mechanics, *Improper integrals in abstract spaces.*

Joseph Milkman, New York, May, *Hermite polynomials, Hermite functionals and their integrals in real Hilbert space.*

C. E. Miller, Chicago, June, *The topology of rotation groups.*

W. G. Miller, Florida, August, minor in economics, *The mathematics of production and consumption in a static economy.*

B. E. Mitchell, Wisconsin, August, *Some properties of matrices under unitary transformation.*

Don Mittleman, Columbia, June, *The unions of trajectorial series of lineal elements generated by the plane motion of a rigid body.*

R. W. Moller, Catholic University, June, minors in physics and philosophy, *Symmetric functions of the numbers belonging to a divisor of $p-1$ where p is a prime.*

W. K. Moore, Kansas, June, minor in physics, *The characterisations of a class of transformations and of a class of differentiable functions.*

Cathleen S. Morawetz, New York, February, *Contracting spherical shocks treated by a perturbation method.*

Sister M. Jesse Morrison, Catholic University, June, minors in physics and education, *Magic rectangles modulo m.*

R. B. Murphy, Princeton, October, *On tests for outlying observations.*

G. G. Murray, Jr., Harvard, June, *Measures in Boolean algebras.*

William Nachbar, Brown, June, *Initial motion problems in plane strain of a plastic-rigid material.*

G. G. O'Brien, Maryland, June, *Theorems on stability and convergence in numerical solutions of partial differential equations.*

H. W. Oliver, Chicago, August, *Differential properties of real functions.*

Ingram Olkin, North Carolina, June, *On distribution problems in multivariate analysis.*

Barrett O'Neill, Massachusetts Institute of Technology, September, minor in English and history, *Some fixed point theorems.*

Daniel Orloff, Chicago, June, *Invariant measures.*

W. R. Orton, Jr., Illinois, June, *Representation of functions of a complex variable and related integral equations.*

D. B. Owen, University of Washington, March, *A two sample test procedure.*

W. D. Peeples, Jr., Georgia, June, *Elliptic curves and rational distance sets.*

M. P. Peisakoff, Princeton, October, *Transformation parameters.*

G. O. Peters, Georgia, August, *Bernoulli polynomials of the second kind of higher order. Euler polynomials of the second kind of first and higher orders. Boole polynomials of the first and second kinds of higher order. Bernoulli, Euler, and Boole series and functions of negative degree.*

G. M. Petersen, Toronto, June, minor in quantum mechanics, *Methods of summation of divergent series and means of Fourier constants.*

W. A. Pierce, Harvard, March, *Extensibility foundations of Euclidean, hyperbolic, and spherical geometry in the plane.*

F. A. E. Pirani, Carnegie Institute of Technology, June, *On the quantization of the gravitational field of general relativity.*

F. V. Pohle, New York, January, *The Lagrangian equations of hydrodynamics: Solutions which are analytic functions of the time.*

H. O. Pollak, Harvard, June, *Some estimates for the extremal distance.*

Ruth L. Potter, Washington University, May, *Oscillation and boundedness of solutions of a second-order linear differential equation.*

F. P. Pu, Syracuse, January, *Some inequalities in certain non-orientable Riemannian manifolds.*

Philip Rabinowitz, Pennsylvania, June, *Normal coverings and uniform spaces.*

K. G. Ramanathan, Princeton, June, *The theory of units of quadratic and Hermitian forms.*

R. F. Reeves, Iowa State, June, minor in physics, *Force fields in which centers of gravity can be defined.*

Daniel Resch, Syracuse, January, *Some Baecklund transformations of partial differential equations of second order.*

T. L. Reynolds, North Carolina, June, *On the impossibility of an odd perfect number not divisible by five with six different prime divisors.*

D. D. Rippe, Michigan, June, *Statistical rank and sampling variation of the results of factorization of covariance matrices.*

G. B. Robison, Cornell, September, minor in education, *Invariant integrals over a Banach space.*

R. L. Roderick, Brown, June, I. *On the measurement of ultrasonic attenuation in solids by the pulse technique and some results in steel.* II. *The radiation pattern from a rotationally symmetric stress source on a semi-infinite solid.*

I. H. Rose, Harvard, June, *On the cohomology theory of associative algebras.*

I. A. Rosenberg, Chicago, August, *Subrings of simple rings with minimal ideals.*

H. L. Royden, Jr., Harvard, June, *Harmonic functions on open Riemann surfaces.*

J. H. Sampson, Princeton, June, *The Dirichlet problem in the large.*

James Sanders, Syracuse, June, *Some classes of partial differential equations of the fourth order.*

W. K. Saunders, California, Berkeley, July, *Existence of the solution of the exterior problem of the electromagnetic field.*

J. W. Sawyer, Missouri, August, *A study of metric torsion.*

Albert Schild, Pennsylvania, February, *On a problem in conformal mapping of schlicht functions.*

L. L. Scott, Illinois, October, *Finite metabelian groups and planes of* Σ_{14}.

Raymond Sedney, Carnegie Institute of Technology, June, *On the hydrodynamical theory of lubrication.* I. *The Reynolds lubrication equation with smooth outflow.* II. *The finite length journal bearing with high eccentricity.*

J. B. Serrin, Jr., Indiana, June, minor in physics, *The existence and uniqueness of flows solving four free boundary problems.*

B. W. Shaffer, Brown, June, *Analysis of chip formation in the turning operation.*

J. A. Silva, Duke, June, minor in physics, *Representation of arithmetic functions in* $GF[p^n, x]$ *with values in an arbitrary field.*

Maurice Sion, California, Berkeley, June, *On the existence of functions having given partial derivatives on Whitney's curve.*

K. T. Smith, Wisconsin, August, *The L-B topology in locally convex spaces.*

L. J. Snell, Illinois, October, minor in physics, *Applications of martingale system theorems.*

Milton Sobel, Columbia, January, *An essentially complete class of decision functions for certain standard sequential problems.*

S. M. Spencer, Jr., Duke, June, minor in philosophy, *Transcendental numbers over certain function fields.*

J. P. Stanley, Toronto, June, *The theory of propagation of very long radio waves in the ionosphere.*

M. L. Stein, California, Los Angeles, January, *On methods for obtaining solutions of fixed end point problems in the calculus of variations.*

Fritz Steinhardt, Columbia, June, *On distance functions and on polar series of convex bodies.*

R. L. Sternberg, Northwestern, June, *Non-oscillation theorems for systems of differential equations.*

R. A. Struble, Notre Dame, June, *Metric and measure in locally compact groups.*

R. B. Talmadge, California Institute of Technology, June, minor in physics, *The representation of Baire functions.*

W. F. Taylor, California, Berkeley, January, *On tests of hypotheses and best asymptotically normal estimates related to certain biological tests.*

M. E. Terry, North Carolina, June, *Some rank order tests which are most powerful against specific parametric alternatives.*

J. E. Thompson, Minnesota, June, minor in mechanics, *Studies of the behavior of the mapping function on the boundary in conformal mapping.*

J. A. Tierney, Maryland, June, *An approximation to transonic flow of a polytropic gas.*

F. H. Tingey, University of Washington, August, minor in meteorology, *Extension of Kolmogoroff statistic to more than one dimension.*

S. H. Tsao, Michigan, February, *On groups of order* $g = p^2 g'$.

G. E. Uhrich, University of Washington, December, minor in physics, *On the solutions of systems of generalised hyperbolic differential equations.*

Bethume Vanderburg, Chicago, August, *Linear combinations of Hausdorff summability methods.*

D. E. Van Tijn, Indiana, June, minor in physics, *An interpretation for free variables.*

H. C. Volkin, Carnegie Institute of Technology, June, *Rotating and accelerated reference systems.*

D. H. Wagner, Brown, June, *On free products of groups.*

J. H. Wahab, North Carolina, June, *Some new cases of irreducibility for Legendre polynomials.*

A. M. Wedel, Iowa State, July, minor in physics, *Volterra transforms of some hypergeometric series.*

L. M. Weiner, Chicago, August, *Lie admissible algebras.*

Harold Weintraub, Harvard, June, *Borel monogenic functions.*

H. J. Weiss, Carnegie Institute of Technology, June, *Structural stability of thin-walled open sections.*

John Wermer, Harvard, June, *On the harmonic analysis of certain groups and semigroups of operators.*

R. A. Worsing, Iowa State, July, minor in physics, *Integral equation solutions of elastic plate problems.*

Arthur Wouk, Johns Hopkins, June, *Difference equations and J-matrices.*

J. B. Wright, Michigan, February, *Metaprojective geometry.*

F. H. Young, Oregon, June, *A matrix transformation of Fourier coefficients.*

The following four doctorates were conferred in 1950, but were not included in the list in the preceding volume of this BULLETIN (vol. 57, pp. 211–219).

Geraldine A. Coon, Rochester, November, minor in physics, *The double Laplace transform and its application to partial differential equations.*

L. E. Moses, Stanford, September, *An iterative construction of the optimum sequential decision procedure when the cost function is linear.*

C. J. Pipes, Oklahoma, August, minor in physics, *Properties of sets and functions relative to exceptional sets.*

W. A. Rutledge, Tennessee, August, *The Hurwitz integral quaternions as a principal ideal domain.*

NEW PUBLICATIONS

Artin, E. *Algebraic numbers and algebraic functions.* I. New York, Institute for Mathematics and Mechanics, New York University, 1951. 2+5+345+5 pp. $3.50.

Benac, T. J. See Waismann, F.

Bieberbach, L. *Einführung in die Funktionentheorie.* 2d ed. Bielefeld, Verlag für Wissenschaft und Fachbuch, 1952. 12.60 DM.

Bilimović, A. *Racionalna mehanika.* II. *Mehanika sistema.* Belgrade, Naučna Knjiga, 1951. 7+405 pp.

Burkill, J. C. *The Lebesgue integral.* (Cambridge Tracts in Mathematics and Mathematical Physics, no. 40.) Cambridge University Press, 1951. 8+87 pp. 12 s. 6 d.

Carnap, R. *The continuum of inductive methods.* University of Chicago Press, 1952. 5+92 pp. $3.50.

Carrier, G. F. See *Foundations, aerodynamics of high speed.*

Cartan, É. *Leçons sur la géométrie des espaces de Riemann.* 2d ed. Paris, Gauthier-Villars, 1951. 8+378 pp. 1800 fr.

Cebyšev, P. L. *Polnoe sobranie sočinenii P. L. Cebyševa.* Vol. 5. *Pročie sočineniya. Biografičeskie materialy.* Moscow, Izdatel'stvo Akademii Nauk SSSR, 1951. 474 pp., 1 plate.

Curry, H. B. *Leçons de logique algébrique.* Paris, Gauthier-Villars, 1952. 163 pp. 1600 fr.

Doig, P. *A concise history of astronomy.* New York, Philosophical Library, 1951. 11+320 pp. $4.75.

Dorfman, R. *Application of linear programming to the theory of the firm, including an analysis of monopolistic firms by non-linear programming.* Berkeley, University of California Press, 1951. 9+98 pp. $3.50.

Dörrie, H. *Einführung in die Funktionentheorie.* Munich, Oldenbourg, 1951. 559 pp. 48 DM.

Dörrie, H. *Unendliche Reihen.* Munich, Oldenbourg, 1951. 11+725 pp. 58 DM.

Einstein, A. See Lorentz, H. A.

Foundations, aerodynamics of high speed. With a bibliography compiled by G. F. Carrier. New York, Dover, 1951. 10+286 pp. $3.50.

Gillespie, R. P. *Partial differentiation.* New York, Interscience, 1951. 8+107 pp. $1.95.

Goodstein, R. L. *Constructive formalism. Essays on the foundations of mathematics.* Leicester, University College, 1951. 91 pp. 15 s.

Gumprecht, R. O., and Sliepcevich, C. M. *Tables of functions of first and second partial derivatives of Legendre polynomials.* Ann Arbor, Engineering Research Institute, University of Michigan, 1951. 12+310 pp. $3.50.

Gumprecht, R. O., and Sliepcevich, C. M. *Tables of light-scattering functions for spherical particles.* Ann Arbor, Engineering Research Institute, University of Michigan, 1951. 15+574 pp. $6.50.

Gumprecht, R. O., and Sliepcevich, C. M. *Tables of Riccati Bessel functions for large arguments and orders.* Ann Arbor, Engineering Research Institute, University of Michigan, 1951. 16+260 pp. $3.50.

Hodge, W. V. D., and Pedoe, D. *Methods of algebraic geometry.* Vol. 2. Cambridge University Press, 1951. 9+387 pp.

Jaeger, J. C. *An introduction to applied mathematics.* Oxford, Clarendon, 1951. 13+445 pp. $7.00.

Jeffery, G. B. See Lorentz, H. A.

Kempthorne, O. *The design and analysis of experiments*. New York, Wiley, 1952. 19+631 pp. $8.50.

Landau, L., and Lifšic, E. *Statističeskaya fizika* (*klassičeskaya i kvantovaya*). Moscow, Gosudarstvennoe Izdatel'stvo Tehniko-Teoreticeskoi Literatury, 1951. 479 pp. 14.75 rubles.

Lifšic, E. See Landau, L.

Lobačevskiĭ, N. I. *Polnoe sobranie sočineniĭ*. Vol. 3. *Sočineniya po geometriĭ*. Moscow, Gosudarstvennoe Izdatel'stvo Tehniko-Teoreticeskoi Literatury, 1951. 535 pp., 2 plates.

Lorentz, H. A., Einstein, A., Minkowski, H., and Weyl, H. *The principle of relativity*. A collection of original memoirs on the special and general theory of relativity. With notes by A. Sommerfeld. Trans. by W. Perrett and G. B. Jeffery. New York, Dover, n.d. 8+216 pp. $3.50.

Lösch, F., and Schoblik, F. *Die Fakultät* (*Gammafunktion*) *und verwandte Funktionen mit besonderer Berucksichtigung ihrer Anwendungen*. Leipzig, Teubner, 1951. 6+205 pp. $4.03.

Maxwell, J. C. *Matter and motion*. New York, Dover, n.d. 13+163 pp., 1 plate. $2.50.

Minkowski, H. See Lorentz, H. A.

Murnaghan, F. D. *Finite deformation of an elastic solid*. New York, Wiley, 1951. 5+140 pp. $4.00.

Nehari Z. *Conformal mapping*. New York, McGraw-Hill, 1952. 8+396 pp. $7.50.

Newman, M. H. A. *Elements of the topology of plane sets of points*. 2d ed. Cambridge University Press, 1951. 7+214 pp. $4.75.

Öner, Y. *Metafizik şuur ve matematik-metafizik* [*hakkinda*]. İstanbul, Ateş, 1951. 80 pp.

Pedoe, D. See Hodge, W. V. D.

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Proceedings of the International Congress of Mathematicians, Cambridge, Mass., 1950. Providence, American Mathematical Society, 1952. Vol. 1, 6+769 pp. Vol. 2, 4+461 pp. $15.00.

Quine, W. V. O. *Mathematical logic*. Rev. ed. Cambridge, Harvard University Press, 1951. 12+346 pp. $4.75.

Repertorio di matematiche. Ed. by M. Villa. Padova, Cedam, 1951. 18+731 pp. 5000 lire.

Rios, S. *Introduccion a los metodos estadisticos*. I. Madrid, Diez, 1951. 205 pp. 60 pesetas.

Robinson, A. *On the metamathematics of algebra*. Amsterdam, North-Holland, 1951. 9+195 pp. 18.00 florins.

von Sanden, R. *Praktische Mathematik*. Leipzig, Teubner, 1951. 120 pp.

Sauer, R. *Einführung in die theoretische Gasdynamik*. 2d ed. Berlin, Springer, 1951. 8+174 pp.

Schoblik, F. See Lösch, F.

Scorza Dragoni, G. See Severi, F.

Selections illustrating the history of Greek mathematics. With an English translation by I. Thomas. Vol. 1. *From Thales to Euclid*. Vol. 2, *From Aristarchus to Pappus*. Cambridge, Harvard University Press, 1951. Vol. 1, 16+505 pp. Vol. 2, 10+683 pp.

Severi, F., and Scorza Dragoni, G. *Lezioni di analisi*. Vol. 3. *Equazioni differenziali* *ordinarie e loro sistemi, problemi al contorno relativi, serie trigonometriche, applicazioni geometriche*. Bologna, Zuffi, 1951. 6+255 pp.

SLIEPCEVICH, C. M. See GUMPRECHT, R. O.

SOMMERFELD, A. See LORENTZ, H. A.

Tables of the error function and of its first twenty derivatives. By the Staff of the Computation Laboratory. Cambridge, Harvard University Press, 1952. 28+276 pp. $8.00.

TARSKI, A. *A decision method for elementary algebra and geometry*. 2d ed. Berkeley, University of California Press, 1951. 3+63 pp. $2.75.

The theory of electromagnetic waves. A symposium held under the auspices of the Washington Square College of Arts and Science and the Institute for Mathematics and Mechanics of New York University and the Geophysical Research Directorate of the Air Force Cambridge Research Laboratories, June 6–8, 1950. New York, Interscience, 1951. 8+393 pp. $6.50.

THOMAS, I. See *Selections illustrating the history of Greek mathematics*.

VILLA, M. See *Repertorio di matematiche*.

WAISMANN, F. *Introduction to mathematical thinking. The formation of concepts in modern mathematics*. Trans. by T. J. Benac. New York, Ungar, 1951. 11+260 pp. $4.50.

WEYL, H. See LORENTZ, H. A.

WHITTAKER, E. *A history of the theories of aether and electricity*. Vol. 1. *The classical theories*. New York, Philosophical Library, 1951. 14+434 pp. $12.00.

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This journal is the official organ of the Society. It reports official acts of the Society and the details of its meetings, with abstracts of all research papers presented. It contains some of the officially invited addresses presented before the Society. It contains also announcements and notes on matters of current interest, and reviews of advanced mathematical books.

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Invited addresses offered for publication should be sent to G. B. PRICE, 215 Strong Hall, University of Kansas, Lawrence, Kan.; other articles to W. T. MARTIN, Room 2-255, Massachusetts Institute of Technology, Cambridge 39, Mass.; book reviews to R. P. BOAS, JR., Lunt Hall, Northwestern University, Evanston, Ill.; and items for Notes to the BULLETIN OF THE AMERICAN MATHEMATICAL SOCIETY, 80 Waterman St., Providence 6, R.I. All other communications to the editors should be addressed to the Managing Editor, G. B. PRICE, 215 Strong Hall, University of Kansas, Lawrence, Kan.

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Transactions of the American Mathematical Society

This journal is devoted entirely to research in pure and applied mathematics, and includes in general longer papers than the PROCEEDINGS.

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Papers in algebra, number theory, logic and foundations should be sent to SAUNDERS MACLANE, University of Chicago, Chicago 37, Ill.; in probability, theory of measure, real variables to J. L. DOOB, Department of Mathematics, University of Illinois, Urbana, Ill.; other papers in analysis and applied mathematics to L. V. AHLFORS, Department of Mathematics, Harvard University, Cambridge 38, Mass.; in geometry and topology to G. T. WHYBURN, School of Mathematics, University of Virginia, Charlottesville, Va. All other communications to the editors should be addressed to the Managing Editor, G. T. WHYBURN, School of Mathematics, University of Virginia, Charlottesville, Va.

Memoirs of the American Mathematical Society

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Mathematical Reviews

This journal contains abstracts and reviews of the current mathematical literature of the world. It is sponsored by thirteen mathematical organizations, located both in the United States and abroad.

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THE APRIL MEETING IN NEW YORK

The four hundred seventy-ninth meeting of the American Mathematical Society was held at Columbia University, New York City, on Friday and Saturday, April 25–26, 1952. The meeting was attended by about 225 persons, including the following 207 members of the Society:

C. R. Adams, M. I. Aissen, A. A. Albert, A. C. Allen, Warren Ambrose, R. D. Anderson, R. L. Anderson, Joseph Andrushkiw, R. G. Archibald, Sholom Arzt, Joshua Barlaz, J. H. Barrett, R. G. Bartle, W. R. Baum, Enrique Bayo, R. J. Beeber, E. G. Begle, Stefan Bergman, S. D. Bernardi, Lipman Bers, D. W. Blackett, Jerome Blackman, A. L. Blakers, J. H. Blau, I. E. Block, H. W. Bode, Samuel Borofsky, E. H. Boyle, J. L. Bricker, H. W. Brinkmann, F. E. Browder, A. B. Brown, Bailey Brown, R. G. Brown, R. H. Brown, D. W. Bushaw, Jewell H. Bushey, Eugenio Calabi, P. W. Carruth, Y. W. Chen, F. E. Clark, E. A. Coddington, L. W. Cohen, R. M. Cohn, Philip Cooperman, T. F. Cope, Richard Courant, A. B. Cunningham, M. L. Curtis, R. B. Davis, C. R. DePrima, Avron Douglis, D. M. Dribin, Nelson Dunford, Samuel Ellenberg, C. C. Elgot, M. P. Epstein, M. E. Estill, R. M. Exner, A. L. Fass, J. M. Feld, William Feller, R. M. Foster, L. G. Fourès, J. B. Freier, Gerald Freilich, Bernard Friedman, Orrin Frink, I. S. Gál, A. S. Galbraith, David Gale, G. N. Garrison, B. H. Gere, Abolghassem Ghaffari, H. A. Giddings, B. P. Gill, Leonard Gillman, Samuel Goldberg, J. K. Goldhaber, J. W. Green, L. W. Green, Emil Grosswald, Felix Haas, Marshall Hall, Jr., Carl Hammer, Harish-Chandra, F. S. Hawthorne, E. V. Haynsworth, K. E. Hazard, C. M. Hebbert, G. A. Hedlund, M. H. Heins, Einar Hille, Banesh Hoffmann, L. A. Hostinsky, J. L. Howell, C. C. Hsiung, L. C. Hutchinson, Eugene Isaacson, W. S. Jardetzky, Fritz John, M. L. Juncosa, Shizuo Kakutani, Aidá Kallah, Hyman Kamel, J. F. Kiefer, H. S. Kieval, J. W. Kitchens, George Klein, E. R. Kolchin, B. O. Koopman, Jack Kotik, P. G. Kvick, R. H. Kyle, A. W. Landers, C. W. Langley, Solomon Leader, Joseph Lehner, Benjamin Lepson, M. E. Levenson, Howard Levi, D. C. Lewis, E. R. Lorch, Eugene Lukacs, N. H. McCoy, Brockway McMillan, L. A. MacColl, G. W. Mackey, H. M. MacNeille, Wilhelm Magnus, Irwin Mann, A. J. Maria, M. H. Maria, L. F. Markus, W. T. Martin, A. N. Milgram, K. S. Miller, Don Mittleman, E. E. Moise, E. F. Moore, Marston Morse, W. J. Nemerever, D. J. Newman, I. L. Novak, F. G. O'Brien, A. M. Ostrowski, J. C. Oxtoby, A. J. Penico, I. D. Peters, R. M. Peters, E. L. Post, Walter Prenowitz, W. W. Proctor, M. H. Protter, H. W. Raudenbush, G. E. Raynor, Helene Reschovsky, Moses Richardson, J. H. Roberts, M. S. Robertson, S. L. Robinson, David Rosen, Maxwell Rosenlicht, H. D. Ruderman, Walter Rudin, J. P. Russell, Jacob Samoloff, Arthur Sard, A. T. Schafer, R. D. Schafer, J. A. Schatz, Abraham Schwartz, I. E. Segal, D. B. Shaffer, Daniel Shanks, H. N. Shapiro, I. M. Sheffer, K. M. Siegel, Annette Sinclair, I. M. Singer, James Singer, J. J. Sopka, G. Y. Sosnow, S. K. B. Stein, C. F. Stephens, R. L. Sternberg, F. M. Stewart, Walter Strodt, R. L. Swain, R. L. Taylor, J. M. Thomas, M. L. Tomber, A. W. Tucker, Annita Tuller, A. H. Van Tuyl, B. W. Volkmann, H. V. Waldinger, G. C. Webber, J. V. Wehausen, H. F. Weinberger, Alexander Weinstein, David Wellinger, Franc Wertheimer, Albert Wilansky, Arthur Wouk, J. A. Zilber, Leo Zippin.

At a general session on Friday Professor Marston Morse of the Institute for Advanced Study delivered an invited address on *The* *generalized Fréchet variation and theorems of Riesz-Young-Hausdorff type*. Professor Nelson Dunford presided. An invited address on *Galois theory of differential fields* was delivered at a general session on Saturday by Professor E. R. Kolchin of Columbia University. Professor Orrin Frink presided.

The sessions for contributed papers on Friday were presided over by Professors Lipman Bers and Marshall Hall, Jr. Professor Einar Hille, D. C. Lewis, N. H. McCoy, and Arthur Sard presided over the sessions on Saturday.

The Trustees met at the Faculty Club at 6:00 P.M. on April 25.

The Council met at the Faculty Club at 8:00 P.M. on April 25.

The Secretary announced the election of the following one-hundred eleven persons to ordinary membership in the society:

Dr. Allan George Anderson, Instructor, Oberlin College;
Sister Madeleine Rose Ashton, President, College of the Holy Names, Oakland, California;
Mr. Stanley Edward Asplund, Teaching Fellow, Cornell University;
Miss Phyllis Hunt Belisle, Instructor, Bennett Junior College, Millbrook, New York;
Mr. Charles Bernard Bell, Jr., Research Laboratory Analyst, Douglas Aircraft Company, Inc., Santa Monica, California;
Mr. James Clyde Bradford, North Texas State College, Denton, Texas;
Mr. Wallace Franklin Branham, Mathematician, Long Range Proving Ground, Cocoa, Florida;
Mr. Frederick Arnold Brown, Chief, Computers Branch, T. T. D., Headquarters Air Force Missile Test Center, Melbourne, Florida;
Assistant Professor John Lawrence Brown, Jr., Pennsylvania State College;
Mr. Robert Eugene Bryan, Assistant in Instruction, Yale University;
Mr. Roger Thomas Businger, Chemical Engineer, Pittsburgh Plate Glass Company, Pittsburgh, Pennsylvania;
Mr. Paul Leo Butzer, Research Assistant, University of Toronto;
Mr. Jean-Francois Canu, Mathematician, U. S. Naval Proving Ground, Dahlgren, Virginia;
Mr. John W. Caulder, Assistant Director, Mech. Test Laboratory, Wright Field, Dayton, Ohio;
Mr. Wayne Hwa-wei Chen, Pre-doctoral Associate, University of Washington;
Mr. Heron Sherwood Collins, Teaching Fellow, Tulane University;
Mr. Virgil Lee Collins, State Teachers College, Troy, Alabama;
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Mr. Wesley Lathrop Nicholson, Research Assistant, University of Oregon;
Assistant Professor Francis George O'Brien, Fordham University;
Mr. Morris A. Oliver, Head of Mathematics Department, Bennington College, Bennington, Vermont;
Assistant Professor Ingram Olkin, Michigan State College;
Mr. Eric M. Olson, Columbia University;
Mr. Raymond Edward Ozimkowski, Instructor, Fordham University;
Mr. Flemming Per Pedersen, Lecturer, University of Southern California;
Mr. Tullio Joseph Pignani, Instructor, University of North Carolina;
Mr. Morris Plotkin, Electronic Scientist, Naval Air Development Center, Johnsville, Pennsylvania;
Mr. Valdemars Punga, Associate in Mathematics, George Washington University;
Mr. James Milton Reynolds, Chairman, Alabama State College;
Mr. Louis Robinson, Fellow, Syracuse University;
Mr. James Purcell Rodman, Instructor, Mount Union College, Alliance, Ohio;
Mr. Lawrence Rosenfeld, Mathematician, Cambridge Research Center, Cambridge, Massachusetts;
Mr. J. Paul Roth, Research Assistant, Engineering Research Institute, Mathematics Group, University of Michigan;
Mr. Paul Theodore Schaefer, Graduate Assistant, University of Rochester;
Associate Professor Dan R. Scholz, Southwest Louisiana Institute, Lafayette, Louisiana;
Mr. Jerome Saul-Shipman, Mathematician, Laboratory for Electronics, Inc., Boston, Massachusetts;
Mrs. Sedalia McAfee Sims, Instructor, Wiley College, Marshall, Texas;
Mr. Abe Sklar, 4834 North Troy Street, Chicago, Illinois;
Mr. Paul Slepian, Graduate Assistant, Brown University;
Mr. James Celestine Smith, Jr., Instructor, Loyola University of Los Angeles;
Mr. Judson Calkins Smith, Fellow, University of Washington;
Mr. Joseph M. Stein, Manager of Nuclear Engineering, Western Electric Corporation, Atomic Power Division, Homestead, Pennsylvania;
Mr. Warren Bernard Stenberg, Lecturer, University of California, Berkeley, California;
Miss Dorothy May Swan, Instructor, Monticello College, Godfrey, Illinois;
Rev. Robert John Teed, Instructor, Fournier Institute of Technology, Lemont, Illinois;

Miss Helen Jeane Terry, Instructor, University of Idaho;
Associate Professor Samuel Lothrop Thorndike, College of Emporia, Emporia, Kansas;
Mr. Norman Olcott Tiffany, Instructor, Moravian College for Women, Bethlehem, Pennsylvania;
Mr. Howard Gregory Tucker, University of California, Berkeley, California;
Dr. Dennis Peter Vythoulkas, Research Associate, University of Chicago;
Dr. Jack Warga, Mathematician, Reeves Instrument Corporation, New York, New York;
Mr. Paul Weiss, Mathematician, General Electric Company, Syracuse, New York;
Associate Professor Wayne Grant Wild, Buena Vista College, Storm Lake, Iowa;
Assistant Professor George Francis Will, Siena College, Londonville, New York;
Mr. Joseph Boyd Williams, Anderson College, Anderson, Indiana;
Mr. Mack Lester Williams, Teacher, North Texas State College, Denton, Texas;
Assistant Professor Ralph Arthur Willoughby, Georgia Institute of Technology.

It was reported that the following one-hundred fifty persons had been elected as nominees of institutional members as indicated:

University of Alabama: Assistant Professor Wayman L. Strother.
Brooklyn College: Messrs. Anatole Beck, George William Booth, and Alvin Hausner.
Brown University: Messrs. Donald Burke Lehman, Emin Turan Onat, Richard Thorpe Shield, Robert Norman Tompson, and Assistant Professor Harry J. Weiss.
California Institute of Technology: Mr. Don Elton Edmondson.
University of California, Berkeley: Messrs. Julius R. Blum, Edwin O. Elliott, William Robert Gaffey, Paul J. Kooeis, Emanuel Parzen, Reese T. Prosser, and Joseph Putter.
University of California, Los Angeles: Mr. Peter Swerling.
Case Institute of Technology: Mr. Harold King Crowder.
University of Chicago: Messrs. Louis Auslander, Isidore Fleischer, Louis A. Kokoris, Bertram Kostant, Hazleton Mirkil, George William Morgenthaler, Ronald J. Nunke, Victor Lenard Shapiro, and Fred Boyer Wright.
City College of New York: Messrs. Jerome Harold Neuwirth and Howard Young.
Columbia University: Messrs. Hugh Gordon, Daniel Kocan, Adolf Nussbaum, Daniel Leo Slotnick, and Clifford Spector.
Cornell University: Messrs Douglas Page Baird, William A. Newcomb, and Steven Orey.
Duke University: Mr. Frank Roland Olson.
Harvard University: Mr. Marion Bayard Folsom, Miss Elsie Louise Goedeke, Professor Wassily W. Leontief, Messrs. Peter Evans Martin, William Francis Reynolds, Richard Steven Varga, and Calvin Hayden Wilcox.
Haverford College: Mr. W. Taylor Putney, III.
University of Illinois: Messrs. Arno Cronheim, Joseph Edward Flanagan, Lester R. Ford, William Alexander Michael, Jr., Charles Brusle Rheams, Kung-Sing Shih, Ray Frederick Spring, and Mrs. Joyce White Williams.
Indiana University: Mr. Byron Howard McCandless,
Institute for Advanced Study: Messrs. Nesmith C. Ankeny, Arrigo Finzi, Ernest Rufener, Jacques Leon Tits, and Leonce Guy Fourès.
Iowa State College of Agriculture and Mechanic Arts: Mr. Leslie D. Gates.
The Johns Hopkins University: Messrs. Solomon Wolf Golomb, Basil Gordon, Justin Gregory MacCarthy, Shirley Nickerson Mills, Jr., Bernard Mushinsky, Charles C. Oehring, Frank Chappell Ogg, John Barr O'Toole, John Thomas Robinson, and Mrs. Helena Long Watts.

Massachusetts Institute of Technology: Messrs. Felix Haas and Jack Kotik.

University of Michigan: Messrs. James Patrick Jans, Joachim Kaiser, Jr., William Jean Byrne, Myrle V. Cross, Jr., William Cassidy Fox, John William Jewett, Charles Christopher Kilby, Jr., Geert C. E. Prins, Ralph Alexis Raimi, Joseph Robert Schoenfield, and Drury William Wall.

University of Missouri: Messrs. Charles H. Cunkle, Joe D. Hankins, and Theral Orvis Moore.

Northwestern University: Messrs. Richard James Driscoll, Glenn Jay Kleinhesselink, and Frank Brooke Sloss.

Ohio State University: Mr. Pat Holmes Sterbenz.

Oklahoma Agricultural and Mechanical College: Mr. John Rolfe Isbell.

University of Oregon: Mr. Chia Kuei Tsao.

University of Pennsylvania: Messrs. John Carl Mairhuber and William Gideon Spohn, Jr.

Princeton University: Messrs. Arthur Charles Allen, Charles Harold Bernstein, Patrick Paul Billingsley, Humberto T. Cardenas, Courtney Stafford Coleman, Richard Henry Crowell, Karel Deleeuw, John Harry Gay, Louis Norberg Howard, Alan Treleven James, Joseph Bernard Kruskal, Jr., Thomas Eugene Kurtz, Solomon Leader, Eugene Alfred Maier, Hartley Rogers, Jr., Herbert Scarf, Oliver King Smith, Rene Thom, Bodo Walter Volkmann, and David M. G. Wishart.

Purdue University: Messrs. James Walter Armstrong, Jack Kenneth Hale, Arnold Herman Koechmann, and Carl James Sinke.

Queens College: Miss Roslyn Braverman.

University of Rochester: Mr. William Andrew Small.

College of St. Thomas: Mr. Richard Paul Goblirsch.

Syracuse University: Messrs. Howard Floyd Beckafort, Charles Robert Hicks, and Adnah Gould Kostenbauder, Jr.

University of Toronto: Messrs. Louis Lorne Campbell, Bruce John Kirby, Kerry Brian McCuthchon, Albert Wallace Walker.

Vanderbilt University: Mr. B. F. Bryant.

University of Washington: Mr. William Charles Guenther.

University of Virginia: Messrs. Roger Durgin Johnson, Jr., and Alexander Morton Maish.

Williams College: Mr. John Henry Hoelzer.

University of Wisconsin: Messrs. Chris Carl Braunschweiger, Robert Cleland Carson, Wayne Russell Cowell, Edmund Harry Feller, Jack Waring Hollingsworth, Donald Wright Miller, Lee Albert Rubel, Earl William Swokowski, and Albert Wayne Wymore.

Yale University: Messrs. John Young Barry, John Courtenay Holladay, Miss Marie Lesnick, Messrs. George Benham Seligman, Robert Barr Smith, and Herbert Aaron Steinberg.

The Secretary announced that the following had been admitted to the Society in accordance with reciprocity agreements with various mathematical organizations: French Mathematical Society: Professor Roger Apery, University of Caen, Caen, France, Mr. Claude Berge, Centre National de la Recherche Scientifique, Nevilly, France, Mr. Rene Benjamin Chambaud, Consulting Engineer of Raffineries

Francaises de Petrole de l'Atlantique, Paris, France, Mr. Pierre Ernest Dolbeault, Attaché de recherches au Centre National de la Recherche Scientifique, Melakoff, France, Professor François Alain Limouzin, Retired, Poitiers, France, Professor Felix Pollaczek, Centre National de la Recherche Scientifique, Paris, France, Professor Yves Rene Thiry, Institut des Hautes Etudes, Tunis, Tunisia, Mr. Robert Gaston Velleneuve, Centre National d'Etudes des Telecommunications, Paris, France; London Mathematical Society: Dr. Jan Kalicki, Visiting Assistant Professor, University of California, Berkeley, and Dr. William Barry Pennington, Benjamin Pierce Instructor, Harvard University; Wiskundig Genootschap: Visiting Assistant Professor Johan H. B. Kemperman, Purdue University.

The following actions taken by mail vote of the Council were reported: election of Professors G. A. Hedlund and R. L. Wilder as members of the Executive Committee of the Council for a period of two years beginning January 1, 1952; acceptance by the Council of an invitation from the North Carolina State College of Agriculture and Engineering to hold a sectional meeting at Raleigh on November 28–29, 1952; and approval of the Council to hold the 1952 Annual Meeting in St. Louis in conjunction with the annual meeting of the AAAS.

The following appointments to represent the Society were reported: Professor C. B. Read at the dedication of Mossman Hall at Southwestern College, Winfield, Kansas, on March 22, 1952, and Professor F. J. Murray at a meeting of the Instrument Society of America on March 24, 1952.

The following additional appointments by the President were reported: Professors A. E. Heins (Chairman), P. Chiarulli, P. Gustafson, G. H. Handelman, L. E. Malver, R. C. Meacham, D. Moskovitz, E. A. Whitman, and W. M. Whyburn as a committee on arrangements for the Fifth Symposium on Applied Mathematics to be held at the Carnegie Institute of Technology, Pittsburgh, June 16–17, 1952; Professors J. S. Frame (Chairman), J. H. Bell, H. M. Gehman, P. Herzog, L. Hatz, E. A. Nordhaus, H. E. Stelson, B. M. Stewart, and J. W. T. Youngs as a committee on arrangements for the Summer Meeting of 1952 to be held at East Lansing, Michigan, on September 3–6, 1952; Professor J. M. Clarkson (Chairman), R. C. Bullock, Jack Levine, C. G. Mumford, J. M. Thomas, and W. M. Whyburn as a committee on arrangements for the meeting to be held at North Carolina State College of Agriculture and Engineering at Raleigh on November 28–29, 1952; Professor R. R. Middlemiss (Chairman), Dr. H. Margaret Elliott, Professors T. L. Downs,

H. M. Gehman, and J. W. T. Youngs as a committee on arrangements for the Annual Meeting to be held in St. Louis on December 27–29, 1952; Professors E. G. Begle (Chairman), C. B. Allendoerfer, A. E. Meder, Jr., G. B. Price, and A. W. Tucker as a committee to consider problems of controversial questions both as to procedures of the Council and as to the membership of the Society; Dr. H. M. MacNeille (Chairman), E. G. Begle, Jekuthiel Ginsburg, A. E. Meder, Jr., and J. V. Wehausen, as a committee to advise the Council and Board of Trustees concerning the problem of exchanges of the Society's publications; Professor Marshall Hall as a member of the committee to select hour speakers for Western Sectional Meetings (committee now consists of J. W. T. Youngs (Chairman), P. R. Halmos, and Marshall Hall); Professors M. H. Heins (Chairman), A. A. Albert, Richard Arens, Dr. Mina Rees, and Professor J. M. Thomas as a committee to nominate officers and members of the Council for 1953; Professor W. T. Martin (Chairman), Dean W. L. Ayres, and Professor J. C. Oxtoby as a committee to consider the advisability of continuing the Society's reprinting program.

The following items were reported for the information of the Council: selection of G. B. Price as Managing Editor of the Bulletin Editorial Committee; R. L. Wilder as Chairman of the Colloquium Editorial Committee; William Feller as Chairman of the Mathematical Reviews Editorial Committee; W. T. Martin as Chairman of the Mathematical Surveys Editorial Committee; G. A. Hedlund as Chairman of the Proceedings Editorial Committee; and G. T. Whyburn as Managing Editor of the Transactions and Memoirs Editorial Committee; as new members of the Policy Committee for Mathematics: S. S. Cairns and J. S. Frame (the committee now consists of J. R. Kline (Chairman), Marston Morse, W. T. Martin, and S. S. Cairns for the American Mathematical Society; H. M. Gehman, Saunders MacLane, and J. S. Frame for the Mathematical Association of America; Henry Scheffé for the Institute of Mathematical Statistics; E. H. C. Hildebrandt for the National Council of Teachers of Mathematics. (The representative of the Association for Symbolic Logic has not yet been appointed.)); acceptance by Antoni Zygmund of an invitation to deliver the Colloquium Lectures at the Summer Meeting of 1953; acceptance by Marston Morse of an invitation to deliver the 26th Josiah Willard Gibbs Lecture at the Annual Meeting of 1952; Committees to Select Hour Speakers have invited Dr. G. E. Forsythe to deliver an address at the Eugene, Oregon, meeting on June 21; Professor E. E. Moise, Summer Meeting at East Lansing; J. Leray, Yale University, New Haven, on October 25, 1952; Professor A. M. Gleason at the Annual Meeting of 1952; Professor

W. L. Chow for the February 1953 meeting in New York City; Professors Emil Artin and W. S. Massey at the April 1953 meeting in New York City.

It was reported that the First General Assembly of the International Mathematical Union was held in Rome, March 6–8, 1952. The United States was represented by Professors M. H. Stone, J. R. Kline, Einar Hille, Saunders MacLane, and G. T. Whyburn. Professor Stone was elected President of the Union.

The Council voted to approve the following dates of meetings of the Society: November 28–29, 1952, Purdue University; February 28, 1953, Hunter College; April 24–25, 1953, New York University. The Council voted to approve the week of August 31 for the time of the Summer Meeting of 1953.

The Council voted to approve the substitution of Professor Irving Kaplansky for Professor Reinhold Baer as a representative of the Society on the Board of Editors of the American Journal of Mathematics for the period June 1952 to September 1953.

The Council voted to approve the report of the Committee on Exchanges, which recommended that the policy of the Society be to discontinue all exchanges, after we have fulfilled our present obligations to the University of Georgia, except that the Executive Editor of Mathematical Reviews be authorized to enter into exchanges of the Society's publications for the benefit of Mathematical Reviews and that the Secretary be authorized to negotiate exchanges in order to further the scientific objectives of the Society, through widening the distribution of our publications.

The Council voted to recommend to the membership of the Society that the by-laws be amended as follows: (1) to eliminate the post of Librarian; (2) to create a separate elected Editorial Board for the Memoirs and an elected Committee on Printing and Publishing; (3) to revise Article IX of the by-laws to eliminate the initiation fee and the reduced rates for early years of membership and to eliminate fractional years of membership; (4) to eliminate obsolete sections of Articles IV and VII.

The Council voted to request the President to appoint a committee to consider the problems involved in nominating officers and members of the Council by petition from the membership of the Society.

The Council referred to the Executive Committee, with power to act, current problems concerning our publications.

The Board of Trustees met Friday evening. The Board voted that institutional dues for the period July 1952 to June 1955 be computed on the basis of a page rate of $5.00. The Board voted to set the page charge at $10 for institutions which are not institutional members.

Abstracts of the papers presented are listed below, those with a "*t*" after their numbers having been read by title. The joint papers numbered 340, 372, 373, 376, and 381 were read by Dr. Singer, Professor Miller, Mr. Crispin, Dr. Juncosa, and Dr. Lukacs, respectively.

ALGEBRA AND THEORY OF NUMBERS

319. A. A. Albert: *New power-associative algebras and their derivations.*

It is now known that there exists a central simple commutative power-associative algebra S_p of degree two and dimension $4p$ over a field F of characteristic p. It contains a central simple algebra T_p of degree two and dimension $3p$. The algebra S_p contains a nontrivial idempotent u and is u-stable. However T_p is unstable and so S_p is also unstable. The algebra T_p contains the polynomial algebra $B=F[1, k]$, where $k^p=0$ and the derivation algebra of B is the well known algebra W_p of Witt. We show that the derivation algebra of T_p is the direct sum of W_p and the three-dimensional simple Lie algebra, and that this is the same as the derivation algebra of S_p. We also construct a new class of central simple commutative power-associative algebras of degree two and dimension $(n+3)p^n$ with corresponding central simple subalgebras of dimension $(n+2)p^n$. (Received March 10, 1952.)

320. Joseph Andrushkiw: *Homoreal rings.* Preliminary report.

A ring is real if it is a domain of integrity and zero is not expressible in it as a sum of squares of elements different from zero. Since the field of quotients of a real ring is also real, any real ring can be ordered. A ring R is termed "homoreal" if there exists a real ring $\bar{R}$ homomorphic with it. Dependent upon whether in R either both conditions of reality or only first or only second or none of them are satisfied, four kinds of the homoreal rings can be distinguished. A ring is homoreal, if and only if there exists in it a prime ideal E, different from the null ideal, with the following property: a sum of squares $\sum_1^n e_i^2 \in E$ if and only if all $e_i \in E$. If R is a homoreal ring of the first or second kind, each element of E different from zero is transcendental in the field of quotients of R. Any homoreal ring R of the first kind can be ordered in non-Archimedean way. There exists an order of R in which $me<a$ where $e \in E$, a is an element of any positive residue class of the ordered ring R/E, and m any positive integer. (Received March 10, 1952.)

321*t*. Leonard Carlitz: *A note on common index divisors.*

Let the prime $p\equiv 1 \pmod 3$ and let C_3 denote the cubic subfield of $Z=k(e^{2\pi i/p})$. By a well known theorem of Hensel, 2 is a common index divisor of C_3 if and only if $p=a^2+27b^2$. In this note we derive several theorems of a similar kind. We quote the following results. Let $p\equiv 1 \pmod 4$ and let C_4 denote the quartic subfield of Z. 1. Then 2 is a common index divisor of C_4 if and only if $p\equiv 1 \pmod 8$. 2. Let $p=a^2+b^2$, $a\equiv 1$, $b\equiv 0 \pmod 2$. Then 3 is a common index of C_4 if and only if (i) $3|b$ for $p\equiv 1 \pmod 8$, (ii) $3|a$ for $p\equiv 5 \pmod 8$. Like results are obtained for C_6, C_8, C_{12}. (Received January 11, 1952.)

322*t*. Leonard Carlitz: *A theorem of Dickson on irreducible polynomials.*

In connection with the study of modular invariants Dickson (Trans. Amer.

Math. Soc. vol. 12 (1911) pp. 1–18) determined the number of irreducible polynomials $x^3-x^2+ax+b^2$, where a, $b\in GF(p^n)$, $p>2$. In the present paper the author first finds an asymptotic formula for the number of irreducible polynomials of a given degree $Q(x)=x^m+c_1x^{m-1}+\cdots+c_m$, $c_i\in GF(p^n)$, with assigned first coefficient c_1 and last coefficient c_m. Secondly he finds exact formulas connected with the number of irreducibles with assigned first coefficient and last coefficient equal to a square of the field, thus generalizing Dickson's result. Finally he determines the number of irreducibles with assigned first coefficient. (Received January 11, 1952.)

323t. Leonard Carlitz: *Note on irreducibility of the Bernoulli and Euler polynomials.*

The following special results are obtained. 1. Let p denote an odd prime. Then $B_{m(p-1)}(x)$ is irreducible for $1\leqq m\leqq p$; also $B_m(x)$ is irreducible for $m=2^r$ and $m=k(p-1)p^r$, $1\leqq k<p$, $r\geqq 1$. 2. $B_{2m+1}(x)/x(x-1/2)(x-1)$, where $2m=k(p-1)$, $k\leqq p$, is either irreducible or has an irreducible factor of degree $\geqq 2m+1-p$. 3. If $p\equiv 3 \pmod 4$, $r\geqq 1$, then $E_{p^r}(x)/(x-1/2)$ is irreducible. 4. $E_{kp}(x)/x(x-1)$ is irreducible or has an irreducible factor of degree not less than $p+1$. 5. If $B_m^{(k)}$ denotes the Bernoulli number of order k in Nörlund's notation, then $B_{p-1}^{(x)}/x$ is irreducible. (Received March 12, 1952.)

324t. Leonard Carlitz: *Some formulas of Oltramare.*

Dickson (*History of the theory of numbers*, vol. 1, p. 277) reproduces the following congruences due to G. Oltramare: $1+(m!)^4\equiv -2\{(1/3)^2+(1\cdot 5/3\cdot 7)^2+(1\cdot 5\cdot 9/3\cdot 7\cdot 11)^2+\cdots\} \pmod{4m+1}$, $2^s+(m!)^4\equiv -2^s\{(3/1)^2+(3\cdot 7/1\cdot 5)^2+(3\cdot 7\cdot 11/1\cdot 5\cdot 9)^2+\cdots\} \pmod{4m+3}$, where the moduli are primes. In the present note it is indicated how many congruences of this sort can be obtained by means of known formulas involving generalized hypergeometric series. (Received February 15, 1952.)

325t. Leonard Carlitz: *Sums of primitive roots of the first and second kind in a finite field.*

Let $e_i|p^{nm}-1$, $k_i|p^{nm}-1$, $A(x)|x^m-1$, where $A_i(x)\in GF[p^n, x]$, and let $a_i(x)$ be the linear polynomial corresponding to $A_i(x)$. Let β_i denote a number of $GF(p^{nm})$ that simultaneously belongs to the exponent e_i and to the polynomial $a_i(x)$ (as defined by Ore). Extending the results of a previous paper we here consider the following problems. 1. The number of solutions $\beta_1, \cdots, \beta_r$ of $\alpha=\alpha_1\beta_1+\cdots+\alpha_r\beta_r$, where $\alpha_i\neq 0$. 2. The number of solutions $\beta_1, \cdots, \beta_r, \xi_1, \cdots, \xi_s$ of $\alpha=\alpha_1\beta_1+\cdots+\alpha_r\beta_r+\delta_1\xi_1^{k_1}+\cdots+\delta_s\xi_s^{k_s}$ $(\xi_i\in GF(p^{nm}))$. Asymptotic results are obtained in both problems. 3. In the special case $r=1$, s even, a simple exact formula is obtained. (Received February 15, 1952.)

326. P. W. Carruth: *Products of ordered systems.*

Some miscellaneous results concerning ordered products of ordered systems are given (see Day, Trans. Amer. Math. Soc. vol. 58 (1945) pp. 1–43 for definitions of terms involved). For example, it is shown that the only products that are complemented or relatively complemented lattices are cardinal products. Conditions for a product of lattices to be a lattice and for a product of partially ordered sets to satisfy the ascending chain condition are given. In the proofs use is made of results of Day, loc. cit. (Received February 20, 1952.)

327*t*. Eckford Cohen: *A finite analogue of the Goldbach problem.*

If $m>1$ is an integer with factorization $m=p_1^{\lambda_1}\cdots p_r^{\lambda_r}$, then the elements of the residue class ring R_m can be expressed in the form $p_1^{a_1}\cdots p_r^{a_r}\xi$ where $0\leqq a_i\leqq\lambda_i$, $(\xi, m)=1$. This factorization is unique except for the unit ξ, and the totality of elements of the form $p_i\xi$ are the prime elements of R_m (cf. Vandiver, Trans. Amer. Math. Soc. vol. 13 (1912) p. 293). The following theorem is proved: *There exists a number $g(m)$ such that every number of R_m is expressible as a sum of $g(m)$ primes in R_m if and only if m has at least two distinct prime factors. For such m, the minimum value k of $g(m)$ is given by $k=2$ if m is odd, by $k=3$ if m is even and has at least two distinct odd prime factors or if m is twice an odd prime power, and by $k=4$ if m is of the form $m=2^\lambda p^\mu$, $\lambda>1$, $\mu\geqq 1$, p odd.* Several related theorems are also proved. (Received March 10, 1952.)

328*t*. Eckford Cohen: *Arithmetic functions in algebraic fields.*

If A is an arbitrary ideal of an algebraic number field F, and K is a field of characteristic 0 containing all the roots of unity, then a single-valued function f is (A, K) arithmetic if, for every integer $\alpha\in F$, $f(\alpha)\in K$, and $f(\alpha)=f(\alpha')$ for $\alpha\equiv\alpha'$ (mod A). It is shown that every such function is uniquely expressible as a sum of exponential functions of the type introduced by Hecke (*Vorlesungen über Zahlentheorie*, 1923, p. 220). It is then shown how exponential formulas for a large class of additive congruence problems in algebraic fields can be constructed. The set of all (A, K) functions forms a commutative semi-simple algebra with multiplication defined by "Cauchy composition" (cf. Carlitz, Duke Math. J. vol. 14 (1947) p. 1121). (Received March 10, 1952.)

329*t*. Eckford Cohen: *Congruence representations in algebraic number fields.*

Applications of the ideas in the preceding abstract are developed. The arithmetic function of Rademacher (Math. Zeit. vol. 27 (1928) p. 332) and the Hecke generalization of the Gauss sums (cf. preceding abstract) are discussed and applied. In particular, the number of representations of an algebraic integer as a sum of products, relative to an ideal modulus, is found. Explicit formulas for the number of solutions ξ_i of $\rho\equiv\sum_1^s\alpha_i\xi_i^2$ (mod P^λ), where P is an odd prime ideal, $\lambda\geqq 1$, and ρ, α_i are algebraic integers, $(\alpha_i, P)=1$, are derived. From this result it is shown that every algebraic integer can be expressed as a sum of three squares in the ring formed by the residue classes of algebraic integers modulo an arbitrary odd ideal. (Received March 10, 1952.)

330*t*. Anne C. Davis: *Cancellation theorems for products of order types*. II. Preliminary report.

For notational conventions see Bull. Amer. Math. Soc. Abstract 58-1-77; in addition, "$\alpha\leqq\beta$" means (Fraïssé, C. R. Acad. Sci. Paris vol. 226, p. 1330) that there exists an order-preserving mapping of an ordered set A of type α onto a subset of an ordered set B of type β. Lemma I. *If $\alpha\ \beta=\gamma\cdot\delta$, then at least one of the following conditions must hold*: (i) $\beta=\delta$, (ii) *there exist order types μ_1, μ_2, ρ such that $\alpha=\mu_1+\gamma\cdot\rho+\mu_2$*, (iii) *there exists order types μ_1, μ_2, ρ such that $\gamma=\mu_1+\alpha\cdot\rho+\mu_2$*, (iv) *there exist finite order types ν and ϕ, $\nu\neq\phi$, such that $\alpha\cdot\nu=\gamma\cdot\phi$.* Lemma I implies all cancellation theorems of Bull. Amer. Math. Soc. Abstract 58-1-77. A further consequence of Lemma I is Corollary II. *If α is the type of an ordered set without gaps and without first and last elements, and if $\alpha^2=\beta^2$, then $\alpha=\beta$.* Theorem III. *For every positive integer*

n, $\alpha+\alpha \leqq \alpha \leftrightarrow \alpha^n+\alpha^n \leqq \alpha^n$. By use of Lemma I, in conjunction with Theorem III, one obtains Corollary IV. *If* $\alpha^n=\beta^n$, *n a positive integer, and if* $\sim(\alpha+\alpha \leqq \alpha)$, *then* $\alpha=\beta$. It is easily seen that Corollary IV is applicable, in particular, if α, or β, is a scattered type. (Received March 10, 1952.)

331. J. K. Goldhaber: *Special types of linear mappings of algebras.*

Let $\mathfrak{M}$ be a total matric algebra over a field whose characteristic is different from 2, and let Φ be a linear mapping of $\mathfrak{M}$ into $\mathfrak{M}$. Suppose that Φ preserves idempotents and nilpotents of index two, that is, for $A \in \mathfrak{M}$ if $A^2=A$ then $[\Phi(A)]^2=\Phi(A)$ and if $A^2=0$ then $[\Phi(A)]^2=0$. Then either $\Phi(A)=0$ for all $A \in \mathfrak{M}$ or Φ is an automorphism or an anti-automorphism of $\mathfrak{M}$. This theorem does not extend to simple algebras. However, the following theorem is established: Let $\mathfrak{A}$ be a central simple algebra over a field of characteristic different from 2, and let Φ be a linear mapping of $\mathfrak{A}$ into $\mathfrak{A}$ such that Φ maps the identity of $\mathfrak{A}$ into the identity of $\mathfrak{A}$ and such that for $A \in \mathfrak{A}$, A and $\Phi(A)$ have the same characteristic equation, then Φ is either an automorphism or an anti-automorphism of $\mathfrak{A}$. If $\mathfrak{A}$ is a semi-simple separable algebra, then Φ induces either an automorphism or an anti-automorphism on every simple component of $\mathfrak{A}$. (Received March 10, 1952.)

332. Marshall Hall, Jr.: *Subgroups of free products.*

This is a new proof of the theorem of Kurosch that every subgroup of a free product G is itself a free product of factors which are either free groups or conjugates of subgroups of the free factors of G. It depends upon defining a semi-alphabetical well ordering for the elements of G, and in terms of this finding a set of elements generating the subgroup. This set exhibits the free factors explicitly and the proof consists in showing that these are indeed free factors. (Received March 11, 1952.)

333t. C. E. Rickart: *Spectral permanence for certain Banach algebras.*

Let A be a commutative Banach algebra which is algebraically embedded in a second such algebra B, and let ϕ be a homomorphism of A into the complex numbers. If the embedding of A in B is an isometry, then Šilov [Uspehi Matematičeskih Nauk N.S. vol. 1 (1946) pp. 48–146] has shown that certain ϕ can be extended to B. It is proved here that, if A is regular in a sense defined by Šilov [Travaux de l'Institut Mathématique Stekloff vol. 21 (1947)], then ϕ can always be extended to B. This implies that elements of A have the same spectra in A as in B and includes a result of Kaplansky [Duke Math. J. vol. 16 (1949) pp. 399–418] concerning the minimal character of the norm in a Banach algebra of all continuous functions on a compact space. Some results are also obtained here for the noncommutative case. For example, let A be a B^*-algebra [cf. Rickart, Ann. of Math. vol. 47 (1946) pp. 528–550] which is algebraically embedded in an arbitrary Banach algebra B. Then elements of A always have the same spectral radii in A as in B. If in addition the involution in A can be extended to B, then elements of A have the same spectra in A as in B. If the norm in B is invariant in A under the involution, then the norm in A cannot be greater than the norm in B for elements of A. (Received March 10, 1952.)

334. Maxwell Rosenlicht: *Generalized Jacobian varieties.*

A semilocal subring of the function field of an algebraic curve defines an equivalence relation among the divisors of the curve (cf. Bull. Amer. Math. Soc. vol. 57 (1951) p. 269). A generalized Jacobian variety corresponding to this equivalence relation can be constructed; this is a commutative algebraic group variety whose points are in a natural one-one correspondence with divisor classes of degree zero on the curve. If two equivalence relations, one of which is stricter than the other, are given on the same curve, then there is a natural rational homomorphism of one generalized Jacobian onto the other. The kernel of this homomorphism is a rational group variety whose structure can be explicitly described. A treatment of the classical generalized Jacobi inversion problem is included in this work. (Received March 10, 1952.)

335. R. D. Schafer: *The Casimir operation for alternative alegbras.*

Let $\mathfrak{A}$ be a semisimple alternative algebra of characteristic 0, and $\mathfrak{B}$ be an irreducible alternative module for $\mathfrak{A}$. For any bilinear mapping $f(a, b)$ of $\mathfrak{A}$ into $\mathfrak{B}$, define $F(a, b, c) = f(a, b)c + f(ab, c) - af(b, c) - f(a, bc)$, $f^+(a, b) = f(a, b) + f(b, a)$. If $F(a, b, c) = F(b, c, a) = -F(b, a, c)$, then there exists a linear mapping $g(a)$ of $\mathfrak{A}$ into $\mathfrak{B}$ such that $f^+(a, b) = a \cdot g(b) + g(a) \cdot b - g(a \cdot b)$, where $a \cdot b$ and $a \cdot v$ denote the operations in $\mathfrak{A}^+$ and $\mathfrak{B}^+$. This lemma, which is proved by means of a Casimir operation Γ associated with any representation (S, T) of $\mathfrak{A}$, has the force of the Wedderburn principal theorem for the Jordan algebra attached to any alternative algebra, and allows for some simplification in the proof of the Wedderburn principal theorem for alternative algebras of characteristic 0. (Received March 12, 1952.)

336t. Daniel Shanks: *The quadratic reciprocity law.*

The reciprocity law concerning two odd primes, $P = 2p+1$ and $Q = 2q+1$, is proven directly from the Gauss Lemma in perhaps the simplest and most natural way. Consider the pq numbers: $F(x, y) = xQ - yP$ with $x = 1, 2, \cdots, p$ and $y = 1, 2, \cdots, q$ and assume that there are m solutions of $-p \leqq F < 0$ and n solutions of $0 < F \leqq q$. There is no solution of $F = 0$ and the solutions of $F < -p$ and of $F > q$ may be put into 1-1 correspondence by the symmetrical relations: $x_1 + x_2 = p + 1$ and $y_1 + y_2 = q + 1$. Therefore $m + n \equiv pq$ (mod 2). Since, by the lemma, $(Q/P) = (-1)^m$ and $(P/Q) = (-1)^n$, we thus obtain $(Q/P)(P/Q) = (-1)^{mn}$. Landau [*Aus der elementaren Zahlentheorie*], by considering $F < 0$ and $F > 0$, proves $pq = \sum_{s=1}^{p} [sQ/P] + \sum_{s=1}^{q} [sP/Q]$. By dividing F into twice as many parts, twice as much is proved. The partition and proof given above may also be seen geometrically on a modified Eisenstein lattice. (Received March 12, 1952.)

337. M. L. Tomber: *Lie algebras of type* F.

Let $\mathfrak{L}$ be a Lie algebra over an arbitrary field Φ of characteristic 0, of type F, in the sense that $\mathfrak{L}_\Omega$ is the exceptional simple Lie algebra F_4 where Ω is the algebraic closure of Φ. It is proved that $\mathfrak{L}$ is the derivation algebra $\mathfrak{D}(\mathfrak{J})$ of an exceptional Jordan algebra $\mathfrak{J}$ over Φ and $\mathfrak{D}(\mathfrak{J}_1) \cong \mathfrak{D}(\mathfrak{J}_2)$ if and only if $\mathfrak{J}_1 \cong \mathfrak{J}_2$. The proofs employ the methods of N. Jacobson (Duke Math. J. vol. 5 (1939) pp. 775–783) and the result that any automorphism of F_4 (over Ω) has the form $D \to S^{-1}DS$ for a unique automorphism S of the exceptional Jordan algebra over Ω. (Received March 11, 1952.)

338. Bodo Volkmann: *On the digits of real numbers.* Preliminary report.

Let $g \geqq 2$ be an integer and $F = (f_1, f_2, \cdots, f_i)$ be any block of digits in the g-adic system. This paper discusses the set $K_{F,g}$ of real numbers in the unit interval in whose g-adic expansion the block F does not occur. If $\mathfrak{P}(F) = \{p_1, p_2, \cdots, p_s\}$ denotes the (possibly empty) set of those integers p with $0 < p < i$ for which the block $(f_1, f_2, \cdots, f_p)$ equals the block $(f_{i-p+1}, f_{i-p+2}, \cdots, f_i)$, let $\gamma_{F,g}$ be the greatest real root of the equation $\xi^i - g\xi^{i-1} + \sum_{\nu=1}^{s} \xi^{i-p_\nu} - g\sum_{\nu=1}^{s} \xi^{i-p_\nu-1} + 1 = 0$, the sums being zero if $\mathfrak{P}(F)$ is empty. The Hausdorff dimension of $K_{F,g}$ turns out to be $\log \gamma_{F,g}/\log g$. This result was proved by E. Best (J. London Math. Soc. (2) vol. 47 (1942) pp. 436–454) in the special case $i=1$, $g\geqq 2$ and by the author (forthcoming in J. Reine Angew. Math.) in the special case $g=2$, $i\geqq 2$, $F=(1, 1, \cdots, 1)$. (Received March 11, 1952.)

ANALYSIS

339. A. C. Allen: *The closure of the translations of sets of functions of L_1, and certain applications to harmonic functions.* Preliminary report.

N. Wiener (*The Fourier integral*, chap. 2) gave a series of theorems on the closure of the translations of a function of L_1, the significant class of functions being considered being those whose Fourier transforms do not vanish for any real values of the argument. In this paper the author gives analogous theorems for $n\times n$ matrices whose elements are functions of L_1; "translation" being interpreted as "folding" with a certain translation-matrix; and the significant class of function-matrices considered being those in which the determinant, whose elements are the Fourier transforms of the elements of the matrix, does not vanish for any real value of the argument. As an application of the above, certain generalizations of the Phragmén-Lindelöf type theorem for harmonic functions and of Montel's theorem for bounded analytic functions are derived. The methods used, however, are applicable to a wider class of functions than those which satisfy Laplace's equation, and so have an advantage over methods of proof which depend on conformal mapping. (Received March 12, 1952.)

340. Warren Ambrose and I. M. Singer: *L_2-matrices.*

It is shown that some H-systems are matrix-like. (Received March 12, 1952.)

341. J. H. Barrett: *Differential equations of nonintegral order.*

Let α be a complex number; n the integer such that $0<R\alpha+n\leqq 1$; $f(x)$ a real function, a a real number. $I\{\alpha, a, x, f\}=\int_a^x f(t)((x-t)^{\alpha-1}/\Gamma(\alpha))dt$, $R\alpha>0$; $I\{\alpha, a, x, f\}=D_x^n I\{\alpha+n, a, x, f\}$, $R\alpha\leqq 0$. For $(x-t)^{\alpha-1}$ use the principal value. This is called the Riesz-Holmgren transform. An extension of a result of Hardy's (Messenger of Mathematics vol. 47 (1918) pp. 145–150) is if $X>a$, $f(x)$ is Lebesgue summable on $a\leqq x\leqq X$, $R\alpha>0$, $R\beta>0$, then $I\{\alpha, a, x, I(\beta, a, x, f)\}=I\{\alpha+\beta, a, x, f\}$ and $D_x I\{\alpha+1, a, x, f\}=I\{\alpha, a, x, f\}$ almost everywhere on $a\leqq x\leqq X$. Let $f(x)$ be a Lebesgue summable function on $0<x\leqq X$. $y(x)$ is called a solution of $I\{-\alpha, 0, x, y\}+y(x)=f(x)$, $R\alpha>0$, $0<x<X$, provided that $y(x)$ is Lebesgue summable, $I\{1-\alpha, 0, x, y\}$ exists and is absolutely continuous in x, and $y(x)$ satisfies the equation a.e. on $0<x<X$. The differential equation is transformed into an integral equation, which is solved by means of an inverse of the operator: $I\{0, 0, x, y\}-I\{\alpha, 0, x, y\}$. Let $K_0, K_1, \cdots, K_{n-1}$, where $0<n-R\alpha\leqq 1$, be real numbers; then there is a unique solution satisfying the boundary conditions: $I\{n-\alpha-p, 0, 0^+, y\}=K_p$, $p=0, 1, \cdots, n-1$. Let $U_q(x)$ be the solution where $K_p=0$, $p\neq q$, and $K_q=1$ and $f(x)\equiv 0$. The solution of the nonhomogeneous equation is $y(x)=\sum_{i=0}^{n-1} K_i U_i(x)+\int_0^x f(t)U_{n-1}(x-t)dt$, $x>0$. (Received March 12, 1952.)

342*t*. E. F. Beckenbach: *A property of mean values of an analytic function.*

Let $w=f(z)$ be analytic in the circle $|z|<1$, and let the length of the map of each radius of the circle be $\leqq 1$. Bounds are obtained for the lengths of the images of concentric circles $|z|=\rho$, $0<\rho<1$. This extends a result of Nehari (C. R. Acad. Sci. Paris vol. 208 (1939) pp. 1785–1787). (Received February 29, 1952.)

343t. Stefan Bergman: *The coefficient problem in the theory of linear partial differential equations.*

Let ψ be a solution of $\psi_{xx}+\psi_{yy}+F\psi=0$, $\psi_{xx}=\partial^2\psi/\partial x^2, \cdots$, where F is an entire function. Many relations between the properties of the subsequence $\{a_{m0}\}$, $m=0, 1, 2, \cdots$, of coefficients of a real solution $\psi(x, y)=\sum_{m,n} a_{mn}z^m\bar{z}^n$, $z=x+iy$, $\bar{z}=x-iy$, on one side and location and character of singularities of ψ on the other are independent of F. (See, e.g., Trans. Amer. Math. Soc. vol. 57, p. 299 ff.). If $g(z)=\sum_m a_{m0}z^m$ is a function element of an algebraic function, then the ψ can be represented in the form $\psi=\mathrm{Re}(\sum_{n=1}^{\infty} Q^{(n)}g_n)$ where $Q^{(n)}$ are functions which depend only upon F, and g_n, $n=0, 1, 2, \cdots$, are functions which can be represented in a closed form using θ-functions, their derivatives and integrals of the first kind defined on the Riemann's surface of $g(z)$. Similar theorems are derived for solutions of differential equations $\sum_{\nu=1}^{3} (\partial^2\psi/\partial x_\nu^2)+F(r^2)\psi=0$, where $F(r^2)$ is an entire function of $r^2=\sum_{\nu=1}^{3} x_\nu^2$. (Received April 25, 1952.)

344. Jerome Blackman: *Some notes on the backward heat equation.*

Consider the classical solution of the heat equation on the infinite rod with initial condition $\phi(x)$, given by $u(x, t)=\int_{-\infty}^{\infty}\phi(s) \exp (x-s)^2/4t\,ds$. If $\phi(s)=O(\exp s^2)$, this integral may diverge for a certain $t=T$ but converge for $t<T$. The following problems are investigated: (1) Can $\lim u(x, t)$ exist as $t\to T-$? (2) What properties must this limit have if it exists? The analytic character of the limit is demonstrated; an example and a method of determining a class of examples is given. The methods are applicable to the solution of the backward heat equation $u_{xx}+u_t=0$ for the infinite rod, and demonstrate the relation between uniqueness theorems for the heat equation and uniqueness theorems for certain types of summability of trigonometrical integrals. (Received March 5, 1952.)

345t. R. P. Boas: *Growth of analytic functions along a line.*

Results reported earlier (Bull. Amer. Math. Soc. Abstract 58-2-156) are improved as follows. Let $\epsilon(x)$ be positive, increasing, $\epsilon(x)=o(x)$, $\log x=o(\epsilon(x))$, and let $\epsilon(x)$ satisfy some minor conditions of smoothness and monotonicity; let the increasing sequence $\{\lambda_n\}$ of real numbers satisfy $|\lambda_{n+1}-\lambda_n|\geqq\delta>0$ and $|n-\lambda_n|\leqq\epsilon(n)$. Then if $f(z)$ is of exponential type less than π in the right half-plane, $\log |f(\lambda_n)|\leqq O(\epsilon(\lambda_n))$ implies $\log |f(x)|\leqq O(\epsilon(x))$. If in addition (*) $\int^{\infty}x^{-2}\epsilon(x)dx$ converges, if $f(z)$ is an entire function of exponential type less than π, and if $\log |f(\pm\lambda_n)|\leqq -B\epsilon(\lambda_n)$ with a sufficiently large B, then $f(x)$ is bounded. If $f(z)$ is entire and of zero exponential type, (*) converges and $\{f(\pm\lambda_n)\}$ is bounded, $f(z)$ is a constant; if $|n-\lambda_n|\leqq n^\rho$ and $\log |f(\pm\lambda_n)|\leqq A\lambda_n^\rho$, $0<\rho<1$, then $f(z)$ is of order at most ρ. The first result for functions of zero type improves a result of Levinson as far as concerns the size of $\epsilon(n)$ [*Gap and density theorems*, New York, 1940, pp. 129–130]. (Received March 24, 1952.)

346t. F. E. Browder: *Fundamental solutions in the large for linear elliptic equations of arbitrary order.*

Let K be a suitably differentiable linear elliptic differential operator on a bounded domain D of E^n. If the Dirichlet problem for K on D has at most one solution with zero boundary values, then there exists a fundamental solution $e(x, z)$ of K on D. If K is positive, self-adjoint, and bounded at infinity, a similar result may be established for unbounded domains. Differentiability properties as related to the differentiability of the coefficients of K and the relations between various fundamental solutions are studied. (Received March 12, 1952.)

347. F. E. Browder: *Green's function and the kernel function for linear elliptic equations of arbitrary order.*

Let K be a suitably differentiable linear elliptic differential operator on a bounded domain D of E^n. Then there exists a Green's function for K on D if and only if the Dirichlet problem for K on D with zero boundary values has only the zero solution. If K is positive, self-adjoint, and bounded at infinity, this result may be extended to arbitrary unbounded domains. If K is strongly positive and self-adjoint, there exists for each Poincaré domain D (Courant-Hilbert, vol. II, chap. VII) a kernel function $k(x, s)$ defined in terms of a complete family of solutions of the equation $Ku=0$ in D. In the case of second order equations in normal form, $k(x, s)$ reduces to the Bergman-Schiffer kernel function. The solution of the Dirichlet problem with boundary value function g can be represented in a simple closed form in terms of $k(x, s)$. (Received March 12, 1952.)

348t. Y. W. Chen: *Existence of minimal surfaces with given period and with given conditions on the boundary and at infinity.*

Let $z(u, v)=\Re[a\beta(1+\beta^2)^{-1/2}(e^{i\tau}w^{-1}+e^{-i\tau}w)+2Ki \log w]$ for $|w|=|u+iv|\leqq 1$; the real constants K and $\beta\geqq 0$ are given while a and τ are to be determined later. A simple closed Jordan curve C is given in the x, y plane. In order to satisfy the condition that $\partial z/\partial x\rightarrow\beta$, $\partial z/\partial y\rightarrow 0$ at infinity of the x, y plane which corresponds to $w=0$, one puts $x(u, v)=\Re[a(1+\beta^2)^{-1/2}(e^{i\tau}w^{-1}-e^{-i\tau}w)]+X(u, v)$ and $y(u, v)=\Re[-ai(e^{i\tau}w^{-1}+e^{-i\tau}w)+2K\beta(1+\beta^2)^{-1/2} \log w]+Y(u, v)$. The problem is to find the constants a, τ and two regular harmonic functions $X(u, v)$, $Y(u, v)$ in $|w|<1$ with the following properties: (1) X, Y maps the positive oriented circle $|w|=1$ continuously and monotonically into the negative oriented curve C, (2) $x(u, v)$, $y(u, v)$, $z(u, v)$ represents a minimal surface. The problem is analogous to the classical problem of finding by conformal transformations flows of incompressible fluids around a profile C, with given circulation K and uniform speed β at infinity. Here the minimal surface takes the place of the conformal mapping. The author uses the direct method of the calculus of variations to establish the existence proof for which a sufficient condition (*) is needed to guarantee the nondegeneracy of the limit functions. Various applications of the condition (*) are made to study the ranges of the values of K and β for which the problem has a solution. The main point of the paper is to construct the integral expression to be minimized. (Received February 18, 1952.)

349t. E. A. Coddington: *On ordinary self-adjoint differential operators.* I.

Let L be the formal operator $L=p_0(d/dx)^n+p_1(d/dx)^{n-1}+\cdots+p_n$, where the p_k are complex-valued functions of class C^{n-k} on an open (possibly unbounded) interval (a, b), and $|p_0|\neq 0$. Assume $L=L^+$, where $L^+=(-1)^n(d/dx)^n(\bar{p}_0\cdot)+(-1)^{n-1}(d/dx)^{n-1}(\bar{p}_1\cdot)+\cdots+\bar{p}_n$. Let $\mathfrak{D}$ be the set of all $u\in\mathfrak{L}^2(a, b)$ having continuous derivatives up to order $n-1$, $u^{(n-1)}$ absolutely continuous on every closed subinterval, $Lu\in\mathfrak{L}^2(a, b)$. Define T by $Tu=Lu$, for $u\in\mathfrak{D}$. For $u, v\in\mathfrak{D}$, let $[uv](a)=\lim [uv](x)$, $x\rightarrow a$, where $[uv](x)$ is the skew-hermitian semi-bilinear form appearing in Green's formula. Similarly define $[uv](b)$, and let $\langle uv\rangle=[uv](b)-[uv](a)$. If $\mathfrak{D}_0=\{u\in\mathfrak{D}\,|\,\langle uv\rangle=0 \text{ for all } v\in\mathfrak{D}\}$, define T_0 by $T_0u=Lu$, for $u\in\mathfrak{D}_0$. The operator

T_0 is closed, symmetric, and $T_0^* = T$. For any fixed c, $a<c<b$, and any complex number l, $\mathfrak{I}l\neq 0$, let $\mathfrak{E}_a(l) = \{u \mid Lu = lu \text{ and } u\in \mathfrak{L}^2(a,c)\}$. Similarly define $\mathfrak{E}_b(l)$. Then $\dim \mathfrak{E}_a(l) \geqq \nu$, $\dim \mathfrak{E}_b(l)\geqq \nu$, where $n=2\nu$ or $2\nu+1$ according as n is even or odd. If $\tau(l) = \dim \mathfrak{E}_a(l)+\dim \mathfrak{E}_b(l)-2\nu$, then the deficiency indices (ω^+, ω^-) of T_0 satisfy $\tau(l)+2\nu-n\leqq \omega^+\leqq\tau(l)$ $(\mathfrak{I}l>0)$, $\tau(l)+2\nu-n\leqq\omega^-\leqq\tau(l)$ $(\mathfrak{I}l<0)$. (Received March 12, 1952.)

350*t*. E. A. Coddington: *On ordinary self-adjoint differential operators*. II.

Suppose the deficiency indices (ω^+, ω^-) of the formally self-adjoint differential operator L are both equal to ω (cf. the preceding abstract). If $\alpha_1, \cdots, \alpha_\omega$ are ω functions in $\mathfrak{D}$ which are linearly independent mod $\mathfrak{D}_0$, the boundary conditions $\langle u\alpha_j\rangle = 0$ $(j=1, \cdots, \omega)$ for $u\in\mathfrak{D}$ are self-adjoint if $\langle \alpha_j\alpha_k\rangle = 0$ $(j, k=1, \cdots, \omega)$. Two sets of ω self-adjoint boundary conditions are equivalent if they determine the same subset of $\mathfrak{D}$. There is a 1-1 correspondence between self-adjoint extensions of T_0 and equivalence classes of ω self-adjoint boundary conditions. In case $\omega=0$, T_0 is self-adjoint, and has a spectral representation $T_0=\int\lambda dE(\lambda)$. In this case let δ be a closed interval interior to (a, b), and let $\mathfrak{D}_\delta$ be the set of all $u\in\mathfrak{L}^2(a, b)$ having continuous derivatives up to order $n-1$ on δ, $u^{(n-1)}$ absolutely continuous on δ, $Lu\in\mathfrak{L}^2(\delta)$, and u satisfies n self-adjoint boundary conditions on δ. Define H_δ by $H_\delta u(x) = Lu(x)$, $x\in\delta$, and $H_\delta u(x)=0$, $x\notin\delta$. Then H_δ is self-adjoint, $H_\delta=\int\lambda dE_\delta(\lambda)$, and if $\delta\to(a, b)$, $E_\delta(\lambda)\to E(\lambda)$, if λ is not an eigenvalue of T_0. This result implies the uniqueness of the (suitably normalized) matrix appearing in the generalized Plancherel theorem for T_0. (Received March 12, 1952.)

351. R. M. Cohn: *Essential singular manifolds of difference polynomials.*

It is shown that 0 is not an e.s.m. (essential singular manifold) of the difference polynomial $yy_2^2+y_1^2y_3-\lambda y_1y_3$ if $\lambda=1$, but is an e.s.m. if λ is an element transformally transcendental over the difference field consisting of the rational numbers. Hence there can be no criterion based on the degrees and orders of the terms only for determining whether or not 0 is an e.s.m. of a difference polynomial. This is an unexpected result since the low-power theorem of Ritt provides a criterion of this sort for differential polynomials. It is also shown that such a criterion exists for difference polynomials of second order and for other special classes of difference polynomials. (Received February 25, 1952.)

352. Philip Cooperman: *The multiplier rule for a partial differential equation in two dependent and two independent variables as side condition.*

The object of this paper is to give a proof of the Lagrangian multiplier rule for the case of a general partial differential equation in two dependent and two independent variables as side condition. The method used depends on the fact that the differential equations resulting from the variation of the dependent variables are greater in number than the number of multipliers. Hence, when these equations are regarded as equations in the multipliers, the existence of the multipliers depends on the satisfaction of compatibility requirements. These requirements turn out to be that the dependent variables satisfy certain relations, independent of the multipliers. If there are functions satisfying these relations, as well as the side and boundary conditions, then the multipliers are given by explicit differential expressions in the dependent variables. (Received February 18, 1952.)

353. R. B. Davis: *A new boundary value problem for third-order partial differential equations of composite type.*

Let G be a simply-connected region, with boundary B, for which the classical Green's function (for Laplace's equation) exists and satisfies a certain inequality. Consider the equation (reduced to the canonical form established in a previous paper) (1) $(\Delta u)_s+au_{ss}+bu_{st}+cu_s+du_t+eu=f$, where $a, b, \cdots, f\in C^1$ on G. (The s-curves are the characteristics.) Let $B'\subset G$ extend from B to B, cutting every characteristic in G exactly once, and being tangent to none. Let $g\in C$ on B, and $h\in C^3$ on $G+B$. (There is one further smoothness hypothesis on B'.) The Fredholm alternatives are established for the problem: $u\in C^3$ on G (and C on $G+B$), u satisfies (1) in G, $u=g$ on B, $\Delta u=h$ on B' (i.e., either the problem stated has a unique solution, or else the homogeneous problem $f=g=h=0$ has nonzero solutions). Regions are studied for which the inequality on the Green's function is satisfied. The methods are those of E. E. Levi and the preceding paper of the author. (Received March 10, 1952.)

354. Avron Douglis: *An elementary approach to Cauchy's problem.* Preliminary report.

Any solution u of a linear hyperbolic equation $Lu=\sum_{i,j=1}^{n} a_{ij}(x)\partial^2u/\partial x_i\partial x_j+\sum_{j=1}^{n} b_i(x)\partial u/\partial x_i+c(x)u=f(x)$ in an even number $n\geqq 4$ of independent variables is characterized by an integral relation of Volterra type in terms of its Cauchy data on a space-like initial manifold I. Let C^P: $\phi(x)=0$ designate the characteristic conoid with vertex P determined by the characteristic equation $\sum_{i,j=1}^{n} a_{ij}(\partial\phi/\partial x_i)(\partial\phi/\partial x_j)=0$; let I^P be the intersection of C^P with I, and let A^P be the part of C^P intercepted between P and I^P. Then $u(P)$ can be represented, namely, as the sum of two integrals, the first involving Cauchy data of u and a finite number of their derivatives over the domain of integration I^P, the second involving the solution u over the domain of integration A^P. (Both integrals also contain the coefficients a_{ij}, b_i, c, f and a finite number of their derivatives.) This representation was obtained by a procedure, suggested by Beltrami's method of solving the Cauchy problem for the wave equation, which is based upon integration (in the ordinary sense) over A^P of expressions of the form $F(x)\cdot Lu$. The result is analogous to the integral representations obtained by the method of characteristics for the solutions of hyperbolic equations in two independent variables and bears, like these, on numerous questions relating to initial value problems. (Received April 1, 1952.)

355. Abolghassem Ghaffari: *The corresponding Riccati equations of the hodograph equations.*

The purpose of this paper is to give the corresponding Ricatti equations of the hodograph equations of the compressible subsonic flow (1) $P\psi_\zeta=\phi_\theta$, $Q\phi_\zeta=-\psi_\theta$ and to investigate their solutions. If one defines $t=\int_{\zeta}^{\zeta_s}(PQ)^{-1/2}d\zeta$ and sets $R=(PQ^{-1})^{1/2}$, then the equations (1) may be written in the simplified form (2) $R^{-1}\phi_t=\psi_\theta$, $R\psi_t=-\phi_\theta$. The definitions of ϕ, ψ, ζ, ζ_s, θ, and the expressions of P and Q are given in a previous paper (Bull. Amer. Math. Soc. vol. 58). The first order equations (1) now become (3) $\phi_{tt}+S\phi_t+\phi_{\theta\theta}=0$, $\psi_{tt}-S\psi_t+\psi_{\theta\theta}=0$, where $S=-R^{-1}R'$, where primes denote differentiation with respect to t. The advantage of working with the equations (2) or (3) is that *one has only to consider the one function R and its logarithmic derivative S as* *functions of t instead of dealing with the two unrelated functions P and Q.* Writing ϕ and ψ in the form $\phi=R^{1/2}e^{mt}F_m \cos(m\theta+\epsilon_m)$, $\psi=-R^{-1/2}e^{mt}G_m \sin(m\theta+\epsilon_m)$ and inserting these expressions in equations (3), one gets (4) $F_m''+2mF_m'-\lambda F_m=0$, $G_m''+2mG_m'+\mu G_m=0$, where $\lambda=(1/2)S'+(1/4)S^2$, $\mu=(1/2)S'-(1/4)S^2$. If one puts $F_m'/F_m=f_m$, $G_m'/G_m=g_m$, then *the following Riccati equations for f_m and g_m,* (5) $f_m'+f_m^2+2mf_m=\lambda$, $g_m'+g_m^2+2mg_m=-\mu$, *are obtained, It is shown that $S'\geqq 3S^2$, and also that f_m and g_m verify the conditions* $0<f_{m+1}<f_m<f_{m-1}$, $g_{m-1}<g_m<g_{m+1}<0$. (Received March 5, 1952.)

356. Samuel Goldberg: *The convergence of sequences determined by nonlinear difference equations.* Preliminary report.

Let two real numbers x_0 and x_1 be given and suppose that the terms of the sequence (x_n), $n=2, 3, \cdots$, are determined recursively by the second order nonlinear difference equation with constant coefficients: $x_{n+2}=ax_{n+1}+bx_n^2+cx_n+d$. The problem under consideration is the determination of those values of x_0 and x_1 for which the sequence (x_n) has a finite limit. The special case $a=0$, with the sequence (x_{2n}), is the problem of the iteration of the function bx^2+cx+d. A number of special cases are considered where, with suitable restrictions on the coefficients, the set of values of x_0 and x_1 are given for which $\lim x_n$ exists. The couple x_0, x_1 is said to initiate a finite path to the limiting value L if the terms of the corresponding sequence (x_n) are, from some point on, all equal to L. Conditions are given under which convergence of (x_n) can occur only in this special way. (Received March 11, 1952.)

357. J. W. Green: *On the spherical means of α-potentials.*

By an α-potential is meant in the present paper a potential of the sort investigated by M. Riesz, O. Frostman, and others, based upon the function $1/r^\alpha$, $1\leqq\alpha<3$, instead of the Newtonian $1/r$. Only positive mass distributions are considered. An α-potential, $u(P)$, is subharmonic outside the mass, but in general neither subharmonic nor superharmonic on it. Frostman has shown that it satisfies an inequality analogous to the super-mean property; namely, $A(u, P, r)\leqq A(\alpha)u(P)$, where $A(u, P, r)$ is the volume average of u over the sphere of radius r and center P, and $A(\alpha)$ is a number depending only on α, but greater than 1 for $\alpha>1$. In the present paper, the factor $A(\alpha)$ is studied in some detail, bounded above and below, and its asymptotic behavior near $\alpha=3$ found. Among other inequalities it is shown that $3(3/\alpha)^\alpha\{1/(3-\alpha)+2^{-1}\log(1-\alpha/3)\}\leqq A(\alpha)\leqq 3/(3-\alpha)$, and so $A(\alpha)\sim 3/(3-\alpha)$. The corresponding factor in the case of spherical surface means is determined exactly. (Received March 10, 1952.)

358. George Klein: *On a polynomial inequality of Zygmund.*

The principal result of this paper is that if $S_n(x)$ is a trigonometrical polynomial of order n, then $\|S_n'\|_p\leqq A_p n\|S_n\|_p$ for $0<p<1$, A_p being a constant depending only on p, and where $\|S_n\|_p=\{(1/2\pi)\int_0^{2\pi}|S_n(x)|^p dx\}^{1/p}$. This is an extension of the result for $p\geqq 1$, where $A_p=1$, due to Zygmund (Proc. London Math. Soc. (2) vol. 34 (1932) pp. 394–400) whose proof yields the theorem for $p>1/2$. Bernstein's well known inequality is the case $p=\infty$. The theorem for power polynomials corresponding to Bernstein's theorem is due to A. Markoff and can be obtained from the result of Hille, Szegö, and Tamarkin (Duke Math. J. vol. 3 (1937) pp. 729–739) that for $p\geqq 1$, $\|f_n'\|_p\leqq B_p n^2\|f_n\|_p$. Here f_n is a polynomial of degree n, B_p depends only on p, and $\|f_n\|_p=\{(1/2)\int_{-1}^{1}|f_n(x)|^p dx\}^{1/p}$. This result is also extended to the cases $p>0$. (Received March 11, 1952.)

359*t*. Walter Leighton: *On a function of Ramanujan.*

The author of this note presents an alternate proof that the function of Ramanujan $(1-x-x^4+x^7+x^{13}-\cdots)/(1-x^2-x^3+x^9+x^{11}-\cdots)$ has the unit circle as a natural boundary. (Received March 12, 1952.)

360. L. F. Markus: *Global structure of ordinary differential equations in the plane.*

Two real differential systems $\mathfrak{S}_i$: $\dot{x}=f_i(x, y)$, $\dot{y}=g_i(x, y)$, $i=1, 2$, with $f_i, g_i \in C^{(1)}$ and with isolated critical points $(f_i^2+g_i^2=0)$ in the plane E are o-equivalent in case there exists an orientation preserving homeomorphism of E onto itself which carries the solution curve family of $\mathfrak{S}_1$ onto that of $\mathfrak{S}_2$. The topological analysis of a differential system $\mathfrak{S}$ is effected by certain intrinsic curves known as separatrices. A separatrix S is essentially a solution curve of $\mathfrak{S}$ which cannot be embedded in a plane neighborhood bounded by exactly two solution curves of $\mathfrak{S}$ plus the limit sets of S, and which admits a transversal. The separatrices have two important properties: firstly, they decompose $\mathfrak{S}$ in E into canonical regions each of which admits a transversal and is o-equivalent to one of three elementary types, and secondly, they furnish a complete set of invariants for the o-equivalence classes of plane differential systems. The first property permits the extension of many local theories (e.g., existence of an integral) of differential equations to global ones. The second property unifies the classical function-theoretic descriptions of differential equations. (Received March 13, 1952.)

361*t*. M. H. Protter: *On the solution of hyperbolic equations by the method of finite differences.*

Consider the mixed initial-boundary value problem for the hyperbolic differential equation (1) $au_{xx}-u_{yy}+bu_x+cu_y+du=0$, $a=a(x, y)>0$. Courant, Friedrichs, and Lewy showed that the solution of the difference equation $au_{x\bar{x}}-u_{y\bar{y}}+bu_x+cu_y+du=0$ converged to the solution of (1) (in the case of the Cauchy problem) under a certain restrictive hypothesis on the ratio of the mesh sizes in the x and y directions. By different selections of the difference operator it is possible to prove convergence under less restrictive hypotheses on the ratio of the mesh sizes. Let h, k be the mesh sizes in the x and y directions, respectively. One of the operators that can be considered in this way is the von Neumann operator which replaces $u_{x\bar{x}}(x, y)$ by $\alpha u_{x\bar{x}}(x, y+k)+(1-2\alpha)u_{x\bar{x}}(x, y)+\alpha u_{x\bar{x}}(x, y-k)$ and which leaves $u_{y\bar{y}}$ unchanged. (Received March 7, 1952.)

362. J. H. Roberts: *A nonconvergent iterative process.*

In studying a certain heat transfer problem, Mann and Wolf were led to a consideration of the integral equation $y(t)=\int_0^t(G[y(x)]/[\pi(t-x)]^{1/2})dx$, where $G(y)$ is continuous and strictly decreasing and $G(1)=0$. On a fixed closed interval $[0, T]$ let $y_0(x)=0$, and let $y_1(x)$, $y_2(x)$, $\cdots$ be determined from the equation by the usual iterative process. Under the additional assumption that $|\Delta G/\Delta y|$ is bounded they proved that the sequence $y_0, y_1, y_2, \cdots$ converges uniformly to a solution. The present paper shows by a counter example that the assumption that G satisfies a Lipschitz condition is not superfluous. On the other hand the assumption is not necessary, since the desired result follows for all convex G. (Received April 25, 1952.)

363. David Rosen: *On the behavior of automorphic functions near the boundary.*

The Fuchsian groups $\Gamma(\lambda)$ generated by $z' = z+\lambda$, $z' = -1/z$ are considered, where $\lambda > 0$. It is known, when $\lambda < 2$, that $\Gamma(\lambda)$ is properly discontinuous if and only if $\lambda = 2 \cos \pi/q$, $q = \text{integer} \geq 3$; then $\Gamma(\lambda)$ is of the first kind. Otherwise, for arbitrary $\lambda > 2$, $\Gamma(\lambda)$ is of the second kind. Specialized continued fractions, called λ-fractions, are introduced whose first term is λr_0, $r_0 = \text{integer}$; the general term is $e_i/r_i\lambda$, where $e_i = \pm 1$, $r_i = \text{positive rational integer}$ $(i = 1, 2, \cdots)$. For $\lambda > 2$, every λ-fraction converges to a real number, whereas for $\lambda = 2 \cos \pi/q$, a sufficient condition for convergence is that $r_i + e_{i+1} < 1$ occurs for no more than $[(q-1)/2]-1$ consecutive values of i. It is shown that a substitution $z' = (az+b)/(cz+d) \in \Gamma(\lambda)$ if and only if a/c is a finite λ-fraction. Also, a/c is a parabolic point if and only if it is a finite λ-fraction where a, c are polynomials in λ. The set of limit points of $\Gamma(\lambda)$ is the real axis when $\Gamma(\lambda)$ is of the first kind; as expected, every real number has a λ-fraction representation whose elements are computed by a "nearest integer" algorithm explicitly given. When $\Gamma(\lambda)$ is of the second kind, $\lambda > 2$, the limit points form a perfect nowhere dense set; the real numbers on this set, and only those, have λ-fraction representations. On approach to a parabolic point in a Stolz angle, an automorphic function on $\Gamma(\lambda)$ has a unique limit. If the point is an infinite λ-fraction, an automorphic function does not admit a unique limit even on vertical approach. Similar results hold for automorphic forms. (Received March 10, 1952.)

364*t*. I. M. Sheffer: *On certain entire functions.*

An analytic function $f(z)$ will be said to have property $\mathcal{J}$ at a point z_1 if the sequence of derivatives $\{f^{(n)}(z_1)\}$, $n = 0, 1, \cdots$, takes on only a finite number of distinct values. Entire functions of form (1) $f(z) = Q(z) + \sum_{j=0}^{m-1} A_j \exp \{\omega^j z\}$, where Q is a polynomial and $\omega = \exp \{2\pi i/m\}$, have property $\mathcal{J}$ at all points. Some theorems are given in which a function $f(z)$, having property $\mathcal{J}$ at one point and satisfying supplementary conditions (e.g., having property $\mathcal{J}$ at a second point), is shown to be a function of form (1). These theorems link up with a result of Szegö on power series whose coefficients take on only a finite number of distinct values. (Received February 14, 1952.)

365*t*. Seymour Sherman: *On the roots of a transcendental equation.*

N. D. Hayes [*Roots of the transcendental equation associated with a certain difference-differential equation*, J. London Math. Soc. vol. 25 (1950) pp. 226–232] has given necessary and sufficient conditions on real a_1 and a_2 that the zeros of $\tau(s) = se^s - a_1e^s - a_2$ lie to the left of $R(s) = 0$. In a physical application where $a_1 < 0$ and $a_2 < 0$, D. F. Gunder and D. R. Friant [*Stability of flow in a rocket motor*, Journal of Applied Mechanics vol. 17 (1950) pp. 327–333] give an equivalent condition. It is the purpose of this note to illustrate the idea of H. I. Ansoff and J. A. Krumhansl, *A general stability for linear oscillating systems with constant time lag*, Quarterly of Applied Mathematics vol. 6 (1948) pp. 337–341, by applying it to the case where a_1 and a_2 may be complex. (Received February 11, 1952.)

366. Annette Sinclair: *The zeros of an analytic function of arbitrarily rapid growth.*

Suppose $S_1, S_2, \cdots$ is an infinite sequence of simply connected regions whose closures are nonintersecting and whose only sequential limit point is the point at infinity. In this paper it is proved that, for any preassigned sequence of positive constants $\{M_i\}$, there exists a nonvanishing integral function $f(s)$ such that $|f(s)| > M_i$ when $s \in S_i$. More generally, necessary and sufficient conditions are obtained on a "Q_E-set" S and a set E in $C(S)$ in order that to every function $M(s)$ which is component-wise bounded on S, there corresponds a function $f(s)$ analytic and nonvanishing in $C(E)$ and such that $|f(s)| > |M(s)|$ when $s \in S$. (A set S is a "Q_E-set" if (1) its components are closed regions, and (2) its sequential limit points lie in $C(S)$. Actually, the theorem is proved with a weaker condition substituted for (1).) (Received March 14, 1952.)

367*t*. D. B. Sumner: *A convolution transform admitting an inversion formula of integro-differential type.*

The inversion operator $E(D)$ for the convolution transform $f(x) = \int_{-\infty}^{\infty} G(x-t)\phi(t)dt$ is given by $[E(s)]^{-1} = \int_{-\infty}^{\infty} \exp(-st)G(t)dt$. It is shown by means of the example $G(x) = [\exp(-2x) + 2 \cos \pi\beta \cdot \exp(-x) + 1]^{-1}$, $0 < \beta < 1$, that cases where $E(s)$ is meromorphic arise from kernels little more complicated than that of Stieltjes' transform, and that the order in which the differentiating factor and the integrative factor of the inversion operator are applied is material. The two factors $\sin \pi D/\pi$ and $\sin \pi\beta/\sin \pi\beta(1-D)$ of the invertor are represented as integrals, and the methods of the complex variable are employed. (Received January 25, 1952.)

368. Albert Wilansky: *The inset of a summability matrix.*

Let A be a summability matrix, a_k the column limit $\lim_n a_{nk}$. The inset $I(A)$ is the set of sequences summable A such that $\sum a_k x_k$ converges. If $I(A) = (A)$ (the field of A), say A has maximal inset. If every B with $(B) = (A)$ has maximal inset, say A has PMI (propogation of maximal inset). The regular matrix ($a_{nn} = a_{n,n-1} = 1/2$, $a_{nk} = 0$ otherwise) does not have PMI. The Cesaro and Riesz matrices have PMI. All these matrices are of type M and their fields cannot be distinguished by previously known invariants. A co-regular matrix with maximal inset has an equipotent regular matrix. There is a co-regular normal matrix such that no equipotent matrix has maximal inset. Hence there is no equipotent regular matrix. (Example is due to K. Zeller). The regular normal matrix $a_{nk} = s_n t_k$ has PMI. (Received March 7, 1952.)

369*t*. František Wolf: *Spectral decomposition of a class of operators in Banach space.* Preliminary report.

The spectral decomposition concerned is of a generalized type $f(A) = \int_{\sigma(A)} f(\lambda) d^n E(\lambda)$ where (i) $f(\lambda)$ is of class C^n in an open set containing $\sigma(A)$, (ii) the integral is essentially Bochner's (Bochner, *Fouriersche integrale*, 1932, also F. Wolf, *(C, k) summability of trigonometric integrals*, University of California Press, 1946 and L. Schwartz, *Théorie des distributions*). The class of operators with real bounded spectrum can be characterized in different ways: (i) the resolvent $R_\lambda = O((\lambda - \bar{\lambda})^{-n+1})$, (ii) the polynomials $f(A)$ as functions of f prove to be continuous under the topology of C^n (cf. L. Schwartz). An equivalent class can be characterized by $\|A^k\| \leq Mk^{n-1}$, $k = 0, 1, -1, \cdots$. This is an extension of E. R. Lorch's class of "weakly almost periodic" operators. It includes nilpotent operators of order not more than $n-1$. The generalized "decomposition of the identity" $E(\lambda)$ is outside of $\sigma(A)$ a polynomial of order $n-1$ and $\int f(\lambda) d^n E(\lambda) \cdot \int g(\lambda) d^n E(\lambda) = \int f(\lambda) g(\lambda) d^n E(\lambda)$ which yields a corresponding orthogonality property for $E(\lambda)$. An eigenspace S_{λ_0}, defined as the annihilator of $g(A)$ where $\lambda \neq \lambda_0 \rightarrow g(\lambda) \neq 0$ and $g(\lambda_0) = g'(\lambda_0) = \cdots = g^{(n-1)}(\lambda_0) = 0$, is such that $A - \lambda_0 I$ is nilpotent in S_{λ_0}. If $h'(\lambda_0) = \cdots = h^{(n-1)}(\lambda_0) = 0$, then $h(A)$ is a scalar in S_{λ_0}. There exists an expanding family of manifolds $M(\lambda)$ such that $\sigma(A \mid M(\lambda_0))$ is exactly $\sigma(A) \cap \{\lambda | \lambda \leq \lambda_0\}$ and $M(-\infty) = (0)$, $M(\infty)$ is the whole space. (Received March 13, 1952.)

APPLIED MATHEMATICS

370t. H. G. Bergmann: *The boundary layer problem for certain nonlinear ordinary differential equations.*

This paper considers a problem similar, but simpler in type, to that arising in the theory of buckling of circular plates. One has a pair of nonlinear ordinary differential equations of second order depending upon a parameter, (i) $p_{xx}(x) = q^2(x)/2$, (ii) $kq_{xx}(x) + p(x)q(x) = 0$, $-1 \leq x \leq 1$, with the associated boundary conditions $p(-1) = p_1$, $p(1) = p_2$, $q_x(-1) = q_x(1) = 0$. When the parameter k approaches zero as a limit, the solutions p and q approach limit functions whose existence and uniqueness we prove by variational methods. The nonuniform convergence in the "boundary layer" is studied by an appropriate stretching process. The limit functions satisfy the limit equations, but no longer satisfy all the boundary conditions. Since q_{xx} disappears in the limit equation, it is reasonable that the boundary condition on q should no longer be met; the interesting result is that although the limit equation remains of the same order in p, the boundary condition here may be lost. The paper proves that if p_1 is negative, the limit function does indeed assume this value; but if p_1 is positive, the limit function assumes instead the value $-.47271p_1$. (Received February 13, 1952.)

371. Jack Kotik: *Linear water waves and the equation of the vibrating beam.* Preliminary report.

We consider two-dimensional surface waves on an ocean of infinite depth. The existence problem has been proposed for the following data: given the surface elevation and slope at $x = 0$ for all values of the time, $-\infty < t < \infty$. It is shown (with inessential rigor restrictions) that this problem is impossible unless the prescribed functions are analytic, and (with essential hypotheses) that if the solution exists it is unique, as follows: in deep water the surface condition can be differentiated to yield $\phi_{tttt} + \phi_{xx} = 0$, which is formally the equation for the transverse vibrations of a beam. An integral representation is found for the solution from which the analyticity in t and uniqueness follow. The main difficulty is that the source solutions do not die out at infinity but only oscillate with increasing rapidity. (Received March 11, 1952.)

372. K. S. Miller and F. J. Murray: *Error analysis for differential analyzers.*

The machine solution of a system of differential equations differs from the true solution by satisfying somewhat different differential equations and by sporadic perturbations β during the solution process. The machine differential equations are considered to involve parameters α and λ, such that setting α and λ zero yields the correct equations. λ errors raise the order of the system, α errors do not. For no λ errors, the machine solutions, $u_i(x, \alpha, \beta)$, of practical systems (including nonlinear systems, with broken region of analyticity) depend analytically on α and β. The α and β partial derivatives for each order satisfy a linear system of differential equations, with the same homogeneous part. Thus the Taylor series for u_i yields the solution error accurately without "linearizing" the system. This error analysis for α and β applies to both continuous and digital machines. When λ errors are present, the solution error is analyzed into a long range part, analytic in λ, and a transient part not analytic in λ. The methods thus indicated are combined to yield a discussion of the joint effect of all error types. (Received March 7, 1952.)

373. K. M. Siegel, J. W. Crispin, and R. E. Kleinman: *The zeros of the associated Legendre function of order one and nonintegral degree.*

The consideration of certain problems related to the scattering of electromagnetic waves involves the determination of those values of the degree for which the associated Legendre function with a fixed real argument between -1 and 0 will vanish. This paper presents a method which will yield numerical results more easily than any previously reported method. This method involves expanding $P'_\nu(x_0)$ in a Taylor series about integer and half-integer values of ν, terminating the series after three terms, and equating the expansion to zero. The solution of the resulting algebraic equation leads to the zeros, which are the values of ν that are desired. Although the coefficients in this expansion are themselves infinite series, they are rapidly convergent, and thus can be readily evaluated. The method has been applied to the cases $x_0 = \cos 165°$ and $x_0 = \cos 170°$. In addition, the method provides a means of obtaining quantitative results for the normalizing factors $\int_{x_0}^1 [P_{\nu_i}(x)]^2 dx$ and $\int_{x_0}^1 [P'_{\nu_i}(x)]^2 dx$ with the ν_i such that $P'_{\nu_i}(x_0) = 0$. (Received March 5, 1952.)

374*t*. R. L. Sternberg and Hyman Kaufman: *A general solution of the two-frequency modulation product problem.* I.

Consider an arbitrary modulator whose output versus input characteristic $Y = Y(X)$ is a continuous function of X on a finite closed interval $[-a, a]$. Let the input be $x(t) = P \cos (pt + \theta_p) + Q \cos (qt + \theta_q)$, $0 < P < P + Q \leq 2P \leq a$. Then by extending the method of W. R. Bennett (*The biased ideal rectifier*, Bell System Technical Journal vol. 26 (1947) pp. 139–169) it is shown that the double Fourier series coefficients of the output $y(t) = Y(x(t))$ can be approximated to within an arbitrary 2ϵ by simple linear combinations of the values of four new functions which were introduced by Bennett and are independent of the modulator characteristic, while the double Fourier series having these approximate values as coefficients converges and differs from the output $y(t)$ by less than ϵ for all t. Tables of the four new functions mentioned are in preparation for Part II of the present paper. (Received February 29, 1952.)

375. H. F. Weinberger: *Upper and lower bounds for torsional rigidity.*

By an extension to multiply-connected domains of a minimum principle of J. B. Diaz (Proceedings of the Symposium on Spectral Theory and Differential Problems, Oklahoma A. and M., 1951, p. 289), the following is shown. If a region with torsional rigidity S be split into disjoint subregions with rigidities $S_1, \cdots, S_n$, then $S \geq S_1 + \cdots + S_n$. This is a lower bound for S or, by transposing, an upper bound for S_1. Of particular interest is the region S_1 consisting of the simply-connected region S with holes $S_2, \cdots, S_n$. For the hollow square with sides $2b$ and $2a$, the upper bound $2.2496\,(b^4 - a^4)$ is so obtained. Good lower bounds for certain regions with holes are obtained as follows. If S_1 consists of the simply-connected region S with a hole S_2 and if the torsion function of S has a level line entirely outside S_2 but inside a larger region S'_2, then $S_1 \geq S - S'_2$. For a regular polygon with similar hole, a circle circumscribed about the hole serves as S'_2. In particular, for the hollow square $S \geq 2.2496b^4 - 2\pi a^4$. For small holes, the upper and lower bounds are very near to each other. (Received March 12, 1952.)

376. D. M. Young and M. L. Juncosa: *On the convergence of solutions of difference equations to solutions of the heat equation.*

Let $u(x, t)=\sum_{n=1}^{\infty} a_n \sin n\pi x \exp(-n^2\pi^2 t)$ be a Fourier series solution of the heat equation $u_{xx}=u_t$ in R: $0<x<1$, $t>0$, with boundary conditions $u(+0, t)=u(1-0, t)=0$, $t>0$, and initial condition $u(x, +0)=f(x)$, $0<x<1$. For points (x, t) of R such that Mx and M^2t/r^2 are integers, M an integer, let $U_M(x, t)$ satisfy the difference equation $U_M(x, t+k)-U_M(x, t)=r\{U_M(x+h, t)+U_M(x-h, t)-2U_M(x, t)\}$, where $h=1/M$, $k=rh^2$. Let $\overline{U}_M(x, t)$ be obtained from $U_M(x, t)$ by bilinear interpolation. By the use of a generalization of Fejér's theorem on summability $(C, 1)$ of Fourier series (D. Jackson, *The theory of approximation*, p. 66) the authors prove that if $f(x)$ is continuous in $0\leqq x\leqq 1$ except for a finite number of finite jumps, if the Fourier series for $f(x)$ converges, and if $0<r\leqq 1/2$, then as $M\rightarrow\infty$, $\lim \overline{U}_M(x, t)=u(x, t)$, uniformly for $t\geqq t_0>0$ and for $0\leqq x\leqq 1$. This generalizes previous results; for instance Leutert (Journal of Mathematics and Physics vol. 30 (1952) pp. 245–251) has proved convergence for $0<r\leqq 1/4$. On the other hand, Hildebrand, Bull. Amer. Math. Soc. Abstract 58-2-216, proved convergence for $0<r\leqq 1/2$, assuming bounded variation for $f(x)$. (Received March 12, 1952.)

GEOMETRY

377*t*. T. Minagawa and Tibor Radó: *Infinitesimal rigidity of nonconvex surfaces.*

The theorem stated in a previous communication (T. Radó, *Rigidity theorems for non-convex surfaces*, Bull. Amer. Math. Soc. Abstract 58-4-378) remains valid if the asymptotic condition relative to the boundary curve is dropped. At the same time, differentiability assumptions can be reduced to the point where the results appear in their natural setting. (Received February 11, 1952.)

378*t*. Tibor Radó: *Rigidity theorems for nonconvex surfaces.*

Let R be a simply-connected piece of surface with boundary C. Then R will be termed i.r.b. (infinitesimally rigid for point-wise fixed boundary) if R does not admit of nonidentical infinitesimal deformations leaving the boundary fixed point-wise. Now let S be a simply connected surface satisfying one of the following three conditions. (1) The Gauss curvature K of S is positive. (2) $K<0$, and the asymptotic lines can be taken as parameter curves for S in the large. (3) $K=0$, and the generators and their orthogonal trajectories can be taken as parameter lines for S in the large, and there are no "flat points" present. *Theorem.* If R is a simply connected piece of S whose boundary C has aysmptotic direction at a finite number of points at most, then R is i.r.b. *Remarks.* (a) Case (1) is, essentially, known, and is stated merely for completeness. (b) Analyticity is not assumed. (c) The result has a bearing on the problem of flexible, inextensible membranes. (Received January 28, 1952.)

LOGIC AND FOUNDATIONS

379. Ilse L. Novak: *On direct unions of algebras.* Preliminary report.

The terminology used here is taken from J. C. C. McKinsey, J. Symbolic Logic vol. 8 (1943) pp. 61–76. Given an algebra Γ which is known to be a direct union of algebras Γ_i, where i ranges over a given index set I, and a sentence S which holds in Γ. One wishes to determine what conditions this imposes on the factors of Γ. For certain classes of sentences it is possible to obtain equivalent conditions which hold for all index sets I. If all the factors are isomorphic, more general results can be obtained. If I is finite, one can always find equivalent conditions. (Received March 13, 1952.)

STATISTICS AND PROBABILITY

380*t*. J. C. Kiefer and Jacob Wolfowitz: *Stochastic estimation of the maximum of a regression function.*

Let $F(y|x)$ be a family of distribution functions which depend on a real parameter x, let $\int y dF(y|x) = M(x)$, and suppose that $\int (y-M(x))^2 dF(y|x) \leqq S < \infty$. $M(x)$ is strictly increasing for $x < \theta$ and strictly decreasing for $x > \theta$. Let $\{a_n\}$ and $\{c_n\}$ be sequences of positive numbers such that $c_n \to 0$, $\sum a_n = \infty$, $\sum a_n c_n < \infty$, $\sum a_n^2/c_n^2 < \infty$. Let x_1 be an arbitrary number and define, for all n, $x_{n+1} = x_n + a_n((y_{2n} - y_{2n-1})/c_n)$. Here y_{2n-1} and y_{2n} are independent chance variables with respective distributions $F(y|x_n - c_n)$ and $F(y|x_n + c_n)$. Under weak regularity conditions on $M(x)$ the authors prove that x_n converges stochastically to θ. (Received February 14, 1952.)

381. Eugene Lukacs and Otto Szász: *Analytic characteristic functions.*

We give here a proof of the following theorem: If a characteristic function $\phi(s)$ is analytic in a neighborhood of the origin, then it is also analytic in a horizontal strip and can be represented in this strip by a Fourier integral. This strip is either the whole plane or has one or two horizontal boundary lines. The purely imaginary points on the boundary of the region of convergence (if this region is not the whole plane) are singular points of $\phi(s)$. A number of properties of analytic characteristic functions follow from this theorem: (1) In either half-plane, the singularity nearest to the real axis is located on the imaginary axis. (2) There are no zeros on the segment of the imaginary axis located inside the strip of analyticity. (3) The zeros and the singular points of $\phi(s)$ are located symmetrically with respect to the imaginary axis. (4) If the characteristic function of an infinitely divisible law is analytic, then it has no zeros inside its strip of convergence. (Received March 5, 1952.)

382*t*. Jacob Wolfowitz: *On the stochastic approximation method of Robbins and Monro.*

Let $H(y|x)$ be a family of distribution functions which depend on the parameter x. Let $M(x) = \int y dH(y|x)$, and suppose $|M(x)| < C < \infty$ and $\int y^2 dH(y|x) < S < \infty$. Let $\{a_n\}$ be a sequence of positive numbers such that $\sum a_n = \infty$, $\sum a_n^2 < \infty$. Suppose that either (I) $M(x) \leqq \alpha - \delta$ for $x < \theta$, $M(x) \geqq \alpha + \delta$ for $x > \theta$ or (II) all the following hold: (1) $M(x) < \alpha$ for $x < \theta$, $M(\theta) = \alpha$, $M(x) > \alpha$ for $x > \theta$, (2) for some positive δ, $M(x)$ is strictly increasing if $|x-\theta| < \delta$ and (3) $\inf_{|x-\theta| \geqq \delta} |M(x) - \alpha| > 0$ Let x_1 be arbitrary and define recursively $x_{n+1} = x_n + a_n(\alpha - y_n)$, where y_n is a chance variable with the distribution $H(y|x_n)$. Then x_n converges stochastically to θ. This generalizes results of Robbins and Monro, Ann. Math. Statist. vol. 22 (1951) pp. 400–407. (Received February 25, 1952.)

TOPOLOGY

383. R. D. Anderson: *On extending the domain of a monotone interior mapping.*

The author shows that if M, imbedded in S^n, is a closed set of dimension not greater than $n-2$ and f is a monotone interior mapping of M onto a space Y, then there exists a monotone interior mapping F of S^n onto a space in which Y is imbedded such that $F(S^n-M)$ is homeomorphic to S^n-M and for any point y in Y, $F^{-1}(y)$ is $f^{-1}(y)$. (Received March 12, 1952.)

384*t*. H. W. Becker: *Nonplanar graphs and passive circuits*. Preliminary report.

A passive circuit is a subgraph with two terminal-nodes. A graph which can be mapped on a sphere can be mapped on a plane, but this is not necessarily true of p.c., hence the distinction between planar, global, and nonglobal p.c. Kuratowski's theorem, that all nonplanar graphs contain one of two elementary n.g., the triple wye, or pentagon-pentacle, has this analogy: all nonplanar p.c. contain one of three el. n.p.c., the X (Bloch), Δ (Carvallo), or Y (el. sesquispheric) bridges. In any bridge, all terminal and unambiguous boundary branches are nonbridgers (have no negative terms in the transfer conductance). But in a nonplanar bridge, there is no unique pair of boundary meshes. Consequently n.p.c., like n.g., have no topological duals—but have algebraic duals with square first partial derivatives, like physical p.c., Proceedings of the American Mathematical Society vol. 1 (1950) p. 316. For $9\leqq n\leqq 13$, the fundamental n.g. $(a)^*_n=1, 2, 3, 10, 29$ (a majority of a^*_{13}), and the n.g. $(a)_n=1, 4, 21, 130, 828$. For $8\leqq n\leqq 12$, the fundamental n.p.c. $(b)^*_n=1, 4, 12, 48, 200$ (a majority of b^*_{12}), and the n.p.c. $(b)_n=1, 10, 90, 766, 6251$. So far, $(b)_n\sim 1.5\cdot 8^{n-8}$. These enumerations would all be doubled, admitting algebraic duals. The first graph and bridge-bridges such that neither they nor their duals are physical are $\langle a\rangle^{**}_{14}=1$, $\langle b\rangle^{**}_{12}=4$. Reference is made to classic papers of Hassler Whitney. (Received March 21, 1952.)

385*t*. H. W. Becker: *The enumeration of series-parallel-bridge graphs*. Preliminary report.

R. M. Foster classified s.p.b. graphs by rank and nullity (Transactions of the American Institute of Electrical Engineers, 1932, p. 309) and formulated the atlas of s.p. graphs C_n (International Congress of Mathematicians, 1950). Enumeration of all nonseparable n-branch s.p.b. graphs a_n starts with the fundamental graphs a^*_n, called pure active bridges, Bull. Amer. Math. Soc. vol. 54 (1948) p. 76, and proceeds by the methods there applied to passive s.p.b. circuits b_n. Topological binomial coefficients and products reduce to ordinary ones only for networks of maximum assymmetry, where the branch diversity equals the number of branches; there are no such graphs, $n<12$. Thus the topological power $2^t=5, 6, 7, 8, 9, 10, 12$, or 16 according to the symmetries involved. For $10\leqq n\leqq 13$, the fundamental bridge-bridge graphs $a^{**}_n=1, 2, 3, 11$; for $6\leqq n\leqq 13$, $a^*_n=1, 0, 1, 3, 4, 7, 22, 51$, whence $a_n=1, 2, 11, 47, 197, 837, 3689, 16456>2C_{13}$. With the aid of a table of $s_{n,d}$, s.p. circuits classified by diversity, the analogous $a_{n,d}$ is tabulated. This achieves the cross-check $b_{n-1}=\sum d\cdot a_{n,d}$. For $5\leqq n\leqq 12$, $b_n=1, 6, 37, 195, 1003, 5034, 25231, 127217$. Nonplanarities few, $b_n\sim 1.6\ 5^{n-4}$, additional light on the Riordan-Shannon problem. Journal of Mathematics and Physics vol. 21 (1942) p. 83. The previous procedure started from $b^*_n=1, 0, 2, 5, 12, 34, 113, 388$, $5\leqq n\leqq 12$. The two different procedures enforce a mutual correction and corroboration, in a census exceedingly prone to error. (Received March 21, 1952.)

386. A. L. Blakers: *On the realization of homotopy sequences*.

Let $\mathfrak{S}=(\cdots\to A_n\to B_n\to C_n\to A_{n-1}\to\cdots\to A_1\to B_1\to C_1\to 1)$ be a sequence of sets and single-valued functions, subject to the conditions: (1) All sets are groups, except possibly C_1, which is a set with a unit element. (2) All groups to the left of C_2 are abelian. The groups C_2, A_1, B_1 need not be abelian. (3) All functions except possibly $j_1: B_1 \to C_1$ are homomorphisms. (4) The sequence is exact. In addition if $i_1: A_1 \to B_1$, $b_1, b_1' \in B_1$, and $j_1(b_1) = j_1(b_1')$, then $b_1' b_1^{-1} \in i_1(A_1)$. (5) A_1 acts as a group of operators on all groups of $\mathfrak{S}$ and the homomorphisms of $\mathfrak{S}$ are operator homomorphisms. A_1 acts on itself by inner automorphisms, and if $h_2: C_2 \to A_1$, $c_2, c_2' \in C_2$, then $(h_2(c_2))c_2' = c_2 c_2' c_2^{-1}$. (6) B_1 acts as a group of operators on all groups B_n, $n \geqq 1$; on itself by inner automorphism. (7) The operators from A_1, B_1, on B_n, $n \geqq 1$, are related by $a_1(b_n) = (i_1(a_1))b_n$ $(b_n \in B_n,\ a_1 \in A_1)$. The algebraic properties of $\mathfrak{S}$ are all shared by the homotopy sequence of any 0-connected pair. The main theorem proved about $\mathfrak{S}$ is that it can be realized as the homotopy sequence of a 0-connected C. W. complex-pair. More specifically, there is such a pair (K, L) and an operator isomorphism of the homotopy sequence of (K, L) onto $\mathfrak{S}$. Moreover, all Whitehead products in (K, L) are trivial. (Received March 6, 1952.)

387. M. L. Curtis: *The Poincaré-Lefschetz duality theorem for closed Brouwer manifolds.*

Let S be a closed connected n-manifold (not assumed to be triangulable) and G, I be, respectively, an arbitrary abelian group and the group of integers. Let $\mathcal{H}$, H', H be a cohomology theory and two homology theories with coefficient groups G, I, G. Assume there exists a cap product $\cap$ pairing $\mathcal{H}$ and H' to H. If $H_n'(S)$ is isomorphic with I and z_n is a generator of $H_n'(S)$, let $\phi_r: \mathcal{H}^r(S) \to H_{n-r}(S)$ be defined by $\phi_r(u) = u \cap z_n$. This paper is devoted to showing that under these circumstances each ϕ_r $(r = 0, 1, \cdots, n)$ is an isomorphism onto. (Received March 12, 1952.)

388t. S. P. Diliberto: *Existence of cubical maps.* Preliminary report.

Let K and L be finite Euclidean polyhedra which are subdivided into cubes, and let $f: |K| \to |L|$ be a continuous map. It is shown that there exist cubical maps f_ϵ approximating f (paralleling the existence of simplicial maps when K and L are cut into simplices): for $\epsilon > 0$, there exist subdivisions $\bar{K}$ and $\bar{L}$ and a continuous map $f_\epsilon: |\bar{K}| \to |\bar{L}|$ such that (i) dist $(f, f_\epsilon) < \epsilon$, (ii) f_ϵ is ϵ-homotopic to f, (iii) if $|A^p|$ is a p-cube of $\bar{K}$, there exists a q-cube $(q \leqq p)$ $|B^q|$ of $\bar{L}$ such that $f(|A^p|) = |B^q|$, (iv) f_ϵ induces a natural map $C_{f_\epsilon}: C_p(\bar{K}) \to C_p(\bar{L})$ where $\partial C_{f_\epsilon} = C_{f_\epsilon}\partial$. By standard methods one obtains subdivisions $\tilde{K}$ and $\tilde{L}$ and a *vertex* map f_ϵ satisfying the nearness properties (i), (ii) and such that if a_i are vertices of $|A^p| \in \tilde{K}$, there exists $|B^q| \in \tilde{L}$ such that $f(a_i) \in |B^q|$ (q possibly greater than p). $\bar{K}$ is the $2m \cdot n$th subdivision of $\tilde{K}$, $m = \dim K$ and $n = \dim L$. f_ϵ is extended to $\bar{K}$ so as to have properties (iii) and (iv). (Received March 17, 1952.)

389. Mary E. Estill: *A primitive dispersion set of the plane.*

R. L. Wilder, in his book, *Topology of manifolds*, makes the following definitions. "If M is a connected set and D a subset of M such that $M - D$ is totally disconnected, then D may be called a dispersion set of M. If no proper subset of D is a dispersion set of M, then let us call D a primitive dispersion set of M." And Wilder raises the question of the existence of a primitive dispersion set of the plane. In this paper an example is given of a primitive dispersion set of the plane. (Received January 28, 1952.)

390t. R. H. Kyle: *Calculation of invariants of a quadratic form.*

It is convenient to have methods of calculating the invariants of a quadratic form which do not involve a preliminary reduction to canonical form. Let f be a quadratic form with integer coefficients, and p be an odd prime. Let the elementary divisors of the matrix of f be $r_i = r_i' p^{d_i}$, with $d_i = e_k$ when $r_{k-1} < i \leq r_k$, and $e_k < e_{k+1}$. Put $s_l = \sum_1^l e_k(r_k - r_{k-1})$. Let $\{f^{(r)}\}$ denote the collection of subforms of f obtained by dropping all but r variables of f. Then there exists at least one subform g in $\{f^{(r_k)}\}$ such that $|g| p^{-s_k}$ is an integer prime to p. Moreover if g, h are two such subforms, the Legendre symbols $(|g| p^{-s_k}/p)$, $(|h| p^{-s_k}/p)$ are equal and define a unit $\epsilon_k(p)$. The $\epsilon_k(p)$ are invariants of the class of f, and are calculable without reduction to canonical form. The r_k, e_k, $\epsilon_k(p)$ provide a complete set of local invariants of f at odd primes. The invariants $(|f_k|/p)$ defined in B. W. Jones, *The arithmetic theory of quadratic forms*, are given by $(|f_k|/p) = \epsilon_k(p)\epsilon_{k-1}(p)$ and the Minkowski units $c_p(f)$ in turn are given by $c_p(f) = (-1/p)^{[q/2]} \cdot \prod'(|f_k|/p)$ where $q = \sum'(r_k - r_{k-1})$; $\sum', \prod'$ being taken over those k such that e_k is odd. (Received March 12, 1952.)

391*t*. R. H. Kyle: *Embeddings of the Moebius band in S^3*.

A rectangle $ABCD$ is embedded in S^3 with AD identified with CB. The boundary $ABCD$ of the Moebius band thus constructed forms a knot l, and the line joining the midpoints of AD and BC forms a knot k called the center knot of the band. The linking number $\nu(k, l)$ gives the number of twists. It is shown that given an oriented knot k and an odd integer ν, an oriented knot l can be constructed in a well-defined way, so that l is the boundary of a Moebius band having k for its center knot and ν twists. Conversely if two Moebius bands M, M' have boundaries which are equivalent to each other but not to a circle, then there exists a semilinear mapping of S^3 onto itself sending M' onto M. Thus distinct pairs (k, ν) define distinct knots except in the special case (circle, ± 1), and the invariants of k are also invariants of l. This gives a complete classification of a subclass of the parallel knots, analogous to H. Seifert's classification of doubled knots in *Schlingknoten*, Math. Zeit. vol. 52 (1949). (Received March 12, 1952.)

392. R. H. Kyle: *The quadratic form of a knot.*

The quadratic form of a projection of a knot was defined by L. Goeritz, *Knoten und quadratische Formen*, Math. Zeit. vol. 36 (1932). It was shown that the Minkowski units $c_p(f)$ of this form are invariant under combinatorial transformations, but the question of topological invariance remained unsettled. H. Seifert, in *Der Verschlingungsinvarianten der zyklischen Knotenüberlagerungen*, Abh. Math. Sem. Hamburgischen Univ. vol. 11 (1935), exhibited the connection between the quadratic form and the two-sheeted covering of the knot. He gave an example of two knots which can be distinguished by the linking invariants of this covering but not by the Minkowski units, and he asked whether an example of the opposite case could be found. A negative answer to Seifert's question is given by expressing $c_p(f)$ as a product of certain of the linking invariants The topological invariance of the Minkowski units is an immediate consequence of the topological invariance of the linking numbers. (Received March 12, 1952.)

393*t*. J. C. Moore: *A new proof of the Blakers and Massey triad theorem.*

The theorem states that if $(X; A, B)$ is a triad such that $X = A \cup B$, A and B are open in X, A, B, and their intersection are arcwise connected, (X, A) is m-connected, $m \geq 2$, (X, B) is n-connected, $n \geq 2$, then the triad $(X; A, B)$ is $(m+n)$-connected (Bull.

Amer. Math. Soc. Abstract 57-2-164). Let $x_0 \in A \cap B$. Let D be the space of paths in X which start at x_0 and end in A; let E be the space of paths in X which start at x_0 and end in B, and let $Y = D \cup E$. Then the triads $(X; A, B)$ and $(Y; D, E)$ have isomorphic triad homotopy groups, and Y has trivial homology groups. Therefore it is sufficient to prove the theorem in the special case where X has trivial homology. In this case A is $(m-1)$-connected, and B is $(n-1)$-connected. Let $C = A \cap B$, and let Δ be the diagonal of $A \times B$. Δ is homeomorphic with C. $H_q(A \times B, \Delta) = \{0\}$ for $q \leqq m+n-1$ by the Künneth theorem. Therefore $\pi_q(A \times B, \Delta) = \{0\}$ for $q \leqq m+n-1$. These facts imply $\pi_q(X; A, B) = \{0\}$, $q \leqq m+n$, by standard methods. (Received January 28, 1952.)

394t. J. C. Moore: *On the homotopy groups of spheres.*

Using the concept of generalized covering space of G. W. Whitehead (*Fibre spaces and the Eilenberg homology groups*, Bull. Amer. Math. Soc. Abstract 58-2-267) it is proved that for p a prime the p-primary component of $\pi_{2p}(S^3)$ is a cyclic group of order p, and that $\pi_q(S^3)$ has no elements of order p for $2p < q < 4p-3$. Let Z_p denote a cyclic group of order p. Then using the relative Hurewicz theorem and the results of J. P. Serre (Ann. of Math. vol. 54 (1951) pp. 425–505) it is proved that if $E^2: \pi_q(S^n) \to \pi_{q+2}(S^{n+2})$ is the double suspension homomorphism, then for n odd the induced homomorphism $E^2_p: \pi_q(S^n) \otimes Z_p \to \pi_{q+2}(S^{n+2}) \otimes Z_p$ is an isomorphism for $q < (n+1)p-3$. Combining these results with those of Serre, it is proved that for n odd $\pi_q(S^n) \otimes Z_p = \{0\}$ for $n+2p-3 < q < n+4p-6$, and the p-primary component of $\pi_{n+4p-5}(S^n)$ is a cyclic group of order p. It is also proved that for n even and $p \neq 2$, $\pi_q(S^n) \otimes Z_p = \{0\}$ if $q \neq 2n-1$ and $n+2p-3 < q < \min \{2n+2p-4, n+4p-6\}$, and the p-primary component of $\pi_{n+4p-5}(S^n)$ is a cyclic group of order p. (Received February 7, 1952.)

395t. J. C. Moore: *On the homotopy of the union of two spheres with a point in common.*

Let $x_0 \in S^n$. Then $S^n \vee S^n$ is defined to be the subset $S^n \times \{x_0\} \cup \{x_0\} \times S^n$ of $S^n \times S^n$. Let X be the space of paths in $S^n \times S^n$ which start at (x_0, x_0), and end in $S^n \vee S^n$. $H_q(X)$ is a free abelian group with k generators for $q = (k+1)n - k$, $k \geqq 1$, and trivial for other values of $q > 0$. Then it is proved that $\pi_q(X) = \pi_q(S^{2n-1}) \cdot \pi_q(S^{3n-2} \times S^{3n-2}) \otimes \pi_q(S^{4n-3} \times S^{4n-3} \times S^{4n-3})$ for $q \leqq 5n-6$. This is used to extend some results of G. W. Whitehead on the distributive law for the composition operation (Ann. of Math. vol. 51 (1950) p. 215). (Received February 1, 1952.)

396t. J. C. Moore: *On the relative Hurewicz theorem.*

The formulation of the Hurewicz theorem of J. P. Serre (Ann. of Math. vol. 54 (1951) p. 491) is extended to the relative case. The theorem then reads that if (X, A) is a pair such that X and A are connected, simply connected, uniformly locally connected spaces whose homology groups are finitely generated, k is a field, and $H_q(X, A; k) = \{0\}$ for $0 \leqq q < n$, then $\pi_q(X, A) \otimes k = \{0\}$ for $0 \leqq q < n$, and $\pi_n(X, A) \otimes k$ is isomorphic to $H_n(X, A; k)$. The theorem is then applied to prove that if $E: \pi_q(S^n) \to \pi_{q+1}(S^{n+1})$ is the suspension homomorphism for n odd, and if $\alpha \in \pi_q(S^n)$ is an element such that $E(\alpha) = 0$, then α has order a power of 2. The Blakers and Massey triad theorem is extended to read that if $(X; A, B)$ is a triad such that X, A, B, and $A \cap B$ are connected, simply connected, uniformly locally connected spaces with finitely generated homology groups, (X, A) and (X, B) are 2-connected, k is a field, $H_q(X, A; k) = \{0\}$ for $q \leqq m$, and $H_q(X, B; k) = \{0\}$ for $q \leqq n$, then $\pi_q(X; A, B) \otimes k = \{0\}$ for $q \leqq m+n$. (Received February 6, 1952.)

397*t*. Moses Richardson: *Solutions of irreflexive relations*. Preliminary report.

The concept of solution of an irreflexive binary relation is defined in J. von Neumann and O. Morgenstern, *Theory of games and economic behavior*, Princeton, 1944, where it is proved that if the relation is strictly acyclic, a solution must exist. In a previous note (Bull. Amer. Math. Soc. vol. 52 (1946) pp. 113–116), the present author proved that if the graph of the relation contained no odd cycles modulo 2, solutions exist. The assumption of asymmetry in this note was unnecessary. In the present paper, the investigation of various sufficient conditions for the existence of solutions of irreflexive relations, in which no assumption is made as to symmetry or transitivity, is continued. In particular, it is proved that if the graph is finite and contains no odd oriented cycles, then solutions exist. This result improves on the previous one since intransitivity is no longer required, thus making the result applicable to the theory of games. The proof uses a new method but still employs the theory of linear graphs. Still weaker but more complicated assumptions, as well as extensions to infinite systems, are studied. (Received February 14, 1952.)

398*t*. P. M. Swingle: *Densely looped indecomposable connexes*.

Let S be the plane. C is a densely looped indecomposable connexe if C is connected and for each pair R, R' of mutually exclusive regions where $C\cdot R\neq 0\neq C\ R'$, one of these, R say, is such that its boundary plus $S-C$ contains a continuum which bounds a domain and separates $C\cdot R'$. There exists in S a compact indecomposable connexe (and a compact indecomposable continuum) which is not a densely looped indecomposable connexe. Each composant of a compact indecomposable continuum is itself a densely looped indecomposable connexe. Let C be a densely looped indecomposable connexe contained densely in a domain of S and N be a finite set of limit points of C. Then $C+N$ is a densely looped indecomposable connexe. Let D be a compact connected domain in S, C be a connected subset of D, K' be a composant of an indecomposable connexe in $D\cdot(S-C)$ where both K' and D and C and D have the same closure; then C is a densely looped indecomposable connexe. (Received March 10, 1952.)

399*t*. C. T. Yang: *On cohomology theories*.

It is proved that for fully normal spaces the Alexander-Kolmogoroff cohomology theory agrees with the unrestricted Čech cohomology theory for arbitrary coefficient groups. A natural isomorphism between these two kinds of cohomology groups is actually constructed. The following corollaries result: (1) For compact Hausdorff spaces the Alexander-Kolmogoroff cohomology theory agrees with the Čech cohomology theory (Spanier, Ann. of Math. (1948)). (2) The homotopy axiom for the Alexander-Kolmogoroff cohomology theory holds for fully normal spaces and hence all of the Alexander-Kolmogoroff cohomology groups of convex subsets of linear metric spaces are trivial. (3) The extension, reduction, and hence map excision theorems hold for the unrestricted Čech cohomology theory. (4) If a fully normal space has Lebesgue dimension at most n, then all of its Alexander-Kolmogoroff cohomology groups in dimensions above n vanish. (Received February 22, 1952.)

L. W. Cohen,
Associate Secretary

THE APRIL MEETING IN CHICAGO

The four hundred eightieth meeting of the American Mathematical Society was held at the University of Chicago on Friday and Saturday, April 25–26, 1952. There were about 270 registrations, including the following 229 members of the Society:

W. R. Allen, J. W. Armstrong, K. J. Arnold, Louis Auslander, W. L. Ayres, Reinhold Baer, W. R. Ballard, P. T. Bateman, A. F. Bausch, Henry Beiman, Leon Benson, Gerald Berman, S. F. Bibb, R. H. Bing, R. P. Boas, W. M. Boothby, Raoul Bott, D. G. Bourgin, Joseph Bram, Richard Brauer, R. H. Bruck, R. C. Buck, P. B. Burcham, A. S. Cahn, A. P. Calderón, R. H. Cameron, K. H. Carlson, W. B. Caton, Lamberto Cesari, K. T. Chen, E. W. Chittenden, H. M. Clark, F. M. Clarke, M. D. Clement, Harvey Cohn, E. G. H. Comfort, A. H. Copeland, Sr., V. F. Cowling, E. H. Crisler, C. W. Curtis, M. M. Day, B. V. Dean, John DeCicco, J. C. E. Dekker, W. E. Deskins, Allen Devinatz, Flora Dinkines, W. F. Donoghue, L. A. Dragonette, Roy Dubisch, W. F. Eberlein, B. J. Eisenstadt, H. M. Elliott, Benjamin Epstein, M. H. M. Esser, H. P. Evans, R. L. Evans, Trevor Evans, H. S. Everett, G. M. Ewing, E. R. Fadell, Chester Feldman, R. C. Fisher, Isidore Fleischer, J. S. Frame, Evelyn Frank, L. E. Fuller, R. E. Fullerton, M. P. Gaffney, David Gilbarg, A. M. Gleason, Casper Goffman, H. E. Goheen, Michael Golomb, A. W. Goodman, S. H. Gould, L. M. Graves, R. L. Graves, E. L. Griffin, T. E. Hagensee, Franklin Haimo, P. C. Hammer, Gerald Harrison, H. L. Harter, Charles Hatfield, L. J. Heider, R. G. Helsel, I. N. Herstein, J. J. L. Hinrichsen, I. I. Hirschman, Vaclav Hlavatý, D. L. Holl, T. C. Holyoke, S. P. Hughart, Ralph Hull, J. R. Isbell, W. E. Jenner, Meyer Jerison, R. E. Johnson, G. K. Kalisch, Samuel Kaplan, Irving Kaplansky, William Karush, Chosaburo Kato, M. W. Keller, J. B. Kelly, L. M. Kelly, J. H. B. Kemperman, D. E. Kibbey, Fred Kiokemeister, S. C. Kleene, Erwin Kleinfeld, Fulton Koehler, L. A. Kokoris, Marc Krasner, A. H. Kruse, M. Z. Krzywoblocki, E. P. Lane, R. E. Langer, Leo Lapidus, C. G. Latimer, J. R. Lee, R. A. Leibler, R. B. Leipnik, D. J. Lewis, S. D. Liao, B. W. Lindgren, O. I. Litoff, T. C. Littlejohn, A. J. Lohwater, Lee Lorch, M. F. McFarland, J. D. McKnight, C. C. MacDuffee, Saunders MacLane, Morris Marden, E. P. Merkes, J. M. Miller, J. M. Mitchell, M. A. Moore, G. W. Morgenthaler, E. J. Moulton, H. T. Muhly, S. B. Myers, W. M. Myers, Jr., Zeev Nehari, O. M. Nikodým, Katsumi Nomizu, E. A. Nordhaus, E. P. Northrop, R. J. Nunke, E. J. Olson, E. H. Ostrow, R. R. Otter, M. H. Payne, Sam Perlis, J. K. Peterson, L. E. Pursell, A. L. Putnam, C. R. Putnam, Gustave Rabson, R. A. Raimi, O. W. Rechard, P. V. Reichelderfer, Haim Reingold, R. B. Reisel, Daniel Resch, P. R. Rider, G. S. Ritchie, R. A. Roberts, Alex Rosenberg, P. C. Rosenbloom, Arthur Rosenthal, W. C. Royster, H. J. Ryser, R. G. Sanger, A. C. Schaeffer, H. M. Schaerf, O. F. G. Schilling, Lowell Schoenfeld, J. R. Schoenfield, W. T. Scott, D. H. Shaftman, M. E. Shanks, V. L. Shapiro, S. S. Shu, Edward Silverman, R. J. Silverman, M. F. Smiley, A. H. Smith, E. S. Sokolnikoff, E. H. Spanier, E. J. Specht, George Springer, H. E. Stelson, B. M. Stewart, J. A. Sullivan, C. T. Taam, H. P. Thielman, R. M. Thrall, W. J. Thron, E. F. Trombley, C. K. Tsao, S. M. Ulam, W. R. Utz, N. H. Vaughan, Thirukkannapuram Vijayaraghavan, Bernard Vinograde, G. L. Walker, D. W. Wall, Sylvan Wallach, M. J. Walsh, S. E. Warschawski, D. R. Waterman, M. S. Webster, M. T. Wechsler, André Weil, L. M. Weiner, H. H. Wicke, L. R. Wilcox, K. G. Wolfson, W. D. Wood, F. B. Wright, L. C. Young, P. M. Young, J. W. T. Youngs, Daniel Zelinsky, J. L. Zemmer, Antoni Zygmund.

By invitation of the Committee to Select Hour Speakers for Western Sectional Meetings, Dr. S. M. Ulam of the Los Alamos Scientific Laboratories addressed the Society at 2 P.M. Friday on *Combinatorics and analysis.*

There were a total of eight sessions for the presentation of contributed papers. Three were held on Friday at 10:30 A.M., two on Friday at 3:15 P.M., and three on Saturday at 10:30 A.M. One of the concurrent sessions on Saturday morning was a special session for the presentation of papers which arrived in the Providence office after the deadline. Presiding officers at the various sessions were Professors Richard Brauer, R. H. Bruck, R. H. Cameron, Michael Golomb, L. M. Graves, S. C. Kleene, Zeev Nehari, E. H. Spanier, and J. W. T. Youngs.

Abstracts of the papers read follow. Those having the letter "*t*" after their numbers were read by title. Paper number 403 was read by Professor Copeland, number 411 by Dr. Rosenberg, number 430 by Dr. Hammer, number 441 by Professor Warschawski, number 451 by Professor Ewing, and number 452 by Mr. Cahn. Mr. Blair was introduced by Professor M. F. Smiley, Mr. Feit by Professor R. M. Thrall, Mr. Fettis by Professor P. R. Rider, and Mr. Allen by Professor O. H. Hamilton.

ALGEBRA AND THEORY OF NUMBERS

400. R. L. Blair: *Ideal lattices and the structure of rings.*

A ring A is said to satisfy condition C or D (C_r or D_r) in case the lattice of ideals (right ideals) of A is complemented or distributive, respectively. The principal results are the following. A ring A which is semi-simple in Jacobson's sense (Amer. J. Math. vol. 67 (1945) pp. 300–320) satisfies C (C_r) if and only if A is isomorphic with the discrete direct sum of simple rings (of simple rings with minimal right ideals). If A satisfies C_r, then $A = M \dotplus M^*$, where M is the maximal regular ideal of A (Brown and McCoy, Proceedings of the American Mathematical Society vol. 1 (1950) pp. 165–171) and M^* is the annihilator of M. The rings M and M^* satisfy C_r and M^* is bound to its radical (M. Hall, Trans. Amer. Math. Soc. vol. 48 (1940) pp. 391–404) which is nilpotent of index two. An ideal (right ideal) I of A is *s-irreducible* in case $B \cap C \subseteq I$ for ideals (right ideals) B, C of A implies that $B \subseteq I$ or $C \subseteq I$. (Cf. L. Fuchs, Comment. Math. Helv. vol. 23 (1949) pp. 334–341.) A ring A satisfies D (D_r) if and only if each ideal (right ideal) of A is the intersection of all s-irreducible ideals containing it. Finally, if A is a semi-simple ring which satisfies D_r, then A is isomorphic with a subdirect sum of division rings. (Received March 12, 1952.)

401. K. T. Chen: *A group ring method for finitely generated groups.*

Let G be a finitely generated group, and J a ring with unit element. One may assume $G = F/R$ where R is a normal subgroup of a free group F with free generators $x_1, \cdots, x_m$. Let a factor group M/N, $F \supset M \supset N$, be an invariant of G such that, if G has another presentation, the correspondingly defined factor group is isomorphic with M/N. For example, $M/N=[F, F]/[F, R]$ is such a factor group. Then the quotient ring $JM/(N-1)_M$ is also an invariant of G, where $(N-1)_M$ is the ideal generated by all $u-1$, $u\in N$, in JM. Let $X_1, \cdots, X_m$ be noncommutative indeterminates. Denote by Λ the ring of all formal power series in the form $\sum_{p=0}^{\infty} a_p$, $a_p=\sum\mu_{i_1\cdots i_p} X_{i_1}\cdots X_{i_p}$, $\mu_{i_1\cdots i_p}\in J$. Define ϕ: $JF\to\Lambda$ such that $\phi(x_i)=1+X_i$ and $\phi(\mu a)=\mu\phi(a)$, $\mu\in J$. Then $\phi(JM)/\phi((N-1)_M)$ is an invariant of G. (Received March 13, 1952.)

402. F. Marion Clarke: *Note on quasi-regularity and the Perlis-Jacobson radical.*

The Perlis-Jacobson use of quasi-regularity to characterize the radical of an "arbitrary" algebra or ring R [Bull. Amer. Math. Soc. vol. 48 (1942) pp. 128–132; Amer. J. Math. vol. 67 (1945) pp. 300–320] is shown to require any one of the following properties of weak associativity or commutativity: P1. If x and z are respectively left and right quasi-inverses of a quasi-regular element y of a right quasi-regular right ideal of R, then $(xy)z=x(yz)$; P2. The left and right quasi-inverses of a quasi-regular element of a right quasi-regular right ideal of R are equal and unique; P3. If R has odd characteristic and $R^{(+)}$ consists of the same elements with the same addition as in R but with multiplication defined $r\cdot s=(rs+sr)/2$, then a quasi-regular element of $R^{(+)}$ has the same unique right quasi-inverse in R and in $R^{(+)}$. It is shown that P1, P2, and P3 are equivalent in rings of odd characteristic and that analogous properties are equivalent with reference to any quasi-regular element of R. (Received March 12, 1952.)

403. A. H. Copeland and Frank Harary: *A characterization of implicative Boolean rings.*

An implicative Boolean ring is defined in terms of a cross-product operation and the usual Boolean operations. However it is shown in this paper that such rings can be characterized in terms of familiar ring concepts only. More specifically, a Boolean ring B can contain such a cross-product if and only if it is isomorphic to its quotient rings modulo the non-unit principal ideals. The isomorphisms enable one to set up a semigroup of transformations (not necessarily unique) of B into B. These are one-parameter transformations where the parameter is a nonzero element of the Boolean ring. The product of the transformations defines the cross-product of the parameters. The inverse of one of these transformations, when defined, is an implication which is neither strict nor material. The implication can be extended to elements for which the inverse does not exist and can be given a logical interpretation. (Received March 7, 1952.)

404. W. Feit: *The formula for the degree of the skew representations of the symmetric group.*

By A. Young's well known method, each irreducible representation of the symmetric group on n letters can be associated with a regular diagram containing n nodes, which can be denoted by $(a_1, \cdots, a_m)$ where $\sum a_i=n$, $a_1\geq\cdots\geq a_m\geq 0$; the degree of the associated representation is the number of standard orderings of $(a_1, \cdots, a_m)$. In giving a proof of the Murnaghan-Nakayama recursion formula, G. de B. Robinson considers what he calls skew diagrams and associates with each of these a representation of the symmetric group; such a representation he calls a skew representation. Each skew diagram may be denoted by $(a_1, \cdots, a_m)-(b_1, \cdots, b_m)$ where $a_i \geqq b_i$, $i=1, \cdots, m$, $(a_1, \cdots, a_m)$, $(b_1, \cdots, b_m)$ are regular diagrams. The object of this note is to give the explicit formula for the number of standard orderings of such a skew diagram, which is also the degree of the representation associated with it. The number of orderings of $(a_1, \cdots, a_m)-(b_1, \cdots, b_m)$ is $n! \det (g_{ij})$, where $g_{ij}=1/(a_j-b_i-j+i)!$, $n=\sum a_i-\sum b_i$. The proof of the formula is by induction. (Received March 12, 1952.)

405. J. S. Frame: *Character values for a fixed class in the symmetric groups*. Preliminary report.

To each irreducible representation F_α of the symmetric group S_m corresponds a partition diagram of m nodes arranged in α_1' rows of α_i nodes $(0 \leqq \alpha_{i+1} \leqq \alpha_i)$ and in α_1 columns of α_j' nodes $(0 \leqq \alpha_{j+1}' \leqq \alpha_j')$, where $\sum \alpha_i = \sum \alpha_j' = m$. Define at each node the hook length $h_{ij}=\alpha_i-i+\alpha_j'-j+1$ (that is, one more than the sum of the nodes to the right and below) and the diagonal deviation $d_{ij}=j-i$. Let $h=\prod h_{ij}$ be the product of the m hook lengths and let $s_k=\sum (j-i)^k$ be the sum of the kth powers of the m diagonal deviations d_{ij}. Then the degree of F_α is $f_\alpha = m!/h$. The second main result of the paper is to express as a polynomial in the s_k the common value $\omega_{abc\cdots}$ of the f_α equal characteristic roots of the matrix of F_α that represents the sum of the $g_{abc\cdots}$ elements in that class of S_m whose permutation cycles contain respectively $a+1$, $b+1$, $c+1, \cdots, 1, 1$ symbols. Thus $\omega_0=s_0=m$, $\omega_1=s_1$, $\omega_2=s_2-(m^2-m)/2$, $\omega_2+\omega_{11}=(s_1^2-s_2)/2$, $\omega_3=s_3-(2m-3)s_1$, $\omega_{21}=s_2s_1-4s_3-(m^2-13m+16)s_1/2$, $\omega_3+2\omega_{21}+\omega_{111}=(s_1^3-3s_2s_1+2s_3)/6$, etc. These and similar formulas give the values in each particular class for all the characters $\chi_{abc\cdots}=f_\alpha\omega_{abc\cdots}/g_{abc\cdots}$ of all symmetric groups S_m, and make evident the familiar relations between associated characters. (Received March 13, 1952.)

406. L. E. Fuller: *A canonical form for a matrix over a principal ideal ring modulo m.*

The basic canonical form is for m an integral power of a prime. The elements in these residue class rings are either units or divisors of zero. In the canonical form chosen, every diagonal element is a power of the prime of the modulus and is determined in a prescribed order. Every element is then a multiple of the diagonal of its row; every element above the diagonal is a multiple of the next higher power of the diagonal of its row. Also every element is either "reduced" modulo its column diagonal, or is zero. The Hermite canonical form for a field can be shown to be a special case, although in general this form lacks the triangular property. Uniqueness can be proved by an induction on the order in which the diagonal elements are chosen. To extend this result to the general case, the forms for each of the relative prime factors of m are determined. These are then combined by congruence methods using an extension of the Chinese remainder theorem. (Received February 27, 1952.)

407. I. N. Herstein: *A theorem on rings.*

In his paper *A theorem on division rings* (Canadian Journal of Mathematics vol. 3 (1951) pp. 290–292) Kaplansky proved: Let R be a semi-simple ring with center Z, and suppose that for every x in R some power (depending on x) is in Z $(x^{n(x)} \in Z)$. Then R is commutative. One might well ask what conditions on a ring R with center Z having $x^{n(x)} \in Z$ might prevent commutativity. We prove: Let R be a ring with center Z having $x^{n(x)}$ in Z for all $x \in R$. Then if R is not commutative, the commutator ideal of R must be a nil-ideal. As a consequence we have that if R has no nil-ideals and satisfies $x^{n(x)} \in Z$, then it is commutative. These results contain that of Kaplansky cited above, and in a sense are the best possible. (Received February 25, 1952.)

408. R. E. Johnson: *On ordered domains of integrity.*

Let K^* be the nonzero elements of a domain of integrity K. An element a of K^* is called even if there exist n elements $b, c, \cdots, d$ of K^* such that a is a product of the $2n$ elements $b, c, \cdots, d, b, c, \cdots, d$ in some order. Denote by S the additive semigroup generated by the even elements of K^*. The principal theorem of this paper is that K is orderable if and only if $S \subset K^*$. This extends to a domain of integrity a result of Szele's (*On ordered skew fields*, to appear in the Proceedings of the American Mathematical Society) that a division ring D is orderable if and only if the additive and multiplicative semigroup generated by the nonzero squares of elements of D is contained in D^*. (Received March 3, 1952.)

409. Erwin Kleinfeld: *Simple alternative rings.*

Let R be any alternative ring. If a, b, c are in R and the pairs a, c and b, c are anticommutative, then $(c^2, R)(a, b, c) = 0$. Thus if R is a division ring of characteristic $\neq 2$, squares of associators and commutators are in the center of R. This leads to a relatively short and simple proof that R is either associative or a Cayley-Dickson division algebra over its center, a result previously obtained by R. H. Bruck and the author (Proceedings of the American Mathematical Society vol. 2 (1951) pp. 878–890) and by L. A. Skornyakov (Ukrain. Mat. Zurnal vol. 2 (1950) pp. 70–85), the latter restricting himself to characteristic $\neq 3$. Let S be a simple alternative ring of characteristic $\neq 2, 3$. Then with the aid of the above identity it is shown that S is a Cayley algebra if and only if S contains pairwise anticommutative elements a, b, c such that (a, b, c) is not a divisor of zero. S is a Cayley-Dickson division algebra if and only if S is not associative and $(a, b, S) = 0$ whenever $(a, b) = 0$. The author conjectures that all simple alternative rings are either associative or Cayley algebras. (Received January 17, 1952.)

410. D. J. Lewis: *Cubic Diophantine equations.*

The following theorem is establish: If $F(x_1, x_2, \cdots, x_n)$ is a cubic homogeneous polynomial with rational coefficients and if n is sufficiently large, then $F(x) = 0$ possesses a nontrivial, rational, integral solution. The proof consists in establishing the existence of a nonsingular, integral, linear transformation $x_i = \sum_1^n c_{ij} y_j$ such that $F(x) = G(y)$ and $G(y_1, y_2, \cdots, y_s, 0, \cdots, 0) = \sum_1^s a_i y_i^3$, $n \leq 6s$. L. G. Peck [Amer. J. Math. vol. 71 (1949) pp. 387–402] has shown that such diagonalized polynomials always have nontrivial solutions provided s is sufficiently large. (Received March 7, 1952.)

411. J. E. McLaughlin and Alex Rosenberg: *Zero divisors and commutativity of rings.*

A Zorn ring is an associative ring in which every non-nil left ideal contains a nonzero idempotent. If the left zero divisors in a Zorn ring A form a proper left ideal, that ideal is the radical of A, and A is a division ring modulo its radical. Conversely, if A is a division ring modulo its radical, the set of left zero divisors is either A or its radical. If a Zorn ring A properly contains its center and all the zero divisors are in the center, then A is a division ring or A is a field F modulo its radical. In the latter case the center of A maps onto a field Z modulo the radical, and F is either a purely inseparable extension of Z or has transcendence degree greater than 1 over Z. This generalizes a theorem of Herstein (Proceedings of the American Mathematical Society vol. 1 (1950) pp. 370–371). If the notion of left zero divisor is replaced by left topological zero divisor, similar theorems hold for Banach algebras. (Received March 11, 1952.)

412t. H. W. E. Schwerdtfeger: *Matrices commuting with their own derivative.*

G. Ascoli has shown (Rendiconti Sem. Mat. Torino vol. 9 (1950) pp. 245–250) that a nonderogatory function matrix $X(t)$ which commutes with its own derivative $X'(t)$ has necessarily all its values commutative: $X(s)X(t)=X(t)X(s)$. In the present paper this theorem is proved again for an *analytic* nonderogatory matrix $X=X(t)$ that satisfies the condition $XX'=X'X$. By making use of induction and of the fact that any two matrices commuting with a nonderogatory X are commutative, it is shown that the derivatives of all orders are commutative: $X^{(i)}X^{(j)}=X^{(j)}X^{(i)}$ $(i, j=0, 1, 2, \cdots)$. If moreover the characteristic polynomial of X is separable, then a *constant* matrix T can be found such that TXT^{-1} is diagonal. The supposition that X is nonderogatory may be dropped if X is a 2×2-matrix, or if $X(t)$ is $n\times n$ and quadratic in t. The same is shown for $n=3$ and $X(t)$ cubic in t. (Received March 6, 1952.)

ANALYSIS

413t. H. D. Block and Buchanan Cargal: *Arbitrary mappings.*

This paper consists of generalizations of some of H. Blumberg's results concerning arbitrary functions (*Arbitrary point transformations*, Duke Math. J. vol. 11 (1944) pp. 671–685). In particular those results concerning the existence of (i) a residual set, each point of which is of homogenous inexhaustible functional approach, and (ii) an everywhere dense set, with respect to which the function is continuous, are shown to hold in much more general types of spaces. As might be anticipated, the proofs are considerably simpler in the more general case. (Received March 10, 1952.)

414t. R. P. Boas and R. C. Buck: *Expansion of analytic functions in polynomial series.* II.

To facilitate the study of the representation of analytic functions by means of series of polynomials, the following terminology is introduced. Given a function $\psi(t)=\sum c_n t^n$ regular at zero and with $c_n\neq 0$ for $n=0, 1, \cdots$, a function $f(z)=\sum a_n z^n$ is said to be of ψ type τ if $\limsup |a_n/c_n|^{1/n}=\tau<\infty$. By means of an integral transform, functions $f(z)$ of ψ type τ correspond one-to-one with functions $F(w)$ which are regular in $|w|>\tau$ and vanish at infinity. This enables the authors to extend earlier results on the representation of entire functions to functions of finite ψ type. For appropriate choices of ψ results are obtained for most of the better known basic sets of polynomials, in particular, for the Laguerre, reversed Laguerre, Hermite, and ultraspherical polynomials. (Received March 6, 1952.)

415. R. C. Buck: *Admissible sequences with vanishing differences.*

A sequence $\{a_n\}$ is admissible if there is an entire function $f(z)$ of exponential type, and of type less than π on the imaginary axis such that $f(n)=a_n$ for $n=0, 1, 2, \cdots$. Let $b_n=\Delta^n a_0=(-1)^n\sum_0^n {}_nC_k b_k$. $\{a_n\}$ is admissible if and only if $\sum b_n z^n$ is regular on the interval $-1\leqq x\leqq 0$. The interpolating function $f(z)$ is then (ML)-$\sum_0^\infty {}_zC_n b_n$ where ML denotes Mittag-Leffler summability. Suppose that $\limsup |a_n|^{1/n}\leqq 1$.

Then, if $b_n=0$ for a set of n of density greater than 1/3, $\{a_n\}$ is admissible; if the density is 1/2, the interpolating function $f(z)$ is of zero type. If $b_n\geqq 0$ for all n, $\{a_n\}$ is again admissible, and f of zero type. This is also true if $b_n\geqq\epsilon>0$ for all n except a subsequence $\{\lambda_k\}$ for which $\lambda_{k+1}-\lambda_k\to\infty$. (Received March 14, 1952.)

416. A. P. Calderón: *A general ergodic theorem.*

The validity of the individual ergodic theorem for nonabelian groups of transformations is investigated. Let $\mathcal{E}$ be a measure-space of finite total measure and $\mathcal{G}$ a locally compact group of measure-preserving transformations of $\mathcal{E}$ satisfying suitable measurability conditions. Let N_t be a family of compact open symmetric neighborhoods of the identity in $\mathcal{G}$ depending on $t>0$ and such that $N_tN_s\subset N_{t+s}$. Then if $|N_{2t}|<k|N_t|$ where $|N_t|$ stands for the left invariant measure of N_t, there exists a subset R of the reals of density 1 such that for every function $F(x)$ integrable in $\mathcal{E}$ the averages $|N_t|^{-1}\int_{N_t}F(gx)dg$ converge almost everywhere in x as $t\to\infty$ through R. A dominated ergodic theorem in the form of Pitt holds for these averages. As a consequence of the latter a theorem of Dunford-Zygmund's type is proved. (Received March 12, 1952.)

417t. A. P. Calderón: *A note on invariant measures.*

The following theorem is established. Let $\mathcal{E}$ be a measure space of finite total measure and $\mathcal{G}$ a group of one-to-one transformations of $\mathcal{E}$ preserving measurable sets and sets of measure zero. Then if $\mathcal{G}$ is measurable in the sense of von Neumann, a necessary and sufficient condition in order that there exist a measure invariant with respect to the transformations of the group and absolutely continuous with respect to the given measure μ is that for every measurable set A, $\mu(A)>0$ imply that $\inf\mu(gA)>0$, $g\in\mathcal{G}$. (Received March 12, 1952.)

418. R. H. Cameron: *A generalization of the Poisson formula for the solution of the heat flow equation.*

In this paper a solution is given for the differential equation $\partial^2G/\partial\xi^2-a\partial G/\partial t+\theta(t,\ \xi)G=0$ subject to the boundary conditions $G(t,\ \pm\infty)=0$ and $G(0,\ \xi)=\sigma(\xi)$. Under suitable smoothness, etc., conditions on θ and σ, it is shown that a solution is given by the Wiener integral $G(t,\ \xi)=\int_C \exp\ \{r^2\int_0^1\theta[t(1-s),\ 2rx(s)+\xi]ds\}\cdot\sigma[2rx(1)+\xi]d_wx$, where $r=(t/a)^{1/2}$. This reduces to the Poisson integral solution of the heat flow equation when $\theta=0$. (Received November 13, 1951.)

419. V. F. Cowling: *On the distribution of the values of the partial sums of a Taylor series.*

The object of this paper is to generalize the results of a preceding paper by the author (Proceedings of the American Mathematical Society vol. 2 (1951) pp. 732–738). In the paper referred to it was shown that if the points of affix $a_ka_{k-1}^{-1}z$, $k=1,\ 2,\ \cdots,\ n$ $(a_k\neq 0,\ k=1,\ 2,\ \cdots,\ n)$, lay in a certain region $E=E(V)$ of the complex plane for $z\in Z$, then the values of the partial sums $S_k(z)=a_0+a_1z+\cdots+a_kz^k$, $k=1,\ 2,\ \cdots,\ n$, would be complex numbers such that for $z\in Z$ the points corresponding to these numbers would be contained in a predetermined region V. In the present paper we replace the region E by a number of regions and require only that certain subsets of the quantities $a_na_{n-1}^{-1}z$ fall in these regions for $z\in Z$. The following result is a simple application of these methods. Let $P(z)=a_0+a_1z+\cdots+a_kz^k$ where $a_i\neq 0$, $i=0,\ 1,\ \cdots,\ k$. Suppose $0<r_i\leqq 1$, $i=1,\ 2,\ \cdots,\ k$. Then $P(z)$ has all

its zeros interior to the circle $|z| \leq \text{Max}\,[((r_1+1)/r_2)|a_2/a_1|, ((r_2+1)/r_2)|a_1/a_2|, \cdots, ((r_k+1)/r_1)|a_{k-1}/a_k|]$. (Received December 24, 1951.)

420. Allen Devinatz: *On positive definite functions*. Preliminary report.

Let Q be a convex set in the z-plane which is symmetric with respect to the real axis and contains the origin. Necessary and sufficient conditions are given for a continuous complex-valued function $f(z)$, defined on Q, to be written in the form $f(z) = \int_{a_1}^{b_1}\int_{a_2}^{b_2} \exp(xt_1+iyt_2)\,dV(t_1, t_2)$, where x and y are respectively the real and imaginary parts of z and $V(t_1, t_2)$ is a bounded monotone increasing function. The numbers b_1 and b_2 may take on the value ∞, and a_1 and a_2 the value $-\infty$. The main condition is that $\sum_{i,j=1}^{n} \alpha_i\bar{\alpha}_j f(z_i+\bar{z}_j) \geq 0$ for all finite sets of complex numbers $\{\alpha_1, \cdots, \alpha_n\}$ and all sets $\{z_1, \cdots, z_n\}$ which range over a certain convex subset of Q. The methods employed use the techniques of operators in Hilbert space and the theory of reproducing kernels (N. Aronszajn, Trans. Amer. Math. Soc. vol. 68 (1950)). The results obtained generalize previous results of the author (Bull. Amer. Math. Soc. Abstract 57-1-30), contain a result of M. Krein (C. R. (Doklady) Acad. Sci. URSS. N.S. vol. 26 (1940) pp. 17–22) and contain a corrected version of an incorrect theorem stated by S. Livshitz (C.R. (Doklady) Acad. Sci. URSS. N.S. vol. 44 (1944) pp. 3–7, Theorem 3) for which the author has a counter example. (Received March 13, 1952.)

421. W. F. Donoghue: *Causality and quantum mechanics*. Preliminary report.

A theory is the specification of a linear algebra, R, which is determined by the observables, and a set of states on R. It is physically correct to assume that the linear span, R', of the states is weak-star dense in R^*, the algebraic dual of R, and weak-star bounded there. The Krein-Milman theorem then provides a full set of pure states in R^* if not in R'. The theory is causal (may be embedded in a causal theory) if and only if all of the pure states in R' (R^*) are sharp, in von Neumann's terminology dispersionless. It is established that the causality of a theory (or the possibility of extending it to a causal theory) is equivalent to the commutativity of R. The postulates for quantum mechanics used by von Neumann as well as those suggested by Segal fall into the framework considered, and the celebrated theorem of von Neumann follows from less restrictive hypotheses. Since the causality question reduces to a problem concerning the commutativity of R, it follows that those physical theories which attempt a causal interpretation of quantum mechanics differ from the orthodox formulation of that theory in the specification of R: that is, in the theory of measurement. (Received March 10, 1952.)

422. R. L. Evans: *Solution of linear ordinary differential equations containing a parameter.*

Consider the differential equation (1) $y^{(n)} - \sum_{j=1}^{n} \lambda^{m_j} p_j(\lambda, x) y^{(n-j)} = 0$ where: (i) $p_j(\lambda, x) = \sum_{\nu=[(j-\mu-1)/p]}^{\infty}\sum_{\mu=-pm_j}^{\infty} a_{j,\mu,\nu}\lambda^{-\mu}x^{\nu}$ are such that all $p_j(\lambda, x)/\lambda^{pm_j}$ converge wherever $|x| < x_0$ and $|\lambda| > \lambda_0$, (ii) n and p are positive integers and the m_j's are non-negative integers, (iii) $[-\mu/p]$ is the greatest non-negative integer contained in $(-\mu/p)$, and (iv) x is a complex variable and λ is a large complex parameter. Equation (1) has a regular point at $x=0$ and it may also be said to have a turning point there if one or more m_j exceeds zero. If the given initial conditions define (2) $y^{(j)}(\lambda, 0) = (j!)\sum_{\alpha=0}^{\infty} b_{\alpha,j}\lambda^{-\alpha}$ $(j=0, 1, \cdots, n-1)$ convergent whenever $|\lambda| > \lambda_0$, then (1) and (2) together have a solution of the form (3) $y(\lambda, x) = \sum_{\alpha=0}^{\infty}\sum_{\beta=0}^{\infty} b_{\alpha,\beta}\lambda^{m\beta-\alpha}x^{\beta}$, which converges wherever $|x| < x_0$ and $|\lambda| > \lambda_0$. The formula for computing the unknown $b_{\alpha,\beta}$'s is given. This new application of old results yields a solution of a broad class of initial value problems, including many involving turning points. Examples of such problems are presented. (Received April 10, 1952.)

423*t*. Seymour Ginsburg: *On a relation between sets and elements.*

Let E be any nonempty set and $H = \{h\}$ a family of subsets of E. Let R be a relation such that xRH holds for one and only one element x in each h, and G_α the subset of E consisting of those elements x in E for which xRH holds for at least $\aleph_\alpha$ sets h in H. Denote by $p(A)$ the power of A. If H is the family of all finite subsets of the uncountable set E and $\aleph_\alpha < p(E)$, then $p(G_\alpha) = p(E)$. If $\aleph_\alpha = p(E)$, then G_α may be empty. This modifies a result of Fodor and Ketskemety (Fund. Math. vol. 37). If H is the family of those subsets of E which are of power $\aleph_\gamma$ each, where $\aleph_\gamma \leq p(E)$, and if $\aleph_\alpha = p(E)$, then $p(G_\alpha) = p(E)$. (Received March 3, 1952.)

424*t*. Seymour Ginsburg: *On the number of distinct sums of transfinite series obtained by permuting the elements of a transfinite sequence.*

Let $N(\sum_{\xi<\lambda} a_\xi)$ be the number of distinct sums of all the λ-type series obtained by permuting the elements of the sequence of ordinal numbers $\{a_\xi\}_{\xi<\lambda}$. Generalizing two results of Sierpinski (Fund. Math. vol. 36) it is shown that if ω_ρ is a regular ordinal, then $N(\sum_{\xi<\omega_\rho} a_\xi) \leq \sum_{\xi<\omega_\rho} \aleph_\rho^{\aleph_\xi}$. Furthermore if $\omega_\rho = \omega_{\alpha+1}$, then there exists an ω_ρ sequence $\{a_\xi\}$ such that $N(\sum_{\xi<\omega_\rho} a_\xi) = \aleph_\rho$. If ω_ρ is a regular ordinal and $\{a_\xi\}_{\xi<\omega_\rho}$ is a nondecreasing sequence of ordinals, then $N(\sum_{\xi<\omega_\rho} a_\xi) = 1$. (Received March 3, 1952.)

425*t*. Seymour Ginsburg: *Ordinal inequalities.*

Let $\{\alpha_\xi\}_{\xi<\lambda}$ and $\{\beta_\xi\}_{\xi<\lambda}$ be two increasing sequences of ordinal numbers whose limits are α and β respectively. In terms of the Cantor normal form of α and of β, necessary and sufficient conditions are given for the limit of the sum and the product of the two sequences to equal the sum and the product of the limits of the sequences. (Received March 3, 1952.)

426*t*. Seymour Ginsburg: *Real functions on posets.*

Let P be a poset with no minimal element. A single-valued, real function on P has a limit a if, to each element p_0 of P, and $\epsilon > 0$, there corresponds an element $p_1(p_0, \epsilon)$ of P such that $p_1 \leq p_0$, and $|f(p) - a| < \epsilon$ for $p \leq p_1$. The usual results on sum, difference, multiplication, and quotient of limits hold. In addition, a poset P can be imbedded isomorphically into an everywhere branching poset Q (Day, *Oriented systems*, Duke Math. J. vol. 11) by an isomorphism g. Furthermore, a function h of Q onto $g(P)$ can be found which has the following two properties: (1) $h[g(p)] = g(p)$; and (2) if f is any function on P, then the function f_*, which is defined by $f_*[g(p)] = f(p)$ for p in P, and $f_*(q) = f_*[h(q)]$ for q in Q, has a limit a on Q if and only if f has a limit a on P. Thus, in a certain sense the real functions on everywhere branching posets are the most general as regards limits. (Received March 10, 1952.)

427. R. L. Graves: *The Fredholm theory in Banach spaces.*

It is possible to generalize the qualitative, nondeterminental aspects of the classical Fredholm theory to a completely continuous operator, K, on any Banach space; in particular the resolvent may be expressed as the quotient of two entire functions, say $R(z) = \text{DET}\ (1 - zK)/\det\ (1 - zK)$. It has been pointed out that the coefficients of the particular entire functions used by Fredholm in the classical case may be expressed in terms of the traces of the interates of the operator, where trace $K = \int_0^1 K(x, x)dx$. For an operator, K, on a general space, define trace K as the sum of the characteristic values (with multiplicity) and trace $|K|$ as the sum of the corresponding absolute values. When trace $|K| < \infty$, this definition of the trace makes the Fredholm formulas applicable, and only minor modifications are necessary when trace $|K^p| < \infty$. Further, if $\|K_q - K\| \to 0$, where K_q is finite ranged, and trace $|K_q| \leq M$, then $\det_2 (1 - zK_q) \to \det_2 (1 - zK)$ and $\mathrm{DET}_2 (1 - zK_q) \to \mathrm{DET}_2 (1 - zK)$ uniformly on compact sets. (These entire functions correspond to those used by Carleman.) Again, minor modifications give similar results when trace $|K_q^p| \leq M$. This yields as particular consequences the determinental results of Fredholm and Carleman. (Received March 11, 1952.)

428. E. L. Griffin: *On semi-finite rings.*

Let M be a ring of operators on a Hilbert space H of arbitrary dimension, with M' as commutant. If M and M' are finite rings, then there exists a positive definite operator C belonging to the center of M which generalizes von Neumann's invariant C for factors. Although the trace in M is always strongest continuous, it is strongly (weakly) continuous if and only if C is a bounded operator. If two such rings are *-isomorphic in such a way that their coupling operators correspond under the isomorphism, then there exists a linear isometry implementing the isomorphism. Further, one can prove that: 1. The strongest topology is purely algebraic. 2. The strong and weak topologies are not purely algebraic but depend on the coupling operator C. 3. C is the identity if and only if M and M' are conjugate isomorphic. 4. If C is bounded, then each positive linear functional on M is a finite sum of functionals of the form (Ax, x), $A \in M$, $x \in H$. These results can be extended to semi-finite rings (which are rings such that every nonzero projection contains a finite nonzero projection) by introducing ratios of cardinals as formal scalars. (Received March 17, 1952.)

429t. E. L. Griffin: *Isomorphisms of rings of type* III. Preliminary report.

A ring of operators is of type III if each of its nonzero projections is an infinite projection. This paper extends the results of our previous paper *On semi-finite rings* (see preceding abstract) to rings of type III. In particular, the notion of coupling operator is generalized and one obtains the Theorem: *If an isomorphism between two rings of type* III *links the coupling invariants, then the isomorphism is implemented by a linear isometry.* This result leads to the following results: 1. *The strong and weak topologies are purely algebraic in rings of type* III, 2. *The strongest topology is purely algebraic in general rings, and* 3. *The strong and strongest topologies coincide in rings of type* III. The methods used here are essentially the same as those used in our previous paper *On semi-finite rings.* (Received April 8, 1952.)

430. P. C. Hammer and Andrew Sobczyk: *Monotonic set functions and convex sets.*

Let G be a real-valued nondecreasing set function (including possibly $+\infty$ and $-\infty$ in its range) defined for a class $\mathfrak{S}$ of subsets of a set M. To each point x in M let there be associated a class $\mathfrak{S}(x)$ of sets in the domain $\mathfrak{S}$ of G. The association $(x, \mathfrak{S}(x))$ is called *exclusive* if no set in $\mathfrak{S}(x)$ contains x. The association $(x, \mathfrak{S}(x))$ is called *farflung* if y in the complement of $x \in \mathfrak{S}(x)$ implies always that there exists a set $Y \in \mathfrak{S}(y)$ such that $Y \supset X$. A functional $g(x)$ is defined as the supremum of $G(X)$

for all $X\in\mathfrak{S}(x)$. Let $C(r)$ be the set of points of M such that $g(x)\leq r$. For a farflung and exclusive association $(x, \mathfrak{S}(x))$, $C(r)$ is either the entire space M, or $C(r)$ is the intersection of all sets $Y\in\mathfrak{S}(y)$ for all y such that $G(Y)>r$. In particular, if M is a linear topological space and each element of $\mathfrak{S}(x)$ is closed and convex for every x, then $C(r)$ is a closed convex set. Appropriate properties of continuity and of regularity at infinity are defined for set functions to yield boundedness, closure, and other desired properties of the associated families of convex sets. (Received March 12, 1952.)

431t. P. C. Hammer and Andrew Sobczyk: *Subconvex and quasi-convex functionals.*

A real-valued functional $f(x)$ defined for all points x in a linear space M is said to be subconvex if and only if $f(x_1)=f(x_2)$ implies always that $f(ax_1+(1-a)x_2)\leq f(x_1)$, $0<a<1$. A functional $f(x)$ is said to be *quasi-convex* if and only if $f(ax_1+(1-a)x_2)\leq\max\,[f(x_1), f(x_2)]$, $0<a<1$. The following results are obtained. (1) Every convex functional is subconvex and quasi-convex. (2) Every quasi-convex functional is subconvex. (3) A monotonic nondecreasing function of a quasi-convex functional is quasi-convex functional. (4) Every continuous subconvex functional is quasi-convex. (5) The set of points x such that $f(x)\leq r$ is a convex set, called the *r-truncated* set of $f(x)$, if $f(x)$ is quasi-convex. (6) Given any one-parameter family of convex sets $C(r)$ subject to certain hypotheses, there exists a quasi-convex function $f(x)$ such that each $C(r)$ is the r-truncated set of $f(x)$. Generalized forms of convexity are defined, and properties of corresponding functionals derived. N.B. Professor W. Fenchel has written that he had also defined *quasi-convexity* in a manuscript yet to be published. (Received March 12, 1952.)

432. I. I. Hirschman: *Some density theorems obtained from the theory of quasi-analytic functions.*

Let $f(x)\in L_p(-\infty, \infty)$ $(1\leq p<\infty)$ and let X be a subset of the real line R. We define $T_p(f, X)$ as the closure in $L_p(-\infty, \infty)$ of the finite linear combinations of the functions $f(x-\xi)$, $\xi\in X$. A sufficient condition is given for $T_p(f, X)=T_p(f, R)$. For example if $\|f^{(n)}(x)\|_p\leq n!(\log(n+e))^n$, $n=0, 1, \cdots$, and if $\limsup_{t\to\infty}(2/\pi t)\log\log X(t)>1$ where $X(t)$ is the number of elements of X in $[-t, t]$, then $T_p(f, X)=T_p(f, R)$. Let $M_p(\phi, X)$ be the closure in $L_p(-\infty, \infty)$ of the finite linear combinations of the functions $\phi(x)\exp(ix\xi)$, $\xi\in X$. A sufficient condition is obtained for $M_p(\phi, X)=M_p(\phi, R)$. As an example, if $\phi(x)\exp(|x|/\log(|x|+e))\in L_p(-\infty, \infty)$ and if $\limsup_{t\to\infty}(2/\pi t)\log\log X(t)>1$, then $M_p(\phi, X)=M_p(\phi, R)$. These results depend upon the distributions of the zeros of quasi-analytic functions. (Received January 7, 1952.)

433t. Ernest Ikenberry and W. A. Rutledge: *Convergence of expansions in the Hermite polynomials $H_n(hw)$.*

Hille has shown that an expansion of the form $f(s)=\sum_{n=0}^{\infty}f_nH_n(s)$, $s=x+iy$, converges in a well defined strip $-\tau<y<\tau$ in the s-plane (Duke Math. J. vol. 5 (1939) pp. 875–936). In this paper, expansions of the form $g(w)=\exp(-kh^2w^2)\times\sum_{n=0}^{\infty}g_n(h, k)H_n(hw)$, where h and k may be complex numbers, are considered. The region of convergence in the w-plane is a strip of width $2\tau/|h|$, where $\tau=\tau(h, k)=-\limsup\{(1/(2n)^{1/2})\log(2^n\Gamma[n/2]|g_n(h, k)|)\}$. The equation $\tau(h, k)=0$, for a given k, defines a curve in the h-plane which divides this plane into two regions: C, the region in which $\tau>0$, giving values of h for which there exists a strip of convergence in the w-plane, and D, the region in which $\tau<0$, giving values of h for which there exists no strip of convergence. For the class of functions $P(w)$ exp $(-aw^2)$, where $P(w)$ is any entire function of order less than two, or of minimal type if of order two, the region C is the region in the k-plane for which $R(k-1-a/k^2)<-1/2$. An optimum value of k, for given k, is obtained by maximizing a function $K(k)$. (To be published in the Journal of Mathematics and Physics.) (Received March 10, 1952.)

434. J. R. Lee: *Addition theorems in abstract spaces.*

This paper discusses the existence and properties of solutions of addition theorems of the form $f_i(\xi_1+\xi_2)=G_i[f_1(\xi_1), \cdots, f_n(\xi_1); f_1(\xi_2), \cdots, f_n(\xi_2)]$, $i=1, \cdots, n$; where $f_i(\xi)$ are functions having values in an arbitrary Banach algebra with unit element, and G_i are analytic or, in some cases, rational functions. The existence of solutions depends on known scalar solutions $\phi_i/(\zeta)$, holomorphic in some neighborhood of $\zeta=0$ and with values in a domain Δ where $G_i[\alpha_1, \cdots, \alpha_n; \beta_1, \cdots, \beta_n]$ are holomorphic. The case for $n=1$ was discussed by N. Dunford and E. Hille [Bull. Amer. Math. Soc. vol. 53 (1947) pp. 799–805] where, also, the assumption that $\phi(0)$ belongs to the domain of holomorphism of $G(\alpha, \beta)$ was essential. Certain addition formulas that were excluded from consideration previously may now be handled by making use of a set of n formulas. (Received March 11, 1952.)

435. R. B. Leipnik: *Extension of sizes.*

If F is a set of subsets of A, let $\pi F=\bigcap_{B\in F} B$, $\sigma F=\bigcup_{B\in F}B$, $\delta F=\sigma F-\pi F$. Let R be the reals (or a lattice ordered abelian group). A function μ upon subsets of A to R is called a size if and only if $0\leq\mu(\delta F)\leq\sum_{B\in F}\mu(B)$ whenever $\delta F\in$domain μ and $F\subset$domain μ. By restricting suitably the domain of μ, the concepts measure, outer measure, dimension, metric, and generalized Menger metric are included. A general extension theorem is proved for sizes, using metric and measure methods. (Received March 14, 1952.)

436t. Josephine M. Mitchell: *The kernel function in the geometry of matrices.* II.

Let D be the domain defined by $Z\bar{Z}'<I$, where Z is a rectangular $m\times n$ matrix of complex numbers, $\bar{Z}'$ its conjugate transpose matrix, and I the identity matrix. We already have shown that the kernel function of the domain D is $K(Z, \bar{Z}')=V^{-1}[\det (I-Z\bar{Z}')]^{-m-n}$ $(Z\in D)$, where V is the Euclidean volume of D (Bull. Amer. Math. Soc. Abstract 57-4-343). Here the asymptotic behaviour of $K(Z, \bar{Z}')$ is studied as Z approaches a point Z_0 on the boundary of D. It is found that if $\det (I-Z_{0j}\bar{Z}'_{0j})=0$ $(r+1\leq j\leq m)$ but $\det (I-Z_{0r}\bar{Z}'_{0r})\neq0$, Z_{0j} being the matrix consisting of the first j rows of Z_0, then $\lim_{Z\to Z_0} d^{m+n}K(Z,\bar{Z}')=V^{-1}[2 \det (I-Z_{0r}\bar{Z}'_{0r})]^{-m-n}$, where d is a certain positive definite quadratic form in Z and $Z\to Z_0$ in a specified manner. The case in which Z_0 is a unitary matrix is also studied. Also an asymptotic form is obtained using the characteristic roots of the matrix $Z\bar{Z}'$ in place of $\det (I-Z_{0}\bar{Z}'_{0})$. Similar results hold for symmetric and skew symmetric matrices. (Received March 11, 1952.)

437. G. W. Morgenthaler: *A distribution theorem.*

Let $\{\phi_n(x)\}$ be any uniformly bounded system of orthonormal, real-valued functions on a finite interval $[a, b]$. Let $\{a_k\}$ be an arbitrary real sequence satisfying $a_N=o(A_N)$ where $A_N=(a_1^2+a_2^2+\cdots+a_N^2)^{1/2}\to\infty$. Then independently of $\{a_k\}$

there exists a subsequence $\{\phi_{n_k}(x)\}$ and a bounded function $f(x)$ such that for any set $E \subset [a, b]$, $m(E) > 0$, the distribution function of $A_N^{-1}\sum_{k=1}^{N} a_k\phi_{n_k}(x)$ on E tends to a limit distribution whose characteristic function is $(m(E))^{-1}\int_E \exp\{-(\lambda^2/2)f(x)\}dx$, as $N \to \infty$. (Received March 17, 1952.)

438*t*. G. W. Morgenthaler: *Theorems on series of Walsh functions.*

Let $\{\psi_n(x)\}$ be the Walsh orthonormal function defined on $[0, 1]$, and let $S_n(x)$ and $b_n(x)$ denote respectively the nth partial sum and $(C, 1)$ mean of the series $(S) \sum_0^\infty a_k\psi_k(x)$. By "$\dotplus$" is meant the operation of addition as defined by N. J. Fine, and $\{n_k\}$ shall always refer to a lacunary sequence $n_{k+1}/n_k > q > 1$, of positive integers. The following results, analogous to well known results for trigonometric series, can be established: (1) If $S_n(\xi) \to s$ and $0 \leq \alpha_n = O(1/n)$, then $S_n(\xi \dotplus \alpha_n) \to s$ provided the sequence $\{(\xi \dotplus \alpha_n)\}$ is free of dyadic rationals from some place on. (2) If (S) is the Fourier series of $f(x)$ in L, and $\bar{s}(x) = [\limsup_n S_n(x)] < \infty$, then $\underline{s}(x) = [\liminf_n S_n(x)] > -\infty$, and $f(x) = 2^{-1}\{\bar{s}(x) + \underline{s}(x)\}$ p.p. (3) If $\sum_0^\infty a_k^2 < \infty$, then there exists a "continuous" $f(x)$ (i.e. $|f(x \dotplus t) - f(x)|$ is small with $t > 0$) corresponding to each sequence $\{n_k\}$ such that at the place n_k the Fourier coefficient of $f(x)$ is a_k. (4) If the partial sum of the series $\sum_{k=0}^\infty a_k\psi_{n_k}(x)$ oscillate finitely at each point of an interval, then $\sum_{k=0}^\infty |a_k| < +\infty$. (5) If $\sum_{k=0}^\infty a_k\psi_{n_k}(x)$ converges (or is only summable by a Toeplitz method) on a set E, $m(E) > 0$, then $\sum_0^\infty a_k^2 < \infty$. (6) The distribution function of $A_N^{-1}\sum_{k=0}^{N} a_k\psi_{n_k}(x)$ relative to a fixed set E, $m(E) > 0$, converges to the Gaussian distribution with mean value zero and unit dispersion, provided $a_N = o(A_N)$, $A_N = (a_1^2 + a_2^2 + \cdots + a_N^2)^{1/2}$. (Received March 17, 1952.)

439. W. M. Myers: *A functional associated with a continuous transformation.*

Let T: $z = t(w)$, $w \in R_0$, be a continuous transformation from a simply-connected polygonal region R_0 in the Euclidean plane into Euclidean three-space. Then the transformation T is a representation for a surface S. T. Radó (*Length and area*, Amer. Math. Soc. Colloquium Publications, vol. 30, 1948) defines a functional $a(T)$, which he shows is independent of the representation T for the surface, and he calls $a(T)$ the lower area of the surface. In defining $a(T)$, Radó uses finite collections of disjoint domains. Let $a_1(T), \cdots, a_5(T)$, respectively, denote the functionals which arise when finite collections of (1) disjoint simply-connected polygonal regions, (2) disjoint finitely-connected polygonal regions, (3) simply-connected Jordan regions, with disjoint interiors, (4) finitely-connected Jordan regions, with disjoint interiors, and (5) disjoint simply-connected domains, are used rather than finite collections of disjoint domains in the definition of $a(T)$. It is shown that $a_j(T) = a(T)$, $j = 1, \cdots, 5$, and, consequently, any of the above types of collections of sets may be used in place of finite collections of disjoint domains in defining the lower area $a(T)$. (Received March 10, 1952.)

440. P. C. Rosenbloom: *Perturbation theory for linear operators in a Banach space.*

Let T_0 be a linear transformation of the Banach space X into itself, let x_0 be an eigen-vector corresponding to the eigen-value λ_0, and let x_0^* be an eigen-vector of the adjoint transformation T_0^* with the same eigen-value. Let X_1 be the set of x such that $x_0^*(x) = 0$, and suppose that $T_0 - \lambda_0 I$, as a transformation of X_1 into itself, has a bounded inverse. It is shown that if U is a bounded linear transformation of X into itself such that $\|U\|$ is sufficiently small, then there exist a unique eigen-vector $x(U)$ and corresponding eigen-value $\lambda(U)$ of the transformation T_0+U which are close to x_0 and λ_0 respectively and such that $x_0^*(x(U))=x_0^*(x_0)$. Effective methods for computing $x(U)$ and $\lambda(U)$ are given and the rate of convergence is estimated explicitly. These functions have Fréchet differentials at $U=0$; the formulas for these contain as special cases those of Born, Heisenberg, and Jordan for Hermitian operators in Hilbert space. Explicit estimates are obtained for the error in using these formulas for approximations to $x(U)-x_0$ and $\lambda(U)-\lambda_0$. These results are related to recent results of L. V. Kantorovich and F. Wolf. (Received March 12, 1952.)

441. P. C. Rosenbloom and S. E. Warschawski: *Polynomial approximation to solutions of variational problems in the complex domain.* Preliminary report.

The mapping function of a simply-connected domain bounded by the curve C can be characterized by the variational problem: to minimize $\int_C|f'(z)|^2ds$ where f ranges over the class of functions in $H(2, C)$ such that $f(z_0)=0$, $f'(z_0)=1$. Let $P_n(z)$ be the extremal polynomial when the competing functions are restricted to the polynomials of degree not greater than n. We obtain estimates for the rate of convergence of P_n to f for a very wide class of curves C. This problem is a special case of the best approximation to a given function $f(z)$ in $H(2, C)$ in the sense of L_2 on C by polynomials of degree not greater than n. We obtain an estimate for the rate of convergence in terms of the rate of convergence of the ordinary Fourier expansion of $f[\psi(e^{i\theta})](\psi'(e^{i\theta}))^{1/2}$ on $0\leq\theta\leq 2\pi$, where $\psi(w)$ maps $|w|<1$ onto the exterior of C. This result can be extended to the class $H(p, C)$, $p>1$. In these results mild smoothness conditions are imposed on C. Analogous results, however, are also derived for polygonal domains. It should be emphasized that all estimates are given explicitly in terms of the geometrical properties of C. (Received March 12, 1952.)

442. W. C. Royster: *Convexity and starlikeness of analytic functions.*

Necessary and sufficient conditions are found such that a function $f(z)$ regular on an arbitrary arc Γ having a continuously turning tangent maps Γ onto a curve which is convex or starlike with respect to a given point. Some consequences of these conditions are obtained for certain families of curves in the z-plane. (Received February 11, 1952.)

443. V. L. Shapiro: *Square summation and localization of double trigonometric series.*

Let $T_1=\sum_{m,n}a_{mn}e^{i(mx+ny)}$ and $T_2=\sum_{m,n}b_{mn}e^{i(mx+ny)}$ be two double trigonometric series whose coefficients a_{mn} and b_{mn} are $o(1/(|m|+1)^\lambda(|n|+1)^\eta)$, $0\leq\lambda+\eta\leq 1$, $\max(\lambda, \eta)<1$, $\lambda\eta\geq 0$. Associate to each series its Riemann function $F_1(x, y)$ and $F_2(x, y)$ obtained by formal double integrations. Thus $F_1(x, y)=a_{00}x^2y^2/4-(y^2/2)\cdot\sum_m'(a_{m0}/m^2)e^{imx}-(x^2/2)\sum_n'(a_{0n}/n^2)e^{iny}+\sum_{m,n}''(a_{mn}/m^2n^2)e^{i(mx+ny)}$. Designate the square partial sums of T_1 by $S_R=\sum_{\max(|m|,|n|)\leq R}a_{mn}e^{i(mx+ny)}$. The principal result of this paper is that if F_1 and F_2 are equal in any closed domain contained in the interior of the fundamental square Ω, then T_1 is square equisummable $(C,(1-(\lambda+\eta)))$ with T_2 uniformly in any interior closed sub-domain. The method of proof is based on the notion of formal products as developed by Rajchman and Zygmund. This paper also shows that equisummability cannot be carried beyond $(C,1)$ through the use of formal products, and consequently that localization with square summation is not comparable to that using circular summation. (Received March 12, 1952.)

444t. V. L. Shapiro: *Summability of double trigonometric integrals.*

Let ϕ be a complex-valued additive function of a set (B) on every figure in the plane. Furthermore let the total variation of ϕ over a circle of radius 1/2 with center at (u, v) be $o((u^2+b^2)^{\gamma/2})$, $\gamma > -1$. The first principal result of this paper is that there exists a trigonometric series $T=\sum_{m,n} a_{mn}e^{i(mx+ny)}$ with coefficients $a_{mn} = o((m^2+n^2)^{\gamma/2})$ such that T is circularly equisummable $(C, \gamma+1)$ with the integral $\int_{E_2} e^{i(xu+yv)}d\phi(u, v)$. The second result concerns itself with square equisummability where the total variation of ϕ over a square of side 1 with center at (u, v) is $o(1/(|u|+1)^\lambda(|v|+1)^\eta)$ and where $a_{mn}=o(1/(|m|+1)^\lambda(|n|+1)^\eta)$, $0\leq\lambda+\eta\leq 1$, $\max(\lambda, \eta)<1$, $\lambda\eta\geq 0$. It is shown that T is square equisummable $(C, 1-(\lambda+\eta))$ with the integral $\int_{E_2} e^{i(xu+yv)}d\phi(u, v)$. The method of proof involves the notions of formal products and of convolutions of set functions and follows the pattern set forth by Zygmund in his paper *On trigonometric integrals* where analogous results for the real line were first obtained. (Received March 12, 1952.)

445t. M. F. Smiley: *Right H^*-algebras.*

Following a suggestion of H. T. Muhly, Ambrose's definition of H^*-algebras (Trans. Amer. Math. Soc. vol. 57 (1945) pp. 364–386) is weakened by requiring that only the right multiplications R_a have adjoints R_{a^*}. The resulting system is called a *right H^*-algebra A*, and A is called *proper* in case A has no nonzero right annihilator and the mapping $x \to x^*$ is continuous. Examples of proper right H^*-algebras which are not H^*-algebras are easily given by introducing an equivalent norm in a (full) matric H^*-algebra. Conversely, minor modifications of the proof of Ambrose show that every proper right H^*-algebra A is obtained from a proper H^*-algebra A_1 by introducing an equivalent norm in each of the simple components of A_1. Hence proper right H^*-algebras are *dual* in the sense of Kaplansky (Ann. of Math. vol. 49 (1948) pp. 689–701). (Received March 13, 1952.)

446t. C. T. Taam: *On nonoscillatory differential equations.*

Let $f(x)$ belong to $L(0, R)$ for every large R. Equation (1) $y''+f(x)y=0$ is called disconjugate in an interval I if each solution $(\neq 0)$ of (1) has not more than one zero in I. If I is either closed or open, and if open it need not be bounded, it is proved in this paper that (1) is disconjugate in I if and only if there exists some function $\lambda(x)$ which is absolutely continuous on every closed interval contained in I and satisfies $\lambda'+\lambda^2 \leq -f(x)$ almost everywhere in I. Using this criterion the following main result can be easily proved: Assuming that $f(x)$ and $g(x)$ belong to $L(a, \infty)$ and letting (1′) be the equation if $f(x)$ in (1) is replaced by $g(x)$, if (1) is nonoscillatory in I $(a<x<\infty)$ and if $\int_x^\infty f(x)dx \geq \left|\int_x^\infty g(x)dx\right|$ for $x>a$, then (1′) is also nonoscillatory in I. This is a generalization of a theorem of E. Hille (*Non-oscillation theorems*, Trans. Amer. Math. Soc. vol. 64 (1948) pp. 234–252). An explicit sufficient criterion is given. Also a self-adjoint differential equation of the third order is discussed. (Received March 11, 1952.)

447. C. T. Taam: *The boundedness of the solutions of a differential equation in the complex domain.*

Consider (1) $W''+Q(z)W=0$, where $Q(z)$ is analytic. Let x be real and write $Q(x)=q_1(x)+iq_2(x)$, $q_1(x)$ and $q_2(x)$ being real-valued. Further set $\phi(x)=\int_0^x[|a-q_1(x)| + |q_2(x)|]dx$, where $a>0$. In this paper it is proved that along the positive real axis every solution $W(z)$ $(\neq 0)$ of (1) satisfies $W(x)=O(\exp[2^{-1}a^{-1/2}\phi(x)])$ and $\limsup |W(x)| \exp[2^{-1}a^{-1/2}\phi(x)]>0$ as $x\to\infty$. Hence every solution $W(x)$ is bounded on the positive real axis if there exists a positive constant a such that $\phi(x)$ is bounded. If $Q(x)$ is real, these reduce to the same results as obtained by N. Levinson (*The growth of the solutions of a differential equation*, Duke Math. J. vol. 8 (1941) pp. 1–10). It is also shown that $W(x)$ behaves like sine function for large x if $\phi(x)$ is bounded. Applying these results to certain regions in the z-plane, sufficient conditions for the boundedness of the solutions $W(z)$ in these regions are then obtained. (Received March 11, 1952.)

448*t*. D. R. Waterman: *A convergence theorem for Dirichlet series.*

It is known that if $g(z)=\sum c_n z^n$ is analytic in $|z|<1$ and maps a domain in the interior of the circle with an arc $(\alpha\leqq\theta\leqq\beta)$ in its boundary into a region of finite area on the Riemann surface of g, then $\sum c_n e^{in\theta}$ is (C, k) summable, $k>0$, almost everywhere in (α, β) if $c_n=o(n^k)$. For $k=0$, this result of Zygmund is seen to be a localization of the celebrated "finite area" principle of Féjer. The condition $c_n=o(1)$ is necessary for the existence of one point of convergence. The same result may be proved for a Dirichlet series $f(s)=\sum a_n e^{-\lambda_n s}$ with the condition on c_n replaced by $\alpha(n)=\max_{0\leq h\leq 1}\left|\sum_{\lambda_n\leq\lambda_k\leq\lambda_{n+h}} a_k\right|=o(n^k)$. For convergence the condition $\alpha(n)=o(1)$ is shown to be necessary. The principal tool is Zygmund's theory of formal multiplication of integrals which is used to reduce the problem in the half-plane to the problem in the circle. (Received March 13, 1952.)

449. D. R. Waterman: *On an integral of Marcinkiewicz.*

If $\phi(z)$ is of class H_p, $p>0$, one may define $g(\theta)=\{\int_0^1(1-\rho)|\phi'(\rho e^{i\theta})|^2 d\rho\}^{1/2}$, $g^*(\theta)=\{(1/\pi)\int_0^1(1-\rho)d\rho\int_0^{2\pi}\phi'(\rho e^{it})P(\rho, t-\theta)dt\}^{1/2}$ (P is, of course, the Poisson kernel for the circle), and $s(\theta)=\{\iint_{\Omega_\theta}|\phi'|^2 d\omega\}^{1/2}$ where Ω_θ is the rotation (by angle θ) of a region in the interior of the circle and contained between two chords meeting at $z=1$ to form an angle γ. If $f(\theta)$ is of class L_p, $p>1$, one defines $\mu(\theta)=\{\int_0^\pi([F(\theta+t)+F(\theta-t)-2F(\theta)]^2/t^3)dt\}^{1/2}$ where $F(\theta)$ is any primitive of $f(\theta)$. For $g(\theta)$ Littlewood and Paley showed $\|g\|_p\leq A_p\|\phi\|_p$, $p>0$. For $g^*(\theta)$ they proved an analogous result but for $p\geqq 2$ an integral. This was extended to $p>1$ by Zygmund who also gave this result for the Marcinkiewicz function $\mu(\theta)$ and a result analogous to that on g for $s(\theta)$. Here these results are extended in the same form to functions similarly defined for the Hille-Tamarkin class $\mathcal{H}_p$ in the half-plane. (Received March 13, 1952.)

APPLIED MATHEMATICS

450*t*. H. D. Block: *Laws of attraction having a certain generalised Newtonian property.*

Consider a central force law having an associated potential energy of interaction $V=m_1m_2v(r)$ for two point masses of mass m_1 and m_2 a distance r apart, with $v(r)\in C^2$ for $r>0$. The following question is answered. Which of these force laws have the property that two uniform (or more generally, radially symmetric) spheres of radii a and b have as potential energy of interaction $\phi(a, b)m_1m_2v(r)$, where m_1 and m_2 are the masses of the spheres, r is the distance between the centers, and ϕ is a suitable function of the radii. If one specifies in addition that the energy vanish at infinity, or that it be monotone and bounded at infinity, then one finds that only the Coulomb and the Yukawa meson potentials have the required property. The details will appear in the Journal of Mathematics and Physics. (Received March 10, 1952.)

451. H. D. Brunk and G. M. Ewing: *The approximation of double integrals by means of line integrals.*

Given an integral $I = \iint g(r, \theta) dr d\theta$, it is shown that a certain line integral I_α^* along a suitable spiral Γ_α of parameter α converges to I as $\alpha \to \infty$. Conditions on the set R over which I is taken, on the integrand g, and on the class of spirals Γ_α which will serve are such as to generalize a result of J. E. Wilkins, Jr. (Bull. Amer. Math. Soc. vol. 55 (1949) pp. 191–192) in all of these directions. In case $g(r, \theta) = p(r)q(\theta)$ and the spiral is $\theta = \alpha \int_0^r p(u) du$, useful bounds on the error $|I - I_\alpha^*|$ are obtained. (Received March 11, 1952.)

452. A. S. Cahn and D. S. Saxon: *Modes of vibration of a suspended chain.*

A method is described for calculating the characteristic frequencies of a suspended inextensible chain vibrating with small amplitude in the plane of the catenary forming the equilibrium configuration. An asymptotic solution of the linearized equations of motion is obtained such that the accuracy of the results increases as the mode number increases and/or as the catenary becomes flatter. (Received February 25, 1952.)

453. H. E. Fettis: *Some properties and applications of the incomplete Bessel function.* Preliminary report.

The function $j_p(x, \theta)$ defined by $j_p(x, \theta) = \int_0^\theta e^{ix \cos \phi} \cos p\phi d\phi$ has recently become important in the theory of unsteady air forces in compressible media at subsonic speeds. The Bessel function of integral order, $J_p(x)$, is found as a special case when $\theta = \pi$: $j_p(x, \pi) = \pi(i)^p J_p(x)$. Various properties of the function $j_p(x, \theta)$, such as differential relations and recurrence formulae, analogous to those already known for the Bessel function, are derived. Some additional functions related to $j_p(x, \theta)$ are also discussed, and the application of these functions in nonstationary subsonic airfoil theory is indicated. (Received January 8, 1952.)

454. Fulton Koehler: *Estimation of errors in the Rayleigh-Ritz method for eigenvalue problems.*

Let a positive definite, self-adjoint eigenvalue problem be given by the differential equation $Ly = \lambda y$, with boundary conditions. Consider the two variational problems of minimizing the expressions $\int \phi L \phi dP$ and $\int (L\phi)^2 dP$ where ϕ is a linear combination of the first n admissible functions of a given set $\psi_1, \psi_2, \cdots$, and $\int \phi^2 dP = 1$. The roots of the characteristic equations of these problems are denoted by $\mu_1 \leqq \mu_2 \leqq \cdots \leqq \mu_n$ and $\nu_1^2 \leqq \nu_2^2 \leqq \cdots \leqq \nu_n^2$; and one has $\lambda_k \leqq \mu_k \leqq \nu_k$, $k = 1, 2, \cdots, n$, where λ_k is the kth eigenvalue. Let $G(P, Q)$ be the Green's function for the equation $Ly = 0$ with the given boundary conditions and let $K(P, Q) = \int G(P, X) G(X, Q) dX$. One then has $\lambda_k^{-2} - \epsilon_n \leqq \nu_k^{-2} \leqq \mu_k^{-2} \leqq \lambda_k^{-2}$, where $\epsilon_n^2 = \iint (K - A)^2 dP dQ$, and A is the best approximation to K in the L^2 sense by a sum of the form $\sum_{i,j=1}^n a_{ij} L\psi_i(P) L\psi_j(Q)$. If $f_k(P; n)$ and $g_k(P; n)$ are the approximate eigenfunctions corresponding to the roots μ_k and ν_k^2, each will differ in the mean from an eigenfunction corresponding to λ_k by a constant, depending only on k, times ϵ_n. The same is true in the sense of uniform approximation for the function g_k if $\int G^2(P, Q) dQ$ is bounded; and it is true for the function f_k, with ϵ_n replaced by $\epsilon_n^{1/2}$, if $G(P, Q)$ is bounded. The results stated can be extended in slightly modified form to problems of the type $Ly = \lambda \rho y$, where $\rho > 0$. (Received March 14, 1952.)

455. M. Z. Krzywoblocki: *On isotropic turbulence in magneto-hydrodynamics of compressible fluids.*

Batchelor and next Chandrasekhar originated and developed the theory of isotropic turbulence in magneto-hydrodynamics of incompressible media. In the present paper the author derives the equations of isotropic turbulence in magneto-hydrodynamics of compressible fluids under the assumption that all the stationary random variables are dependent. (Received February 11, 1952.)

456*t*. M. Z. Krzywoblocki: *On particular integrals of a system of nonlinear partial differential equations.*

A system of nonlinear partial differential equations of arbitrary order and degree by means of a suitable transformation function is transformed into a system of ordinary differential equations. The latter one can be solved by means of computing devices. The proposed method theoretically allows to solve any system of such equations. The only limitations from a practical standpoint are due to the capacity of the existing computing devices. (Received February 11, 1952.)

457. Daniel Resch: *Some Baecklund transformations of second order partial differential equations.*

Consider two surfaces $z=z(x, y)$ and $z'=z'(x', y')$ and four given relations among the first-order elements of these surfaces. If it is possible to find a one-to-one mapping between these surfaces so that the given relations are satisfied at corresponding points, then the relations constitute a Baecklund transformation between the surfaces. If the unprimed quantities and alternatively the prime quantities are eliminated from the given relations, it is possible, in general, to obtain also a correspondence between a partial differential equation in $z(x, y)$ and one in $z'(x', y')$. C. Loewner, *A transformation theory of the partial differential equations of gas dynamics*, NACA TN 2065 (1950) extended the problem to show that systems of partial differential equations can be similarly connected. In the present paper, similar methods are used to find extended Baecklund transformations that will connect second order partial differential equations in three independent variables. If the given relations are linear in the dependent variables z and z', and their first-order derivatives, then they constitute, with certain restrictions on the coefficients of the linear expressions, a Baecklund transformation of a class of hyperbolic partial differential equations into the wave equation in two spacial dimensions. The transformation is in general different from a coordinate transformation. (Received March 11, 1952.)

GEOMETRY

458. E. F. Allen: *A nine-point conic of a triangle.*

A generalization of the nine-point circle configuration is obtained by setting $z=x+ry$, instead of $z=x+iy$ in the inversive geometry of Morley and Morley. It is shown that a nine-point ellipse of a triangle inscribed in an ellipse can be obtained by a general interpretation of the coordinate z as well as by a projective transformation. The line of images of a point on a circle is shown to be a special case of the line of images of a point on a central conic. Perpendicular lines are particular cases of conjugate lines. (Received March 10, 1952.)

459. Gerald Berman: *Finite projective plane geometries and difference sets.*

Let S be the set of residue classes of integers modulo $q=m^2+m+1$, let d_i $(i=0, 1, \cdots, m)$ be a difference set modulo q, and let L_i be the subset of S containing the $m+1$ residue classes corresponding to d_j+i $(j=0, 1, \cdots, m)$. James Singer [*A theorem in finite projective geometry and some applications to number theory*, Trans. Amer. Math. Soc. (1938) pp. 377–385] showed that if the elements of S are taken to be points with collinearity defined by the sets L_i $(i=0, 1, \cdots, q-1)$, then S is a projective plane. In this paper the difference sets corresponding to $PG(2, m)$ where $m=p^n$ (p a prime integer) are studied with the aid of Galois fields, and simple constructions are given for them. The following two conjectures made by Singer are proved: (i) The difference set td_i $(i=0, 1, \cdots, m)$, where $(t, q)=1$, is equivalent to d_i $(i=0, 1, \cdots, m)$ if and only if t is a power of p modulo q. (ii) If d_i $(i=0, 1, \cdots, m)$ and d_i' $(i=0, 1, \cdots, m)$ are any two difference sets modulo q, there exists an integer t for which td_i $(i=0, 1, \cdots, m)$ is equivalent to d_i' $(i=0, 1, \cdots, m)$. This shows that there are exactly $\phi(q)/3n$ inequivalent difference sets corresponding to $PG(2, m)$. (Received March 13, 1952.)

460*t*. D. G. Bourgin: *Separation of convex sets.*

Let E_n be a simplex in R_n. Results are obtained for the separation of pairs of simplexes in R_n by hyperplanes parallel to the faces of E_n. (Received March 13, 1952.)

461*t*. V. G. Grove: *On generalized curvatures.*

Springer and Dekker generalized the geodesic curvature of a curve on a surface. These are called the union curvature, and the hypergeodesic and supergeodesic curvatures. These curvatures are made to depend upon the union curves of a congruence, or hypergeodesic or supergeodesic curves according to the kind of curvatures. We show that these curvatures are but special cases of a more generalized curvature which may be developed with no use whatever of union curves, or hypergeodesic curves or supergeodesic curves. In addition a new curvature, called by us the relative curvature K_λ, is found. In particular if the relative curvature of a curve vanishes at a point, the principal normal of the curve coincides with the line λ with respect to which the relative curvature is defined. Moreover this relative curvature, and in particular the curvatures of Springer and Dekker, may be found from the geodesic curvature of the curve and the normal curvature of the surface in the direction of the tangent to the curve by simple projections. (Received February 18, 1952.)

462*t*. V. G. Grove: *The quadric of Lie.*

A geometrical characterization of a generic point g of the quadric of Lie of a surface S at a point x is found as follows. Let the asymptotic curves and tangents be called the u- and v-curves and u- and v-tangents. As is known, reciprocal lines l_1, l_2 may be defined without the use of the quadrics of Darboux. The line l_2 intersects the u- and v-tangents in points r and s respectively. The line l_1 intersects the loci of r and s as x generates the v-curve and u-curve respectively in points s_1, s_2. Let k be the harmonic conjugate of x with respect to s_1, s_2. Let I be the harmonic conjugate of x with respect to the focal points on l_1. The generic point g is the harmonic conjugate of x with respect to k and I. The locus of g is the quadric of Lie. (Received February 18, 1952.)

463*t*. P. C. Hammer and Andrew Sobcyzk: *Semispaces in E_n*.

Let x be a point in E_n, $n \geqq 1$, and let E_n, $E_{n-1}, \cdots, E_0 = x$ be a descending set of linear varieties with dimensions given by the subscripts. Let H_k be one of the two k-dimensional relatively open halves of E_k separated by E_{k-1}, for $k = 1, 2, \cdots, n$. The union $S = H_1 \cup H_2 \cup \cdots \cup H_n$ is called a *semispace at x* or merely a *semispace*. Then the following results are obtained. (1) Every semispace is convex. (2) The reflection of a semispace S at x through x is a semispace, called the *complementary semispace* of S. (3) If x is in the complement of a convex set C, then there is a semispace at x which contains C. (4) Every convex set which is not E_n is the intersection set of a class of semispaces. (Received March 12, 1952.)

464*t*. A. R. Schweitzer: *Theorems in Grassmann's extensive algebra*.

Let $X = x_1E_1 + x_2E_2$ and $Y = y_1E_1 + y_2E_2$ where E_1 and E_2 are perpendicular unit vectors. Then the following theorems are derived from the relation: $\sigma \cdot Q(Y, X) = x_1y_1 + x_2y_2 + (x_1y_2 - x_2y_1)Q(E_2, E_1)$ where $\sigma = x_1^2 + x_2^2$ and $Q(Y, X)$ is an operator on X which transforms X into the vector Y. I. If Y and X have equal lengths, then $Q(Y, X) = \cos(Y, X) + \sin(Y, X) \cdot Q(E_2, E_1)$. That is, $Q(Y, X) = e^{i\theta}$ where $i = Q(E_2, E_1)$ and $\theta = (Y, X)$, the angle measured from X to Y. Also $Q(Y, X)$ is equal to $Q(Y, E_1)$ divided by $Q(X, E_1)$ where $Q(Y, E_1) = y_1 + y_2Q(E_2, E_1)$ and $Q(X, E_1) = x_1 + x_2Q(E_2, E_1)$. II. If $Q(Y, X) = -Q(X, Y)$, then X and Y are perpendicular and have equal lengths. In the preceding it is assumed that the length of a linear vector is positive. (Received March 10, 1952.)

Logic and Foundations

465*t*. A. R. Schweitzer: *A classification of mathematics*.

The content of mathematics is separated into three categories: I. Pure mathematics, including arithmetic of number systems, algebra, and analysis; II. Applied mathematics, including geometry, topology, Grassmann's extensive algebra, physics, dynamics, and relativity theories due to Minkowski, Einstein, and Eddington; III. Speculative mathematics, including foundations, logic, and philosophy. These categories are not mutually exclusive. Thus Grassmann's algebra may be interpreted (1) as a theory on number systems, (2) as relevant to the foundations of geometry, (3) as including vector calculus, the mathematics of engineering, and tensor calculus, the mathematics of relativity. In the above classification "analysis" includes theories on functional equations such as difference equations, differential equations, integral equations, integro-differential equations, and the author's equations in iterative composition of functions (including "quasi-transitive" equations). A type of functional equations is fundamental for the calculus of variations. (Received March 10, 1952.)

466*t*. A. R. Schweitzer: *A link between the history of mathematics and the history of philosophy*.

In ancient philosophy number mysticism is prominent. Thus in the Pythagorean school the "tetractys" was considered sacred (compare Burnet, *Early Greek philosophy*, London, 1908, p. 113). In the philosophy of Empedocles reference is made to the four roots of all things (Burnet, loc. cit., p. 264). Also in the Scriptures the number four is frequently mentioned; see, for example, Genesis 2 (10), 13(14); Exodus 28(16), 20(2); II. Samuel 12(6); Job 11(18), (19); Proverbs 30(15), (18); Ezra 1(5), (6); Daniel 7(2), (3); Revelations 4(6), 6(1)–(8), 7(1), 14(7), 21(16); Ephesians 3(18); Luke 19(8). On the mathematical side P. G. Tait (Encyclopedia

Britannica, 11th ed., vol. XXII, p. 718) remarks, "The word 'quaternion' properly means 'a set of four.' In employing such a word to denote a new mathematical method Sir W. R. Hamilton was probably influenced by the recollection of its Greek equivalent, the Pythagorean Tetractys (τετρακτύς, the number four) the mystic source of all things." (Received March 10, 1952.)

Topology

467*t*. D. G. Bourgin: *Topologies for complexes.*

The simplicial complex K is required to be the union of all its n skeletons. Define the full topology of a product space by an open base composed of products of open sets, one in each space. Let K_1 be the barycentric derived of K. If K is imbedded in a product space of closed intervals one for each vertex and the full topology is supposed for the product space, the induced topology in K may be referred to as the convex topology. If K_1 is similarly imbedded and the topology on K is determined by the natural map from the convex K_1 to K, the induced topology may be shown equivalent to the weak topology. The order of fineness is given by geometric, natural, convex, weak. With the aid of the imbedding the key theorem is established that in the weak topology K is paracompact. (Received March 13, 1952.)

468. E. R. Fadell: *Unessential identifications in singular homology theory.*

Let M denote a Mayer complex and M' a subcomplex of M. Radó (T. Radó, *An approach to singular homology theory*, Pacific Journal of Mathematics vol. 1 (1951) pp. 265–290) terms the subcomplex M' an unessential identifier for M provided the natural homomorphism $\pi: M \to M/M'$ induces an isomorphism onto between the corresponding homology groups. Let R denote the singular complex introduced by Radó (T. Radó, loc. cit.), S the singular complex introduced by Eilenberg (S. Eilenberg and J. A. Zilber, *Semi-simplicial complexes and singular homology theory*, Ann. of Math. vol. 51 (1950) pp. 499–513) and $\sigma: R \to S$ the natural chain mapping from R to S. Furthermore, let β^R, β^S denote the barycentric homomorphisms of R and S respectively. The main purpose of this note is to show that the nuclei of $\sigma\beta^R$, β^S form unessential identifiers for R and S respectively. The first of these results is an improvement over certain results obtained by Radó and Reichelderfer (T. Radó, loc. cit., P. V. Reichelderfer, *On the barycentric homomorphism in a singular complex.* To appear in the Pacific Journal of Mathematics). (Received January 17, 1952.)

469*t*. P. C. Hammer and Andrew Sobczyk: *Generalized closure of sets.* Preliminary report.

Let M be a set of points and let $\mathfrak{S}_1$ and $\mathfrak{S}_2$ be classes of subsets of M neither of which includes the null set. Let T be a transformation with domain $\mathfrak{S}_1$ and range $\mathfrak{S}_2$. A subset C of M is called *T-closed* if and only if $X \in \mathfrak{S}_1$, $X \subset C$ implies always that $T(X) \subset C$. The complement of a T-closed set is called a *T-open set*. The following properties follow: (1) The set M and the null set are T-closed and T-open. (2) The intersection set of a class of T-closed sets is T-closed; the union set of a class of T-open sets is T-open. (3) There exists a unique minimal T-closed set containing any subset X of M. This set is called the *T-closure* of X and is designated $\overline{X}$. (4) For every pair of subsets X, Y of M, $\overline{X \cup Y} \supset \overline{X} \cup \overline{Y}$. With proper limitations on M and definitions of $\mathfrak{S}_1$, $\mathfrak{S}_2$, and T, one may obtain ordinary closure, symmetry of any kind, and convexity as special instances. Relations are studied between T-closure and other methods of defining a semi-topology in a set (for examples, suppression of some of the Hausdorff axioms for open sets or neighborhoods, weakening of the requirements for a partial-order topology). (Received March 12, 1952.)

470. S. D. Liao: *On secondary obstruction of sphere bundles*. Preliminary report.

Let $\mathfrak{B}$ be an orientable sphere bundle with (i) base space B which is a finite connected complex, (ii) total space E, (iii) fiber S^n, and (iv) projection ϕ. Consider the Gysin's sequence: $\cdots \xrightarrow{\rho} H^n(B, Z) \xrightarrow{\phi^*} H^n(E, Z) \xrightarrow{\sigma} H^0(B, Z) \xrightarrow{\rho} H^{n+1}(B, Z) \xrightarrow{\phi^*} \cdots \xrightarrow{\rho} H^p(B, Z) \xrightarrow{\phi^*} H^p(E, Z) \xrightarrow{\sigma} H^{p-n}(B, Z) \xrightarrow{\rho} \cdots$, where $Z =$ the group of integers. Suppose that the primary obstruction of $\mathfrak{B}$ vanishes. Then σ is a homomorphism onto and ϕ^* is an isomorphism into. Let f be a cross section of $\mathfrak{B}$ over B^{n+1} ($=$ the $(n+1)$-skeleton of B). Then the secondary obstruction $Z^{n+2}(f)$ is defined. Let $c \in H^n(E, Z)$ be such that $\sigma(c) =$ the unit of the cohomology ring $H(B, Z)$. Then $\theta(f) = c - \phi^* f^*(c) \in H^n(E, Z)$ is defined independently of the choice of c and is a homotopy invariant of f over B^n. The main aim of this paper is to show that (i) $\phi^* Z^{n+2}(f) = \phi^* w^2 \cup \theta(f) - Sq^2\theta(f)$ for $n > 2$ where w^2 is the 2nd Whitney characteristic class of $\mathfrak{B}$; (ii) $\phi^* Z^{n+2}(f) = \theta(f) \cup T^*\theta(f)$ for $n = 2$ where T is the 2-periodic map of E induced naturally by the 2-periodic map of S^2 that sends each point to its antipodal. The corresponding homotopy classification theorems of cross sections over B^{n+1} are also obtained. (Received March 10, 1952.)

471*t*. G. R. Livesay: *Two theorems on the two-sphere*.

(1) Let $S = \{(x, y, z) \in E_3 \mid x^2 + y^2 + z^2 = 1\}$, and let K be a simplicial subdivision of S such that if σ is a simplex of K, then the image of σ under the map p, a reflection in the center of S, is another simplex of K. Let M and N be closed subcomplexes of K such that if $\sigma_M \in M$, $\sigma_N \in N$, then $p\sigma_M \in M$, $p\sigma_N \in N$, and $M + N = K$. If M has no component containing two diametrically opposite points of S, then there are two diametrically opposite points in a component of N. (2) If θ is any angle such that $0 \leq \theta \leq \pi/2$, and f is a mapping of S into the reals, then there are two diameters of S, forming an angle θ, at the four end points of which f has the same value. This extends a theorem of F. J. Dyson [Ann. of Math. vol. 54 (1951)]. (Received January 18, 1952.)

472. Katsumi Nomizu: *On the cohomology of compact homogeneous spaces of nilpotent Lie groups*.

It is known by A. Malcev that any compact homogeneous space of a connected nilpotent Lie group can be expressed in the form $M = G/D$ where G is a simply connected nilpotent Lie group and D a discrete subgroup. The first purpose of this paper is to prove that the cohomology algebra of M (with real coefficients) is isomorphic with the cohomology algebra of the Lie algebra $\mathfrak{g}$ of G. (The isomorphism of the first and second cohomology groups has been shown by Y. Matsushima, Nagoya Math. J. vol. 2 (1951).) From this theorem some results on the Euler characteristic and Betti numbers of M, which are analogous to the known results on homogeneous spaces of compact Lie groups, are obtained. The second purpose is to show that the complex of G-invariant differential forms on M is isomorphic with the complex of invariant cochains of $\mathfrak{g}$. From these results an example of a compact homogeneous space of a noncompact Lie group for which the theory of invariant integrals of E. Cartan does not hold is obtained. (Received March 11, 1952.)

J. W. T. YOUNGS,
Associate Secretary

THE MAY MEETING IN FRESNO

The four hundred eighty-first meeting of the American Mathematical Society was held at Fresno State College, Fresno, California, on Saturday, May 3, 1952. Approximately 75 persons attended, including the following 59 members of the Society:

H. L. Alder, H. M. Bacon, W. G. Bade, A. K. Bell, L. D. Berkovitz, E. W. Beth, F. C. Biesele, W. W. Bledsoe, L. M. Blumenthal, F. H. Brownell, J. G. van der Corput, P. H. Daus, Douglas Derry, D. J. Ewy, F. D. Faulkner, G. E. Forsythe, N. S. Free, R. M. Hayes, M. R. Hestenes, P. G. Hodge, P. J. Kelly, L. D. Kovach, Cornelius Lanczos, B. J. Lockhart, Charles Loewner, M. W. Maxfield, L. F. Meyers, E. D. Miller, F. R. Morris, T. S. Motzkin, Ivan Niven, L. J. Paige, T. K. Pan, R. S. Phillips, D. H. Potts, W. T. Puckett, F. M. Pulliam, C. H. Rawlins, Jr., R. M. Redheffer, J. B. Robinson, R. M. Robinson, Leo Sario, M. M. Schiffer, I. J. Schoenberg, Abraham Seidenberg, R. G. Selfridge, M. A. Shader, Max Shiffman, W. H. Simons, T. H. Southard, M. L. Stein, Robert Steinberg, C. B. Tompkins, J. L. Ullman, W. R. Wasow, M. A. Weber, František Wolf, H. H. Wolfenden, F. H. Young.

By invitation of the Committee to Select Hour Speakers for Far Western Sectional Meetings, Professor I. J. Schoenberg, of the Institute for Numerical Analysis, National Bureau of Standards, Los Angeles, delivered an address entitled *On smoothing operations and related topics*. Professor Schoenberg was introduced by Professor Ivan Niven. There were sessions for contributed papers in the morning and afternoon, presided over by Professors F. R. Morris, H. M. Bacon, and P. H. Daus. Following the meeting there was a guided tour of Pine Flat Dam, now under construction.

Following are abstracts of papers presented at the meeting. Papers with abstract numbers followed by "*t*" were presented by title. Paper number 490 was presented by Professor Steinberg, number 498 by Professor Kelly, and number 502 by Dr. Forsythe. Dr. Reid was introduced by Dr. J. W. Odle and Mr. Bennett by Professor F. C. Biesele.

ALGEBRA AND THEORY OF NUMBERS

473*t*. D. R. Morrison: *Boolean product rings.*

A *Boolean product* $R \circ B$ is defined for every ring R and every Boolean ring B, differing from the *tensor product* in that the left distributive law, $r \circ (b_1+b_2) = r \circ b_1 + r \circ b_2$, holds only if $b_1b_2 = 0$ or $2r = 0$. Every nonzero element of $R \circ B$ is uniquely expressible in the form $\sum_{i=1}^{n} r_i \circ b_i$, where $r_j \neq r_i \neq 0$, $b_i \neq 0$, $b_ib_j = 0$ for $j \neq i$. Where they are meaningful, the commutative and associative laws and the distributive laws with respect to weak direct sums apply to the Boolean product operation, and $GF(2)$ is a unit with respect to the operation. The idempotent Boolean ring $(R \circ B)^0$

(see Bull. Amer. Math. Soc. Abstract 58-1-52) is isomorphic to $R^* \circ B$. If R is biregular, so is $R \circ B$ and relations are determined between the ideals of R, B, and $R \circ B$. For any Boolean ring B, and any prime, p, $GF(p) \circ B$ is a p-ring. Conversely, for any p-ring R, R^* is a Boolean ring and R is isomorphic to $GF(p) \circ R^*$. This establishes a correspondence between Boolean rings and p-rings, which is biunique up to an isomorphism. (Received March 13, 1952.)

474. T. S. Motzkin: *The multi-index transportation problem.*

Let $x_{\alpha_1\alpha_2\cdots\alpha_n}$, $\alpha_i = 1, \cdots, a_i$ ($a_i > 1$, $i = 1, \cdots, n$), be unknowns. For a given r, $r = 1, \cdots, n-1$, make all choices $k = (1 \leqq i_1 < \cdots < i_{n-r} \leqq n;\ \alpha_{i_1}, \cdots, \alpha_{i_{n-r}})$. It is shown that the extreme solutions of the system $\sum x_{\alpha_1\alpha_2\cdots\alpha_n} = c_k$, $x_{\alpha_1\alpha_2\cdots\alpha_n} \geqq 0$ (the summation running over all α_i with $i \neq i_1, \cdots, i_{n-r}$) lie, for every given system of constants c_k, in the ring generated by the c_k, if and only if $r = 1$ and no more than two a_i exceed 2. (Received March 13, 1952.)

475t. R. M. Redheffer: *Power series and algebraic numbers.*

Suppose $f(z) = \sum a_n z^n$ is rational, is regular at $z = 0$, has algebraic coefficients, and is not a polynomial. Let m be the multiplicity of that root in the denominator of f which has maximum multiplicity. Then (apart from a set of at most m values of z) the value of the function $g(z) = \sum a_n z^n / n!$ is transcendental when the value of z is algebraic. (Received March 17, 1952.)

476. Abraham Seidenberg: *Some basic theorems in differential algebra (characteristic p, arbitrary).*

The theorem of the primitive element, the chain theorem, and the Hilbert Nullstellensatz, all appropriately formulated, are established for differential algebra of arbitrary characteristic. Simplified proofs of the known results in the case of characteristic 0 are included. The theory of transcendency and the theory of separability are also considered. (Received March 13, 1952.)

ANALYSIS

477t. R. E. Bellman: *Some mock hydrodynamical equations.* Preliminary report.

The equation (1) $u_t + uu_x = \mu u_{xx}$ was used by J. M. Burgers, Proc. Acad. Sci. Amsterdam vol. 43 (1940), as a mathematical model to illustrate the theory of turbulence. It was shown by E. Hopf, Comm. Pure and Applied Math. vol. 3 (1950), that this equation could be linearized by means of the change of variable, $u = -2\mu(\log \phi)_x$, ϕ satisfying the equation $\mu\phi_{xx} = \phi_t$. It is shown in the present paper that $u = -2\mu(\log \phi)_x$, $v = -2\mu(\log \phi)_y$ furnishes a solution to the two-dimensional analogue of (1), $u_t + uu_x + vu_y = \mu(u_{xx} + u_{yy})$, $v_t + uv_x + vv_y = \mu(v_{xx} + v_{yy})$, if ϕ satisfies the equation $\mu(\phi_{xx} + \phi_{yy}) = \phi_t$, and similarly for three dimensions. These explicit solutions may now be used, following the one-dimensional treatment of Hopf, to determine rigorously the behavior of the solutions as $\mu \to 0$. If the pressure terms are included in the hydrodynamical equations, the equation for ϕ is $\mu(\phi_{xx} + \phi_{yy}) + p\phi = \phi_t$. (Received March 17, 1952.)

478. F. H. Brownell: *Vector analysis on real separable Hilbert space.* Preliminary report.

Consider complex-valued functions $\psi(f)$ over $f \in l_2$, the square summable real sequences and the prototype of real separable Hilbert space. Following Hille, for suitable functions ψ one can define the notion of a vector derivative $\nabla\psi(f)$, where for ψ real $\nabla\psi(f) \in l_2$ with $\|u\|^{-1} |\psi(f+u) - \psi(f) - (\nabla\psi(f), u)| \to 0$ as $\|u\| \to 0$, $u \in l_2$. This can be interpreted in terms of the l_2 coordinate derivatives, and using basis coordinates the generalized Laplacian ∇^2 can be defined invariant under change of basis. In order to integrate on l_2 and still have an abelian group, it seems l_2 must be reduced to some square summable infinite torus X, $X = \{x \in l_2 \mid -k_n < x_n \leq k_n \text{ each } n\}$ with $\sum^{\infty} (k_n)^2 < +\infty$ required, a modification of Jessen's torus integration. This gives a locally compact abelian topological group, which thus possesses Fourier transforms with inverses, and one can verify the expected relationship of Fourier transforms to partial derivatives with respect to l_2 coordinates, as well as to the generalized Laplacian ∇^2. (Received March 13, 1952.)

479t. R. S. Phillips: *On the generation of semi-groups of linear operators.*

Let $T(t)$ be a semi-group of linear bounded transformations on a Banach space $\mathfrak{X}$ to itself, strongly continuous on $[0, \infty)$ with $T(0) = I$. Set $\omega(\xi) = \log \|T(\xi)\|$ and define $\mathfrak{S}(\omega)$ to be the Banach algebra of set functions α where $\|\alpha\| = \int_0^\infty \exp [\omega(\xi)] |d\alpha|$ and the product is given by convolution. Let $\omega_0 = \inf \omega(\xi)/\xi$ and set $\phi(\lambda, \alpha) = \int_0^\infty \exp (\lambda\xi) d\alpha$ for $\operatorname{Re}(\lambda) \leq \omega_0$. Assume further that $\lim_{\delta \to 0} \sup_{\xi \geq 0} [\omega(\xi) - \omega(\xi+\delta)] < \infty$. Then a necessary and sufficient condition that $\alpha(t, \cdot)$ be a semi-group of monotone nondecreasing functions in $\mathfrak{S}(\omega)$ (with $\phi[\lambda, \alpha(t, \cdot)] \to 1$ as $t \to 0$ uniformly in every bounded subset of $\operatorname{Re}(\lambda) \leq \omega_0$) is that $t^{-1} \log \phi[\lambda, \alpha(t, \cdot)] = m\lambda + \int_0^\infty (\exp [(\lambda - \omega_0)\xi] - 1) d\psi + a$ where $m \geqq 0$, a is real, and $\psi(\xi)$ is monotone nondecreasing on $(0, \infty)$ satisfying the conditions $\int_0^1 \xi d\psi < \infty$ and $\int_1^\infty \exp [\omega(\xi) - \omega_0 \xi] d\psi < \infty$. For a semi-group $\alpha(t, \cdot)$ of this description $S(t) = \int_0^\infty T(\xi) d\alpha(t, \xi)$ is again a semi-group of linear bounded transformations on $\mathfrak{X}$ to itself, strongly convergent to I at $t = 0$. If A is the infinitesimal generator of $T(t)$ and B that of $S(t)$, then for x in the domain of A, $Bx = mAx + \int_0^\infty [\exp (-\omega_0 \xi) T(\xi)x - x] d\psi + ax$. In addition if λ_0 belongs to the spectrum of A, then $t^{-1}\phi[\lambda_0, \alpha(t, \cdot)]$ belongs to the spectrum of B. (Received December 18, 1951.)

480. R. S. Phillips: *Perturbation theory for semi-groups of linear operators.*

Those properties of a semi-group of linear bounded operators strongly continuous to the identity at the origin which persist under a linear bounded perturbation of the infinitesimal generator are called *stable*. It is shown that the property of being a semi-group or a group is itself stable. If A generates $T(s)$ and $A+B$ (with B bounded) generates $S(s)$, then $S(s) = \sum_{n=0}^{\infty} S_n(s)$ where $S_0(s) = T(s)$ and $S_n(s) = \int_0^s T(s-\sigma) B S_{n-1}(\sigma) d\sigma$. If $T(s)$ is uniformly continuous for $s > 0$, then so is $S(s)$. However the property of being eventually uniformly continuous is not stable. If $T(s)$ is holomorphic in a sector and bounded near the origin in every sub-sector, then the same is true of $S(s)$. If the resolvent of A is completely continuous, then so is the resolvent of $A+B$. It is possible to develop a "finite" elementary divisor theory for A when the resolvent of A is completely continuous. If $A(s) = A + B(s)$ where A is the infinitesimal generator of a semi-group of operators and $B(s)$ is a strongly continuously differentiable family of bounded linear operators, then there exists a unique strongly-continuous one-parameter family of bounded linear operators $U(s)$ such that $dU(s)x/ds = A(s)U(s)x$ for all $x \in D(A)$ and such that $U(0) = I$. (Received February 22, 1952.)

481*t*. R. M. Redheffer: *A basis theorem for partial differential equations.*

Let $X=(x, y, z, \cdots, w)$ denote the independent variables, $U=U(X)$ the dependent variable, and $V=(U, U_x, \cdots)$ a finite-dimensional vector with U and some of its partial derivatives as components. A partial differential equation is an equation of form $f(X, V)=0$. If f is continuous and if a linearly closed family of solutions U_α solves $f(X, V)=0$, then there is a finite system of linear homogeneous partial differential equations $A \cdot V=0$, $B \cdot V=0, \cdots$, such that (1) each U_α solves the system and (2) each solution of the system solves $f(X, V)=0$. The result follows from simple properties of finite-dimensional vector spaces. (Received March 17, 1952.)

482*t*. R. M. Redheffer: *Differential equations and the Huygens principle.*

Let $U(x, y)=(1/(2\pi)^{1/2})\int_{-\infty}^{\infty} e^{itx}[\Lambda(t)]^{y}F(t)dt$, where Λ is measurable, $\Lambda \neq 0$, $\Lambda \neq \infty$; and let U satisfy one and the same partial differential equation for all F giving sufficiently strong convergence of the integral. Then $\log \Lambda(t)$ is equal almost everywhere to an algebraic function of t, and the functions U all are solutions of one and the same linear homogeneous partial differential equation with constant coefficients. (Received March 17, 1952.)

483. R. M. Redheffer: *Operators and the Huygens principle.*

Let ϕ_{ab} be a class of operators which transfer a disturbance $f(x, y)$ from $x=a$ to $x=b$: $f(b, y)=\phi_{ab}f(a, y)$. These give an abstract formulation of Huygens' principle in one dimension. Assuming linearity, a closure condition $\phi_{ac}=\phi_{ab}\phi_{bc}$, and boundedness in two senses, representation theorems are given. The simplest result (for "uniform media") is $\phi_{ab}=T^{-1}\Lambda^{b-a}T$, where T is the Fourier transform in the L^2 sense, T^{-1} the inverse transform in the L sense, and $\Lambda(u) \in L^p$ for all p. (Received March 17, 1952.)

484. W. P. Reid: *On the relaxation method of solving differential equations.*

Allen and Severn (Quarterly Journal of Mechanics and Applied Mathematics vol. 4, Part 2 (1951) p. 209) have stated that "It is, in fact, only to problems of jury type that the relaxation method is directly applicable," where jury type means equal number of boundary conditions at each of two points. This statement is incorrect. The relaxation method may be applied directly to solve differential equations of any order with the n boundary conditions distributed in any way among P points $(1 \leq P \leq n)$. Their method of raising the order of the differential equations to make them of jury type is a method of fooling oneself and results primarily in extra work. These remarks, by the way, apply also to problems governed by partial differential equations. Also, the applications of the relaxation method have apparently been restricted so far to bounded regions. This is not necessary. The region may be unbounded, and a condition may or may not be specified at infinity. (Received March 10, 1952.)

485*t*. Leo Sario: *An extremal method on arbitrary Riemann surfaces.* I.

A method is indicated for constructing a function which minimizes a prescribed expression m in a given class F of functions defined on an arbitrary Riemann surface

R. Consider the class $\{P\}$ of analytic functions with the development $P=p+i\bar{p}=1/z+\sum_{\nu=1}^{\infty} a_\nu z^\nu$ in a fixed parameter disc $K:|z|\leq 1$, and with a single-valued real part on R. Write $\alpha=\text{Re }(a_1)$ and denote by β the (ideal) boundary of R. Let λ be a real parameter, $-1\leq\lambda\leq 1$. Then (1) there is in $\{P\}$ a unique principal function $P_\lambda=p_\lambda+i\bar{p}_\lambda=1/z+\sum_{\nu=1}^{\infty} a_{\lambda\nu} z^\nu$ which minimizes the expression $m_\lambda=\lambda\cdot 2\pi\alpha+\int_\beta p\,d\bar{p}$. (2) This minimum is $m_\lambda(p_\lambda)=(\pi/2)[(1+\lambda)^2\alpha_1-(1-\lambda)^2\alpha_{-1}]$ where $\alpha_\lambda=\text{Re }(a_{\lambda 1})$. (3) $m_\lambda(p)-m_\lambda(p_\lambda)=D(p-p_\lambda)$, the Dirichlet integral. (4) $P_\lambda=(1+\lambda)P_1/2+(1-\lambda)P_{-1}/2$. (5) $\alpha_1\leq\alpha\leq\alpha_{-1}$ for functions p with $\int_\beta p\,d\bar{p}\leq 0$. *Definition*: the *P-span* $\sigma_P=\alpha_{-1}-\alpha_1$. (6) $\sigma_P=\min\,(2/\pi)\int_\beta p\,d\bar{p}=(2/\pi)\int_\beta p_0\,d\bar{p}_0$. (7) $1/\sigma_P=\min\,(1/2\pi)D(f)=(1/2\pi)D(f_0)$ where (f) is the class of single-valued harmonic functions f on R with the development $f+i\bar{f}=\sum b_\nu z^\nu$ in K, $\text{Re }(b_1)=1$, and $f_0+i\bar{f}_0=(P_{-1}-P_1)/\sigma_P$. (8) The class $\{P\}$ degenerates to a single function if and only if the P-span vanishes. These results remain valid for the subclass $\{Q\}$ of $\{P\}$, defined by the restriction $\int d\bar{p}=0$ along each dividing cycle on R, if the P-span is replaced by the corresponding *Q-span*. For planar surfaces, $\{Q\}$ consists of single-valued analytic functions and the Q-span coincides with Schiffer's span. Earlier results concerning the span appear as special cases. (Received March 10, 1952.)

486*t*. Leo Sario: *An extremal method on arbitrary Riemann surfaces. II.*

The boundedness of $p_{-1}-p_1$ and $q_{-1}-q_1$ ($Q_{-1}-Q_1$ for planar surfaces) is used to establish a unified approach to classification problems on Riemann surfaces. The following abbreviations are employed: H, harmonic single-valued nonconstant; K, harmonic single-valued nonconstant, such that the conjugate function has no periods along dividing cycles; A, analytic single-valued nonconstant; W, meromorphic nonconstant; B, bounded; D, with a finite Dirichlet integral; E, omitting a set of values with positive area. By combinations of letters, classes of functions are denoted with the corresponding combinations of properties. Let O_F be the class of Riemann surfaces not admitting functions belonging to a given class F. Denote by S_P and S_Q the classes of surfaces with vanishing P-span or Q-span, respectively. Then (1) $O_{HB}\subset O_{HBD}=O_{HD}=S_P$, (2) $O_{KB}\subset O_{KBD}=O_{KD}=S_Q$, (3) $O_{WE}=O_{AB}\subset O_{AD}$. Earlier results, concerning H and A, are included in these relations. The last equality (1) states that there are functions HD on R if and only if the P-span vanishes. The relations (2), applied to planar surface, express existence properties of single-valued analytic functions. (Received March 10, 1952.)

487*t*. Leo Sario: *Construction of functions with prescribed properties on Riemann surfaces.*

Let G be the complement $R-D$ of a compact subdomain D of an open Riemann surface R, D being bounded by a finite set α of analytic Jordan curves. The (ideal) boundary of R is denoted by β. Let v be a single-valued function, harmonic on α, let λ be a real parameter, $-1\leq\lambda\leq 1$. Denote by $\{u\}$ the class of single-valued harmonic functions in G with $u=v$ on α, $\int_\alpha d\bar{u}=0$. There exists in $\{u\}$ a unique function u_λ which minimizes the expression $\int_\beta u\,d\bar{u}+\lambda\int_\alpha u\,d\bar{u}$. The function u_λ is associated with v by a normal linear operator L_λ. This theorem remains valid for the subclass $\{u^0\}$ of $\{u\}$ defined by the restriction $\int d\bar{u}=0$ along every dividing cycle. Both u_λ and u_λ^0 are constructed by the extremal method reported in the above abstracts. The linear operator method (Trans. Amer. Math. Soc. March 1952) leads to the following theorem. Let s be a single-valued real function on D, harmonic on α. The condition

$\int_\alpha d\bar{s} = 0$ is necessary and sufficient for the existence of a function p on R such that (1) $p - s$ is harmonic on D, (2) $\int_\beta u d\bar{u} + \lambda \int_\alpha u d\bar{u}$ is minimized by p among all harmonic functions u in G with $u = p$ on α and $\int d\bar{u} = 0$ along α (or else along every dividing cycle). (Received March 10, 1952.)

488. Leo Sario: *Capacity of the boundary and of a boundary component*. I.

Let $h(z)$ be a harmonic function in $|z| \leqq 1$ with $h(0) = 0$. Write $s = \log |z| + h(z)$. Denote the curves $|z| = e^{-1}$, $|z| = 1$ by a and b, respectively. Lemma: $\min_b s - \min_a s \leqq 1$. On an arbitrary open Riemann surface R with (ideal) boundary β, let $\{S\}$ be the class of analytic functions with the development $S = s + i\bar{s} = \log z + \sum_{\nu=1}^{\infty} a_\nu z^\nu$ in a parameter disc $|z| \leqq 1$, and with a single-valued real part s. There is in $\{S\}$ a unique principal function S_β such that $\min \int_\beta s d\bar{s} = \int_\beta s_\beta d\bar{s}_\beta$. *Definition*: the *capacity of* β is $C_\beta = e^{-k_\beta}$ where $k_\beta = (1/2\pi)\int_\beta s_\beta d\bar{s}_\beta$. For planar domains, C_β coincides with the conventional (logarithmic) capacity. Consider the class $\{U\}$ of analytic functions with a single-valued modulus on R and with the development $U = \sum_{\nu=1}^{\infty} b_\nu z^\nu (b_1 = 1)$ in $|z| \leqq 1$. In $\{U\}$, there is a unique principal function $U_\beta = e^{S_\beta}$ such that $\min D(U) = D(U_\beta) = \pi C_\beta^{-2}$. Suppose now β is analytic. For every analytic function w with single-valued modulus on R such that $|w| \leqq C_\beta^{-1}$, the relations $|w| < |U_\beta|$, $w'(0) < 1$ hold unless $w = e^{i\alpha} U_\beta$. Denote by M the class of nonconstant analytic functions with single-valued modulus on an arbitrary Riemann surface R. The subclasses of functions with bounded modulus or with finite Dirichlet integral are denoted by MB and MD, respectively. There are functions MB or MD on R if and only if $C_\beta > 0$. (Received March 10, 1952.)

489*t*. Leo Sario: *Capacity of the boundary and of a boundary component*. II.

Let γ be a fixed boundary component of an arbitrary Riemann surface R with boundary β. Consider the subclass $\{T\}$ of $\{S\}$ defined by $\int d\bar{s} = 2\pi$ along every cycle separating γ from $z = 0$, and $\int d\bar{s} = 0$ along every other cycle. There exists in $\{T\}$ a unique principal function T_γ such that $\min \int_\beta t d\bar{t} = \int_\beta t_\gamma d\bar{t}_\gamma$. *Definition*: the *capacity of a boundary component* γ is $C_\gamma = e^{-k_\gamma}$ with $k_\gamma = (1/2\pi)\int_\beta t_\gamma d\bar{t}_\gamma$. Let $\{V\}$ be the subclass of $\{U\}$ defined by $\int d \arg u = 2\pi$ along cycles separating γ from $z = 0$, $\int d \arg u = 0$ along other cycles. On a planar R, there is in $\{V\}$ a unique principal function $V_\gamma = e^{T_\gamma}$ such that $\min D(V) = D(V_\gamma) = \pi C_\gamma^{-2}$. There exist on R univalent bounded or univalent Dirichlet bounded nonconstant functions if and only if the capacity of at least one boundary component is positive. (Received March 10, 1952.)

490. Robert Steinberg and R. M. Redheffer: *Simultaneous trigonometric approximation*.

Given real functions $f(x)$ and $g(x)$ in L^2, how small can $I = \int_0^b |f(x) + ig(x) - \sum_{k=0}^{n} a_k^{(n)} e^{ikx}|^2 dx$ be made by a suitable choice of the real $a_k^{(n)}$'s, possibly subject to the constraint $\sum_{k=0}^{n} a_k^{(n)2} \leqq M$? Expressions for the minimizing $a_k^{(n)}$'s and the minimum I are obtained. In the unconstrained case, I can be made arbitrarily small as $n \to \infty$ only for conjugate functions $f(x)$ and $g(x)$ if $b = \pi$, but for arbitrary functions if $0 < b < \pi$. The latter case leads to matric relations analogous to the ordinary Parseval identity. The following limitations on the approximation are obtained: (1) In general, there is no series approximation ($a_k^{(n)}$ independent of n) for which $I \to 0$. If there is a series approximation, it is unique. (2) The following classes of function pairs f, g

are equivalent: (a) Those which admit a polynomial approximation with $I\to 0$ and $\sum_{k=0}^{n} a_k^{(n)2} \le M$. (b) Those which admit a series approximation with $I\to 0$ and $\sum_{k=0}^{\infty} a_k^2 \le M$. Hence, in general, the approximation in (a) is not possible. (Received March 17, 1952.)

491. J. L. Ullman: *Hankel determinants in function theory*. Preliminary report.

Consider the power series $a_0z^{-1}+a_1z^{-2}+\cdots=f(z)$, and let $A(n, k)$ be the Hankel determinant of order k with $a_{n+i+j-2}$ as the element in the ith row and jth column. To each sequence of integers $[k_n]$, associate the number $[k_n]'=\limsup_{n\to\infty} |A(n, k_n)|^{\rho(n)}$, $\rho(n)=1/(k_n(n+k_n-1))$. Hadamard proposed the problem of finding constructive means of determining the singularities of an analytic function from the coefficients of a single Taylor series expansion. In this direction he showed (J. Math. Pures Appl. (4) vol. 8, p. 101) that when the elements of $[k_n]$ are equal, $[k_n]'$ is related to certain poles of $f(z)$. Pólya (Math. Ann. vol. 99, p. 687) considers the case in which $\lim_{n\to\infty} k_n/(n+k_n)=\epsilon>0$, and shows that the $[k_n]'$ are related to the entire set of singularities of $f(z)$. The large gap in these results suggested that $[k_n]'$ for the case $\lim_{n\to\infty} k_n=\infty$, $\lim_{n\to\infty} k_n/(n+k_n)=0$, in which the rate of growth of k_n is intermediate to the other cases, would be related to intermediate properties of $f(z)$. A precise relationship is established for functions $f(z)=\int_0^1 d\mu(t)/(t-z)$, $\mu(t)$ real and nondecreasing on (0, 1). Of independent interest is a generalization of the well known fact that if $A(0, k)$ is ultimately zero, then $\mu(t)$ is a step function; namely, it is shown that if $\lim_{k\to\infty} |A(0, k)|^{1/k^2}=0$, then $\mu(t)$ has zero derivative on (0, 1), except for a set of measure zero. (Received April 15, 1952.)

492. W. R. Wasow: *Asymptotic solution of the differential equation of hydrodynamic stability in a domain containing a transition point.*

The differential equation $u^{(4)}+\sum_{j=1}^{4} a_j(x)u^{(4-j)}+\lambda^2\sum_{k=0}^{2} b_k(x)u^{(2-k)}=0$, with $b_0(0)=0$, $b_0'(0)\neq 0$, $b_2(0)\neq 0$, is studied, in a complex neighborhood S of $x=0$, for large values of the complex parameter λ. The $a_j(x)$ and $b_k(x)$ are supposed to be analytic in S. This problem is important in the study of the onset of turbulence in viscous flows. In a paper in Ann. of Math. vol. 49 (1948) pp. 852–871, the author constructed a fundamental system with known asymptotic behaviour in a ring-shaped domain surrounding the origin. It is now shown—independently of that paper—that the differential equation above possesses a fundamental system which, for large λ, is approximated, everywhere in S, by functions that can be simply and explicitly expressed in terms of solutions of the differential equation $y^{(4)}+\lambda^2(xy''+y)=0$. The asymptotic properties of this latter differential equation are known from another paper by the author (Ann. of Math. vol. 52 (1950) pp. 351–361). Thus it is possible to give a complete asymptotic analysis in S of the more general differential equation, at least as far as the asymptotically leading terms are concerned. (Received March 3, 1952.)

APPLIED MATHEMATICS

493. P. G. Hodge: *Upper and lower bounds on the yield load of a square slab with a centered circular cutout*. Preliminary report.

If an elastic-plastic body is loaded with a set of loads P, the *collapse load* corresponding to P is defined as the smallest loads μP, where μ is a scalar, under which the body will exhibit unrestricted plastic flow. The present paper is concerned with a thin square slab with a centered circular cutout, subjected to uniform tensile stresses on each pair of opposite sides. The *cutout factor* λ is defined as the largest multiplier λ such that if (T_x, T_y) represents any collapse load for the square without cutout, the square with a cutout will not collapse under any load less than $(\lambda T_x, \lambda T_y)$. Upper and lower bounds are then found for the cutout factor. The analysis is based on the assumptions of plane stress and Tresca's yield condition. (The results presented in this paper were obtained in the course of research conducted under Contract N7onr-35810 between the Office of Naval Research and Brown University.) (Received March 10, 1952.)

494. Cornelius Lanczos: *The smallest root of the characteristic equation.*

In vibration problems the absolutely smallest eigenvalues of the generally nonsymmetric and nonreal matrix A are of specific interest. Since the usual iterative procedures suppress the small eigenvalues in favor of the large ones, a preliminary inversion of A is customarily demanded. The present procedure analyzes the problem in the orthogonal reference system of the Hermitian matrix $C=\tilde{A}^*A$. The basic equation is written in the form $Cx-\lambda\tilde{A}^*x=0$. The smallest eigenvector x_0 of C is considered as the zeroth approximation of x. In first approximation $x=x_0+\lambda x_1$ where x_1 is the solution of a linear set of equations. The associated quadratic equation gives a (generally complex) λ which is close to the absolutely smallest eigenvalue of A and which can be further refined. This method, if applied to ordinary algebraic equations, yields a simple and convenient algorithm for approximating the absolutely smallest root of an algebraic equation. (Received March 4, 1952.)

495*t*. R. M. Redheffer: *Antennas with obstructions.*

If T, T^{-1} are direct and inverse Fourier transform operators and $\Lambda(u) = \exp i(k^2-u^2)^{1/2}$, $k=2\pi/\lambda$, then $\Lambda^{-a}TtT^{-1}\Lambda^{a}$ operating on the original far pattern $F(u)$ gives the perturbed far pattern $F_0(u)$ due to an obstruction at $x=a$ characterized by the transmission function $t(y)$. Let an arbitrary obstacle be characterized by $T(u, v)$, the transmitted plane wave in "direction" $v=(2\pi \sin \theta)/\lambda$ given a unit incident plane wave in "direction" u. Then $(2\pi)^{1/2}F_0(v)=\Lambda^{-a}\int\Lambda^{a}T(u, v)F(u)du$, and hence $t(y)$ exists if and only if $T(u, v)=\phi(u-v)$, for some ϕ. A dense set in t, f, or F yields a dense set in F_0, whence are obtained conditions of pattern realizability and methods of computation. The perturbed pattern is approximable as a finite sum of expressions $T\Lambda^a/(u-b)$, the transform being taken as a Sommerfeld contour integral. Another approach gives modified Bessel functions. For estimates of F_0 we have $F_0=F+\Lambda^{-a}H$, where $H \exp(-imu)$, given explicitly, is slowly varying for some m. (Received March 17, 1952.)

496*t*. R. M. Redheffer: *Inhomogeneous dielectric media.*

Let R be the complex reflection of a slab in which the dielectric constant is a continuous function of one coordinate, $\epsilon=\epsilon(x)$. If $z=-i(1+R)/(1-R)$ and $-kz=u'/u$; then u satisfies $(u'/\epsilon)'+(2\pi/\lambda)^2u=0$. Formulas are given for $\Re(z)$ in terms of $\Im(z)$, and for $z(x)$ for every termination from $z(x)$ for one. The polar form $z=re^{i\theta}$ satisfies $rd\theta/dr=\tan\theta(\epsilon r^2-1)/(\epsilon r^2+1)$ when ϵ is real, whence is obtained a condition for realizability of a prescribed z. Regarding the medium as limit of thin uniform slabs gives a physical interpretation to the method of "product integration." (Received March 17, 1952.)

GEOMETRY

497. L. M. Blumenthal: *Boolean geometry*. I.

A Boolean metric space $\mathfrak{B}$ is obtained by associating with each pair of elements p, q of a set an element $d(p, q)$ of a Boolean algebra as distance, so that $d(p, q)=0$ if and only if $p=q$, $d(p, q)=d(q, p)$, and $d(p, q)+d(q, r)\supset d(p, r)$ ($+$, $\supset$ denote Boolean addition and inclusion, in the wide sense, respectively). Boolean geometry studies the invariants of $\mathfrak{B}$ under the group of congruences (i.e., distance-preserving mappings). Any Boolean algebra is a space $\mathfrak{B}$ with $d(p, q)=pq'+p'q$ (juxtaposition and $'$ denoting Boolean product and complementation, respectively) and this paper begins the systematic study of these Boolean geometries by developing linearity notions and the theory of metric segments. Defining these in a suitable manner, they are shown to possess many properties of segments in ordinary metric spaces. Thus, (1) the length of a segment S_a^b is the distance $d(a, b)$ of its end elements, (2) if S_a^b, S_b^c are segments and b is metrically between a, c then their set-union is a segment S_a^c, and (3) for $m>4$, any m-tuple of $\mathfrak{B}$ is on some segment if and only if each triple of the m-tuple has that property. (Received February 26, 1952.)

498. P. J. Kelly and L. J. Paige: *Symmetric perpendicularity in hilbert geometry.*

Define a family F of curves as follows: Every curve C in F is a simple, closed, convex curve. If supporting lines at p and q on C meet at w, and ξ is any secant through w, cutting C at m and n, then supporting lines at m and n exist, intersecting on $p\times q$. The authors prove that the family F consists of all ellipses and all nondegenerate triangles. This result is used to show that the hilbert geometries in which perpendicularity is symmetric are hyperbolic geometries. (Received February 18, 1952.)

499. T. K. Pan: *Characterisation of the Gaussian curvature of a surface.*

Let V_n be a hypersurface immersed in a Euclidean space S_{n+1}. Let v be a vector-field in V_n. Define the mean curvature of v at a point P as the square root of the sum of the squares of its $n-1$ principal curvatures at P (the author's paper on *Normal curvature of a vector-field*, to appear soon in Amer. J. Math.). It is found that the hyperspherical curvature at P of V_n is equal to: (1) the product of the extreme mean curvatures of vector-fields in V_n at P, (2) the square of the product of the extreme mean curvatures of asymptotic vector-fields in V_n at P, (3) the product of the mean curvatures of the n vector-fields at P corresponding to the principal directions for the Ricci tensor R_{ij} when V_n is minimal. New interpretations of the Gaussian curvature of a surface S in an ordinary space are obtained: (1) the Gaussian curvature of S at P is the product of the principal curvatures of any two distinct conjugate vector-fields in S at P, (2) the Gaussian curvature of S at P is the negative of the square of the magnitude of the Gaussian representation of a unit arc along an asymptotic line from P in S. A simple proof of the theorem of Enneper is derived. (Received March 21, 1952.)

LOGIC AND FOUNDATIONS

500. D. W. Bennett: *A set of axioms for the natural numbers.*

Let N be a set for which the following axioms are satisfied: I. There exists a mapping f of the set of all nonempty subsets of N onto N. II. For any nonempty subset P in N, $f(P) \in P$. III. There exists a reversible mapping g of the set N onto the set of all elements of N except $f(N)$. IV. For any subset P of N such that $f(P) \neq f(N)$, $g^{-1}f(P) \notin P$. The axioms are seen to involve no concepts except the familiar idea of mapping one set onto another. They lead immediately after appropriate definitions of "1" and "successor" to Peano's postulates for the natural numbers, and conversely. (Received March 12, 1952.)

501. E. W. Beth: *Existence of complete models for extensions of the first-order predicate calculus.*

Let T be a complete extension of the first-order predicate calculus. Every set S of expressions $A(x)$ containing a free variable x defines, for every model M of T, a subset $M(S)$ of the set of all elements of M. Let S be finite; then, obviously, $M(S)$ will be nonempty if and only if S is consistent with T. For an infinite set S which is consistent with T, the set $M(S)$ will not, in general, be nonempty for every model M of T. But it follows from the theorem of Löwenheim-Skolem-Gödel-Malcev that there is a model M_0 of T which is complete in the following sense: for any set S, $M_0(S)$ will be nonempty if and only if S is consistent with T. It follows that $M_0(S)$ is nonempty if and only if, for some model M of T, $M(S)$ is nonempty. For $k=2, 3, \cdots$ an analogous result can be proved for sets of expressions $A(x_1, x_2, \cdots, x_k)$ containing k free variables. (Received March 12, 1952.)

STATISTICS AND PROBABILITY

502. G. E. Forsythe and J. W. Tukey: *The extent of n random unit vectors.*

Let $W_1, \cdots, W_n$ be random unit vectors which are independently and uniformly (with respect to $n-1$ dimensional measure) distributed on the unit sphere in n-dimensional space. Let Δ_n be the *extent* of these vectors, i.e., the signed n-dimensional volume of the parallelotope built on them. If W_i has cartesian coordinates w_{ij}, then $\Delta_n = \det(w_{ij})$. A short geometrical proof of the following is given: for $n \geqq 2$ the ratio $|\Delta_n/\Delta_{n-1}|$ is the product of $n-1$ independent positive quantities, each of which has the same incomplete beta distribution $dF(u) = \Gamma(n/2)\{\Gamma(n/2-1/2)\Gamma(1/2)\}^{-1} \cdot u^{(n-3)/2}(1-u)^{-1/2}du$; the ratios $\Delta_n/\Delta_{n-1}, \cdots, \Delta_2/\Delta_1$, Δ_1 are independent; the odd moments $E(\Delta_n^{2k+1}) = 0$; the even moments $E(\Delta_n^{2k}) = n!(n+2)! \cdots (n+2k-2)!\{0!\,2! \cdots (2k-2)!\}^{-1}\{n(n+2) \cdots (n+2k-2)\}^{-n}$. It is shown that the moments can also be obtained from the moments for the generalized variance; see S. S. Wilks, Biometrika vol. 24 (1932) pp. 471–494, esp. pp. 476–477. The second moment $E(\Delta_n^2)$ had been given by S. Krochmal [Leningrad. Universitet. Uchenye Zapiski vol. 83, Ser. Mat. Nauk 12 (1941) pp. 150–198; Mathematical Reviews vol. 7 (1946) p. 487]. (Received March 10, 1952).

TOPOLOGY

503t. Barrett O'Neill: *Essential sets and fixed points.*

Let f be a self-map of a compact polyhedron X. A subset A of X is defined to be essential if every map in some neighborhood of f in X^X has a fixed point in A. Criteria are established that a subset be essential and that a subset contain an essential fixed point, as defined by M. K. Fort. The main tool is a fixed point index for sets similar to that of Leray and others. An elementary proof is given of the existence of an index and further properties derived. In special cases the essentiality of A is related to the homological properties of f, A, and X. Maps of locally compact spaces are considered through their compactifications. Using Čech theory, the methods apply to nonpolyhedral spaces. (Received April 24, 1952.)

504t. R. H. Sorgenfrey: *Mappings of plane convex sets.*

It is an almost immediate consequence of a theorem of Borsuk that if C is a compact convex set of width w in a euclidian space and f is a continuous real-valued mapping defined on C, then some inverse set under f has diameter at least as great as w. It is shown that if C is planar, then some inverse set has a component of diameter at least as great as $w(3^{1/2})/3$ in general and, in case C is centrally symmetric, at least as great as $w(3^{1/2})/2$. Simple examples exist to show that these numbers cannot be improved. (Received March 12, 1952.)

W. T. PUCKETT,
Acting Associate Secretary

BOOK REVIEWS

Cours de cinématique. By R. Garnier. Vol. 1, *Cinématique du point et du solide. Composition des mouvements.* 2d ed. Paris, Gauthier-Villars, 1949. 4+235 pp.

Vol. 2, *Roulement et viration. La formule de Savary et son extension à l'espace.* 2d ed. Paris, Gauthier-Villars, 1949. 8+287 pp.

Vol. 3, *Géométrie et cinématique cayleyennes.* Paris, Gauthier-Villars, 1951. 12+376 pp. 3000 fr.

Kinematics, as a domain linking geometry and mechanics, has always been a favorite subject of French geometers. These lectures by Garnier provide a new general presentation following the work of Mannheim and Koenigs, somewhat in the manner of G. Darboux's classical work on differential geometry.

After an introduction to vector analysis the first volume begins with point kinematics with many geometric applications, e.g. to geodesics. In the kinematics of the one-parameter continuous motions of a rigid body the distributions of velocity and acceleration at each instant are investigated in detail. The third chapter covers composition of motions and the theorem of Coriolis. The final chapter of this volume is devoted to the determination of a one-parameter motion by integration of its infinitesimal elements.

In the second volume, chapter 5 discusses plane motions, roulettes and the Euler-Savary formula for the curvature of paths, with applications to cycloids. Chapter 6 discusses spherical motions in a corresponding way. The next chapter discusses one-parameter motions in space, again in infinitesimal form and only as far as they depend only on derivatives of the first order. The last chapter of this volume contains corresponding considerations for derivatives of the second order.

The third volume is especially original, containing kinematical considerations for non-euclidean geometry, introduced either in the projective fashion of Cayley and Klein or by conformal mapping into euclidean space. In the elliptic case Garnier also uses the spherical mapping of Hjelmslev, Fubini, and Study. Chapter 10 is devoted to this geometrical introduction of non-euclidean geometry and its motions. Chapter 11 contains differential-geometric investigations of curves and surfaces, velocity and acceleration distributions. Chapter 12 contains among other things a translation of the Euler-Savary formula. The concluding chapters 13 and 14 contain differential-geometric considerations of first and second order for motions in space, beginning in particular with the study of ruled surfaces and axial planes, then of orbits and envelopes.

There is an appendix containing two notes, one on the concepts of surface area and volume, the other on non-euclidean mechanics of points and rigid bodies. Numerous exercises and figures and a detailed index enliven and facilitate the use of this distinguished work. Many-parameter motions and integrals of kinematics are hardly touched upon.

W. BLASCHKE

Dictionary of mathematical sciences. Vol. 1, German-English. By Leo Herland. New York, Ungar, 1951. 235 pp. $3.25.

This is an ambitious and fairly successful attempt to provide mathematicians with the English equivalents of German mathematical terms. The arrangement, the typography, the elaborate system of cross-references and illustrative phrases are all excellent. In most cases the author has avoided the obvious pitfalls of using the English cognate word instead of the English idiom, or of always using the same translation for the same German word. Thus for example *Fakultät* comes out correctly as "factorial," and *Satz* in compounds is "theorem," "law," or "condition" according to the context. Presumably for the benefit of anyone who has to listen to lectures in German, the spoken use of such words as *hoch* (as in *fünf hoch ein Viertel*) is included. The chief adverse criticisms which have to be made are a lack of completeness and a lack of accuracy, both of which could have been avoided if the author had consulted specialists in several branches of mathematics, as he did consult specialists in commerce and statistics for the technical terms in these fields. The dictionary also covers such fields as logic, physics, and astronomy to some extent, but the comments in this review will be confined to its coverage of mathematics proper.

According to the preface this dictionary centers "about the major subjects of mathematics and geometry" [one wonders what the author considers mathematics to be, if geometry is not a part of it]; it "does not claim completeness, although the aim has been to include all important terms." As far as concerns arithmetic, elementary algebra, the less specialized forms of geometry, calculus, the elements of set theory and of the theory of functions, the coverage is quite thorough; a brave attempt has been made to cover abstract algebra; but topology and applied mathematics are practically omitted. The latter omission seems particularly serious since applied mathematics contains an abnormal number of technical terms whose meanings are not readily guessed from their appearance. To select a few examples at random, one does not find ***hebbar***, and the translations given for ***heben*** do not illuminate the missing word; *Simplex*, *Zelle*, *Spaltung*, *Primende* are all missing; *Deckung*, *Bedeckung*, and *Überdeckung* are not there, although ***decken*** is, in contrast to a spirit of inclusiveness which leads to separate entries for *x-Achse*, *y-Achse* and *z-Achse*; the German words for "layer," "beam," "plate," "buckle," "viscous" are missing, and *Schale*, *Torsion*, *beugen* appear in geometrical or optical senses but not in mechanical ones. In algebra, *Verband* is perhaps the most serious omission; another kind of lattice is, however, represented by *Gitter*. One misses *schlicht*, and the omission cannot be excused on the grounds that the German word is frequently used as its own translation, since on the one hand it is also frequently rendered by "univalent" and on the other hand *Nullstellensatz* is correctly rendered in this dictionary as "Nullstellensatz." Only "convolution" and not "Faltung" is given as the translation of *Faltung*. As technical a term as *totalpositiv* is there, but not *vollmonoton* or *vollstetig*; there is a column of *Quadrat-* words, but not *quadratfrei*.

There are only a few quite serious incorrect translations. The author correctly gives ***vollkommen*** as "perfect," but fails to realize that *perfekt* (in algebra) should be "complete." ***Mehrdeutig*** is translated as "ambiguous," which is incorrect at least when applied to functions. However, too often only one of several common meanings is given: *Ansatz* appears only as applied to the "arrangement of an equation"; *ganz*, as applied to a function, is missing; one can ***mitteilen*** a meaning or a motion, but not a theorem or a paper; *Verzerrung* is "deformation" but not "distortion"; *Einheit* appears as a unit of measurement or a number, but not as a unit in the algebraic sense.

Less serious are translations not conforming to current English usage, such as "permanently convergent series" for ***unbedingt konvergente Reihe***; *Monotonie* = "monotony"; *Quader* = "right parallelepiped"; ***umfassen*** (a set) = "comprehend"; *phasisch* = "phasic." *Natürliche Zahl* is rendered only as "natural number," not as "positive integer"; *semikonvergent*, as "semiconvergent," not as "asymptotic"; and *Teilerkettensatz* comes out quite literally as "divisor chain condition." Some translations such as these would do no worse than lead a German to write unidiomatic English; but others would be likely to confuse anybody who has to consult mathematics in both languages.

Deficiencies such as these seriously limit the usefulness of this dictionary either for the student or the working mathematician. In its present form it can be of considerable help, but it can hardly be considered authoritative. It is to be hoped that a second edition of increased scope and greater accuracy will be prepared. If the deficiencies of the present edition can be remedied and its good qualities retained, it will be of great value to mathematicians.

R. P. Boas, Jr.

The mathematical theory of plasticity. By R. Hill. Oxford, Clarendon Press, 1950. 10+356 pp. $7.00.

Although it is more than eighty years since the foundations of the theory of plasticity were laid by Tresca, Saint Venant, M. Lévy, and others, plasticity is still a very young science. After a first strong wave of interest (about 1913–1930), work in this field has slowly but steadily increased and recent years have seen a marked upsurge of interest; this can best be illustrated by the fact that in 1950 no fewer than four very serious books on the subject appeared, by A. M. Freudenthal (Wiley), by R. Hill, by A. Nadai (McGraw-Hill), and a comprehensive survey report by P. G. Hodge (Brown University Notes); in 1951 followed a textbook by Prager and Hodge (Wiley). Among those works Freudenthal's book differs from the others by its wider scope; plasticity as understood in the other books forms only a chapter, although one of central importance, in Freudenthal's approach, since his viewpoint is primarily that of a physicist and technologist. Hill's book is an advanced and comprehensive text, intended as an orientation for engineering scientists and applied mathematicians rather than as a textbook for students; on the other hand Hodge and Prager's useful and interesting book is planned for students on an intermediate level.

Hill's important book presents those aspects of plasticity which so far have been more or less "mathematicised"; by this word we mean that a rational theory which forms part of a larger scientific unit (the science of mechanics) is formulated in mathematical terms. This is true in particular for that part of plasticity theory which is known today as the theory of the "ideal" or "perfect" plastic body. Such a body is described mathematically by the system of equations at the basis of the theory. We may, however, point out a few features: (a) The equations deal only with stresses and deformations at a fixed moment; after the whole configuration is determined for an instant the investigation may be repeated if necessary for the next moment; (b) thermal phenomena are disregarded; (c) work hardening and related phenomena are in general neglected, etc.

The characteristic features of an ideal plastic material, as contrasted to an elastic one and described in the equations of plasticity, are very roughly the following: (1) While in an elastic material there is a one to one correspondence between strain and stress, a completely plastic material flows under constant stress (if only one stress component σ matters, then, $\sigma=\sigma^*$; or, in general, a certain function of the stress tensor remains constant throughout the plastic flow). (2) A relation is assumed between stress tensor and (total) strain velocity tensor (rather than between stress tensor and (elastic) strain tensor). Mathematically, the whole setup consists of ten equations (nine differential equations and one finite equation) for the following ten unknowns: six components of the symmetric stress tensor, Σ, three components of the flow velocity vector, $\bar{v}$, and a non-negative scalar function $\lambda(x, y, z)$. The equations are (a) three equilibrium equations for Σ, (b) the yield condition, a finite equation which singles out the five-dimensional manifold of admissible stresses, (c) the six relations between stress and strain velocity; in the fully plastic state stresses are proportional to strain rates with the factor of proportionality λ. This looks similar to the basic relation of viscous fluid theory; the decisive difference is, however, that there the proportionality factor μ, the viscosity, is a material constant while our λ is an ***unknown*** function of the space coordinates. Of these last six equations five only are independent since they are such that incompressibility holds, in the form $\operatorname{div} \bar{v}=0$; these equations, generally called the Lévy-Mises or Saint Venant-Mises equations, are strictly applicable only to a fictitious material in which elastic strains are zero. The strain equations of Prandtl (1924) and Reuss (1930) take into account not only plastic but also elastic strains. At any rate the complete general system of ten equations is too complicated for actual applications.

The first three chapters of Hill's work form a preliminary unit, where the physical and mathematical foundations are explained (for the general three-dimensional case) and some general theorems, statements of uniqueness, some extremum principles, etc., are considered. A second group, in Chapters IV and V, concerns some plastic-elastic problems. As long as the loads acting on a metal are sufficiently small, the body is elastic; as the loads are increased plastic regions begin to appear which successively grow in size; the plastic zones are, however, restrained from actual flowing ("contained plastic state") by the remaining still elastic material (think of the expansion of a cavity by internal pressure in a thick-walled shell). Finally the plastic regions may spread and merge to such an extent as to make actual plastic flow possible; then the elastic deformation is negligible compared to the much larger plastic deformation, and the corresponding idealization is a rigid plastic type of material with the Lévy-Mises equations valid. In problems of contained plastic strain the material is plastic-elastic; the complete solution, based on the Prandtl-Reuss equations, involves a calculation of stress and deformation in the elastic as well as in the plastic region; the boundary between these regions is in general unknown (the same is true for the rigid plastic boundary in the other problems). So far only elastic-plastic problems of a simple type have been solved; it may, e.g., happen that all elements of the metal reach the yield limit simultaneously so that no plastic-elastic boundary need be found, as in the combined torsion and tension of a cylindrical bar; in other problems, like that of the torsion of a prismatic bar, elastic and plastic zones exist side by side but other simplifications prevail. Then follow the problems, extensively investigated by several authors, of the expansion of a spherical shell and of a cylindrical tube by internal pressure; the presentation is based on the author's own work (with coauthors).

The central part of the book, formed by Chapters VI–IX, deals, for *plane* motion, under conditions of "plane strain," with problems where a plastic rigid material may be assumed. Here we have a type of problem amenable to mathematical methods. For the fully plastic material there are now six equations and unknowns (or five after elimination of λ); the unknowns are the three components σ_x, σ_y, τ of Σ, the two components v_x, v_y of $\bar{v}$ and λ; the equations are (a) two equilibrium conditions, (b) the yield condition in the form $(\sigma_x-\sigma_y)^2+4\tau^2=\text{const.}$, (c) three relations between the stress and strain rate tensors. Here, (a) and (b) are three equations for σ_x, σ_y, τ alone, so that the five equations fall into two sets, as long as no boundary conditions are considered. The set (a), (b) is nonlinear but of the "reducible" type, i.e., after elimination of one stress-variable by means of (b) there remain two partial differential equations which, as in compressible fluid theory, can be transformed into two linear equations, for example by means of an exchange of dependent and independent variables. The theory of these stress equations has been extensively investigated. If the tensions are known, the last two equations for the velocities are linear; it turns out that the characteristics of the "stress equations," and of the "velocity equations" are always real, coincide with each other, and bisect the angles of the principal stress directions (which are also the principal strain rate directions). The elegant theory of this "slip line field" is clearly and completely exposed in VI, including also some numerical methods based on the finite difference equivalents of the basic equations referred to the slip lines.

Chapters VII–IX deal with applications of the theory to actual problems. Unfortunately the "static determinacy" of the stress equations is in general lost in actual problems, since the boundary conditions concern stresses *and* velocities. While one can always find partial solutions of the stress equations, without considering velocities, the correct stress solution of a specific problem can be found in this way only if there are sufficiently many pure *stress*-boundary-conditions. In this case, after the correct stresses have been found, we may solve the boundary problem of the second group, i.e., of the velocity equations. If, however, the boundary conditions do not fall into two such groups, the whole problem must be dealt with more or less simultaneously. In addition, the boundaries between rigid and plastic regions are not known beforehand so that we have certain conditions along free boundaries. Hence the solution will often be based on a method of systematic trial and error where we experiment with assumed plastic rigid boundaries, or with guessed solutions. In addition to these essentially mathematical difficulties, a great difficulty in boundary problems of plasticity consists often in the definition and setting up of the problem, that is, in the determination of the right boundary conditions. In this difficult task one would much like to complement the mechanical knowledge and intuition by means of mathematical information, which, however, is often not available; we are thinking of general existence theorems as they exist in hydrodynamics or elasticity theory, which would help to decide what groups of stress- and velocity-boundary conditions determine a solution.

However, a considerable number of actual problems have been set up, solved, and discussed, particularly in recent years, and Hill has contributed much to this work. Chapter VII deals with so-called "steady state" problems, i.e., problems where stress- and velocity-field do not vary in time. The history of the changes through which the steady plastic state is reached is not taken into account. The chapter contains much of the author's own work. It would lead too far to talk to mathematical readers about "sheet drawing," "sheet extrusion," etc. *Non*-steady problems are those where stress—and velocity—field are varying in time; this variation may be such as to preserve geometric similarity ("quasisteady"), or no such restriction may hold. Again much of the author's work is incorporated; problems of wedge indentation, of compression of a wedge, etc., belong to this class. If, however, a rectangular block of plastic rigid material is compressed between rough rigid plates (shorter than the block), geometric similarity is not maintained. Finally, in Chapter IX, plane problems are discussed which require the consideration of an elastic-plastic material.

In Chapter XI, which deals with "miscellaneous topics," two further mathematically similar *plane* cases are discussed, "plane strain of a more general plastic material," and "plane stress." Both make it necessary to abandon the particular simple yield condition mentioned above, a fact that greatly complicates the mathematical situation. While the basic equations in VI–IX are everywhere *hyperbolic*, with real *orthogonal* characteristics, the more general equations are in general partly hyperbolic, partly elliptic or parabolic (and no longer orthogonal); since the stress equations are not linear, the regions of this varying behaviour depend on the specific solution considered. Essential parts of the theory are given; some of the results seem to be true under more general conditions than those stated.

Chapter X gives the basic equations and some applications for the case of *axial symmetry*. Here, just as in the plane problem, there are only two independent variables; there are, however, four nonvanishing stress components, and the stress equations, as they are, are not statically determined. Chapter XII offers an introduction to non-isotropic problems.

All through the book there are very complete bibliographical notes, in addition to an author index and a subject index at the end of the book.

The reviewer admires Hill's accomplishment: in addition to reporting so much published work, he has filled out many gaps by original remarks and contributions, thus achieving a unified presentation. It may, however, be said that, occasionally, when the reviewer sought instruction regarding a specific problem it was found very hard to follow the explanations. Returning, in such a case, to the original paper—by Hill—the desired information was readily obtained. Probably such things are unavoidable in a work so rich in content. By the way, the reviewer wonders why the useful tool of graphing a streamline pattern in order to impart in one view a picture of a velocity distribution is not used (in problems of plane strain a simple stream function, $\psi(x, y)$, exists with $\partial\psi/\partial x = v_y$, $\partial\psi/\partial y = -v_x$). There were instances when the reviewer could not approve of a mathematical presentation or of an argument. But at any rate, even apart from the question of taste, such details are of minor interest, if, as in this book, the difficult task of expounding critically and in a unified way so much novel research work has been so successfully accomplished. We join the author in his hope that his work "will attract engineers and applied mathematicians to a field which well rewards study and research."

H. GEIRINGER

Fourier transforms. By I. Sneddon. New York, McGraw-Hill, 1951. 12+542 pp. $10.00.

It is the aim of the author to discuss various types of integral transforms from an elementary mathematical viewpoint and to demonstrate how they may be applied to various boundary value problems which arise in the physical and engineering sciences. Accordingly, some basic aspects of these transforms are discussed in the first three chapters of this text. Chapter one is concerned with the Fourier, Laplace, and Mellin transforms for one variable as well as the multiple Laplace and Fourier transforms. It is unfortunate that the complex form of the Fourier transform was not included here, for then one could see that there is no basic distinction between these transforms. That is, what may then be accomplished by the unilateral transform of Fourier may be equally well accomplished by the unilateral Laplace transform, etc. The second chapter contains a discussion of Hankel transforms (real case) as well as the relation between the real multiple Fourier transform and Hankel transforms. It closes with a discussion of dual integral equations of a special class which has been discussed by Titchmarsh and his collaborators. The closing chapter of this part of the book is devoted to a discussion of the finite Fourier and Hankel transforms. These transforms are infinite series of the Fourier or Fourier-Bessel type which arise naturally in Sturm-Liouville expansion theory. The application of these finite transformations to appropriate boundary value problems simply states that one is aware of the correct form of the expansion in advance.

The remaining seven chapters are concerned with the applications of these mathematical methods to many ordinary and partial differential equations which arise in the physical and engineering sciences. No specialized knowledge of physics is assumed and the remaining background is discussed with the view of supplying the necessary differential equations and their subsidiary conditions. We find, in the second portion of the book, applications drawn from vibration theory, elasticity, hydrodynamics, and heat conduction as well as some problems drawn from atomic and nuclear physics.

The book closes with three appendices. The first one is concerned with some properties of Bessel functions, while the second one discusses the method of steepest descent and some numerical methods.

The last section provides a table of useful transforms of the various types discussed.

Some of the recent work involving the use of the integral equation of the Wiener-Hopf type is conspicuous by its absence. It is in these topics that Fourier methods come to the forefront because, for the most part, there are no other methods available. Such a discussion would also serve to accentuate the importance of the role of function-theoretic methods in the integral transforms discussed in this text.

Albert E. Heins

Vorlesungen über Differential- und Integralrechnung. Vol. II. *Differentialrechnung auf dem Gebiete mehrerer Variablen*. By A. Ostrowski. Basel, Birkhäuser, 1951. 482 pp. 67 Swiss fr.

Volume I of this work appeared in 1945, and was reviewed by the present writer (Bull. Amer. Math. Soc. vol. 52 (1946) pp. 798–799). The first volume was devoted to the structure of differential and integral calculus for functions of a single variable, and to the development of the rules of calculus as they apply to the standard elementary functions. This second volume carries the study of limits and continuity further than was done in the first volume, and deals with a variety of additional topics. There are eight chapters. A third volume is planned to complete the work. It will deal with integral calculus in relation to functions of several variables.

Chapter I is entitled *Infinite sets*. After a discussion of denumerable and nondenumerable sets, the chapter is mainly taken up with the concepts of point set topology for Euclidean space. Chapter II treats the theory of limits and continuity for real functions defined on sets in Euclidean space. Chapter III deals with infinite sequences and series, beginning with the concepts of limits inferior and superior for sequences. A prominent place is given to a useful but apparently little known theorem of Cauchy, which reads as follows: Suppose $0<A_1<A_2<\cdots$, $A_n\to\infty$ as $n\to\infty$, and let $\{a_n\}$ be any sequence. Then

$$\frac{a_{n+1}-a_n}{A_{n+1}-A_n}\to d \quad \text{implies} \quad \frac{a_n}{A_n}\to d.$$

This holds as well if $d=\pm\infty$. There are numerous applications of this theorem, among which is a proof of one of the forms of l'Hospital's rule (in Chapter IV). The treatment of series of constant terms follows standard lines. There is a generalized form of Raabe's test, but the very useful test of Gauss is omitted. The discussion of uniform convergence includes Dini's theorem and a general theorem on the exchange of order in repeated limits (essentially the theorem associated with Moore and Osgood). The chapter concludes with a proof of the Weierstrass approximation theorem by Lebesgue's method. This is the longest chapter in the book. There is a wealth of material in the numerous exercises.

In Chapter IV, entitled *Completion of differential calculus*, the majority of the space is devoted to partial differentiation. The author calls a function $z=f(x, y)$ "differenzierbar" if it has first partial derivatives, and "differentiable" if it has a total differential dz in the sense of an approximation to Δz, linear in Δx and Δy. A notion of being uniformly "differenzierbar" is introduced, and in terms of this notion a theorem on necessary and sufficient conditions for differentiability is stated. This notion is also used to give a sufficient condition that $f_{xy}=f_{yx}$. Chapter V contains a thorough treatment of implicit function problems for one or more equations, the attendant discussion of Jacobians, and some consideration of the theory of extrema. Sufficient conditions in the latter theory are discussed in the two variable case only. The chapter ends with a proof of the Hadamard determinant inequality, using the Lagrange multiplier rule. Chapter VI, on methods of numerical approximation, includes topics in interpolation, numerical differentiation and integration, and iterative methods of solving equations. Formulas for error terms are discussed in all cases, and convergence theorems for Newton's method are given. Chapter VII deals with arc length and, for plane curves, with curvature and related topics. Rectifiability is discussed in terms of continuous functions of bounded variation, but most of the chapter is on a more elementary level. The topics in Chapter VIII are: vectors, the Frenet formulas, envelopes of families of surfaces, and the first fundamental quadratic form for a surface.

The text should prove very valuable as a permanent reference for students taught from it. It could also be used to good advantage as a reference by teachers of advanced calculus in this country, particularly on account of the extensive collection of exercises, which occupy nearly 30% of the page space of the book. A large proportion of the exercises are new in the textbook literature. They contain much information of interest. In contrast to the situation in volume I, the diagrams are of a uniformly high standard of execution.

The only serious adverse criticism which the reviewer has to present deals with the first two chapters. The treatment of point set topology is unsatisfactory in several respects. The author beclouds the concept of a point of accumulation by introducing the notion of a point set whose elements need not all be distinct points and talking about points of accumulation of "sets" of this kind (he regards a sequence $\{P_n\}$ as such a set). Now, a "point set" in this latter sense is in reality a function (a sequence being a function defined on the positive integers). There is no need to introduce the concept "point of accumulation" except in the sense that is customary in topology. Strict adherence to this point of view makes for clarity and for less difficulty on the part of students. All that the author wishes to accomplish can be done by suitable discussion of points of accumulation of a set and their relation to convergent sequences chosen from the set. A second criticism relates to the author's definition of a closed set as one which contains all its points of accumulation *and is bounded*. This departure from the usual definition spoils the duality between open and closed sets, complicates the statements of many theorems, and has no advantages apparent to the reviewer. The topological definition of connectedness is not given; the definition which is given on p. 39 (connectedness by polygonal paths) is unsatisfactory, except for open sets, and the remarks about connected regions on p. 42 puzzled the reviewer. The intermediate value theorem for continuous functions is not presented as a connectedness theorem.

ANGUS E. TAYLOR

Conformal mapping. By Z. Nehari. New York, McGraw-Hill, 1952. 8+396 pp. $7.50.

This is a textbook that will fill two needs. The author has designed the first four chapters to serve as the basis for a one term introductory course in complex variables, while the remainder of the book can be used in a graduate course in conformal mapping. It is claimed in the preface that only a knowledge of advanced calculus is necessary to read this book. (Perhaps a slightly better knowledge of the properties of real numbers is actually assumed than is given in most courses in advanced calculus.)

In Chapter I, the properties of harmonic functions in the plane are developed. The author discusses the solutions of the boundary value problems of the first and second kinds, introducing the Green's and Neumann's functions, and the harmonic measure. The first chapter closes with a derivation of the Hadamard variation formula giving the dependence of the Green's function on the domain.

The complex number system is explained in the first part of Chapter II, culminating in a discussion of sequences and series of complex numbers. After an analytic function is defined to be one which has a derivative, the connection between analyticity, the

Cauchy-Riemann equations, and the Taylor series is demonstrated. The chapter concludes with a study of the elementary functions. Chapter III is devoted to the complex integral calculus, collecting together those topics which relate to the Cauchy integral theorem and formula. To lay the groundwork for the numerous extremal problems to be encountered, Chapter IV takes up normal families of analytic functions.

Chapter V is devoted to the conformal mapping of simply-connected domains, beginning with an extensive study of the linear fractional transformations. Then Schwarz's lemma leads the way to the study of bounded analytic functions. The Riemann mapping theorem is proved together with the continuity of the mapping function on the boundary of a smooth Jordan region. On a more practical level, the functions mapping polygons and domains bounded by circular arcs onto a half-plane are found. Several extremal problems in the class of functions univalent in $|z| < 1$ are solved, including the distortion theorems and certain coefficient problems. The principle of subordination is introduced to solve extremal problems in larger classes of functions, dropping the univalency requirement. The chapter continues with a discussion of the Bergman kernel function for simply-connected domains, and closes with the mapping of nearly circular domains.

In Chapter VI, the mappings produced by certain special functions are studied. Included are the rational functions of second degree, the exponential and trigonometric functions, and the elliptic functions. The last chapter deals with the mapping of multiply-connected domains onto canonical domains. The existence proofs here are based upon extremal problems analogous to the one used in proving the Riemann mapping theorem, lending unity to the book. The Dirichlet problem for multiply-connected domains is then discussed. The book ends with a treatment of several special extremal problems: maximizing outer area or minimizing inner area, the Bergman kernel function, and bounded functions in multiply-connected domains.

The author has ended each section with a long list of carefully selected problems, many of which will present a challenge to the student. The book, written in an informal style, is very readable. In some places, however, this informality leads to carelessness in stating hypotheses. For example, in Chapter I, the author states, "the boundary value problem of the second kind can be reduced to a boundary value problem of the first kind." He then gives a proof of this statement ignoring the fact that the domain must be simply-connected. In fact, in the following section, he notes that connectivity was not referred to in the preceding section; but "these results are true regardless of whether the domains in question are simply-connected or multiply-connected." He does at least point out here that one does have to worry about single-valued conjugate harmonic functions in multiply-connected domains. Another example appears when the Cauchy integral theorem is proved for functions with a derivative on the boundary and later, in proving the symmetry principle, is used for functions only continuous on the boundary. Also in Chapter VII, the author claims "to prove the existence of the solution of the Dirichlet problem in the case of a general domain of finite connectivity," not stating how *general*. In reality, he proves it for domains bounded by smooth Jordan curves. The reviewer regrets that the lucid style of the author was not utilized to give a textbook presentation of the theory of prime ends, which would have been appropriate in a book of this type and is lacking in the literature. It is also rather unfortunate that the author followed the only too prevalent custom of claiming that this work, being a textbook, need not be documented. No references to original sources or to other works are given. This is true even when, in general discussions, the author makes statements he does not prove in the book. The value of this excellent book to the graduate student would have been enhanced considerably if it had also furnished a key to further study.

G. SPRINGER

The theory of functions of a real variable. By R. L. Jeffery. (Mathematical Expositions, No. 6.) University of Toronto Press, 1951. 13+232 pp. $6.00.

This book consists of two distinct parts. The first part (Chapters I-V) gives a general introduction to functions of a real variable, measure, and integration, while the second part (Chapters VI and VII, with Chapter VIII as a kind of appendix) treats the problem of inverting the derivative of continuous functions, leading to the Denjoy integrals, and studies the derivates and approximate derivates of functions of a real variable on arbitrary linear sets. The author himself, who in previous papers has made some valuable contributions to these topics, considers the presentation of this second part as the main purpose of his book. In both parts only functions of one real variable are discussed.

After an introduction concerning the real number system, Chapter I deals with sets, sequences, and functions and Chapter II with metric properties of sets. Here the author considers only the outer measure of a set A, called by him the metric of A and designated by $|A|^0$. Two sets A and B are metrically separated if for $\epsilon>0$ there exist two open sets $\alpha\supset A$ and $\beta\supset B$ with $|\alpha\beta|^0<\epsilon$. If A and its complement are metrically separated, then A is measurable and $|A|^0$ is designated by $|A|$, the measure of A. In the statement of Theorem 2.15 one has to add the condition "provided that $|A_n|$ is finite for some n." Chapter III introduces the Lebesgue integral in the classical Lebesgue manner. Immediately after the definition of the Lebesgue integral, "for purposes of comparison and contrast," the Riemann integral is also defined and discussed in one short section. Later, an analogous representation of the Lebesgue integral is given, depending upon a subdivision of the domain on which the function is defined. Chapter IV discusses properties of the Lebesgue integral, including integrability of sequences and integrals containing a parameter, and finally gives a detailed proof of the ergodic theorem. In Chapter V the Vitali covering theorem is first proved and then applied to a discussion of metric density of sets and to the proof of the theorem that a functon $f(x)$ of bounded variation on a set A is differentiable almost everywhere on A; the summability of this derivative is also proved. This chapter is concluded by a short introduction to functions of sets.

Chapter VI gives a systematic and thorough discussion of the problem of inversion of derivatives. In particular, the case of a finite, but non-summable, derivative of a continuous function $F(x)$ in an interval $[a, b]$ is studied; $F(x)-F(a)$ can here be determined by at most denumerably many operations on $F'(x)$, and so the general Denjoy integral, in the very comprehensive form of W. H. Young, is obtained. Finally, descriptive definitions of the special and general Denjoy integrals are discussed; in this last section of Chapter VI some results are only stated, but not proved.

Chapter VII is devoted to the study of derivates and approximate derivates (or, as the author says, approximate derived numbers). Besides the Weierstrass nondifferentiable function, the Besicovitch function without unilateral derivatives is also discussed, in the interpretation given to it by E. D. Pepper, but with different methods. Then, for arbitrary functions $f(x)$ defined on arbitrary sets, the four principal derivates and their mutual relations, as well as the approximate derivates and their relations, are studied systematically. The author here essentially gives a detailed discussion of the very general results obtained by him in Ann. of Math. vol. 36 (1935) pp. 438–447. He now modifies his definition of the four principal approximate derivates of $f(x)$ on a set A, bringing it into the following more concise and inclusive form: The upper right approximate derivate of $f(x)$ at the point x over the set A,D^+f, is the supremum of the numbers a for which

$$\frac{f(\xi) - f(x)}{\xi - x} \geqq a, \qquad \xi > x, \qquad \xi \in A,$$

for a set ξ of right density greater than zero at x. For the discussion of these approximate derivates the notion (already used by him, loc. cit.) of a function $f(x)$ metrically separable relative to the set A is essential; that is, for every a the two sets $A\,[f(x) < a]$ and $A\,[f(x) \geqq a]$ have to be metrically separated. At the end of this chapter, relations to H. Blumberg's investigations on arbitrary functions [Acta Math. vol. 65 (1935) pp. 263–282] are also outlined.

The last Chapter VIII discusses the Riemann-Stieltjes integral and, in connection with it, linear functionals. The author finally indicates how this leads to the idea of the distributions of L. Schwartz.

So we see that many interesting topics are treated in this book. Moreover, the presentation is very careful and readable.

ARTHUR ROSENTHAL

BRIEF MENTION

Einführung in die Funktionentheorie. By L. Bieberbach. 2d ed. Bielefeld, Verlag für Wissenschaft und Fachbuch, 1952. 220 pp., 43 figs. 12.60 DM.

The present volume is a second revised edition of Bieberbach's well known *Einführung in die Funktionentheorie.* It is intended for the reader who possesses a modicum of classical real analysis. Applications of complex variable methods to hydrodynamics and potential theory are treated. In general, emphasis is put on those aspects of classical complex function theory which are of interest in the applications. In addition to the usual standard material of a first course on the theory of functions of a complex variable, there is a section on practical aspects of conformal mapping which includes a brief account of Bergman's orthogonalization methods.

A number of sections are followed by supplementary material which serves in part as exercises and in part as indications of further developments of the subject matter treated in the corresponding section. In this connection the proof due to Ankeny of the fundamental theorem of algebra which is based directly on the Cauchy integral theorem is worth mention (pp. 85–86).

MAURICE HEINS

NOTES

The First General Assembly of the International Mathematical Union was held in Rome, Italy, March 6–8, 1952. The Unione Matematica Italiana acted as host to the Assembly.

The following eighteen countries were represented by official delegates: Australia, Austria, Belgium, Denmark, Finland, France, Germany, Greece, Great Britain, Italy, Japan, Netherlands, Norway, Peru, Spain, Switzerland, the United States of America, and Yugoslavia. Poland sent two observers, while Portugal sent one such representative. UNESCO was represented by Professor R. Berker of the Department of Natural Sciences, while the International Council of Scientific Unions was represented by Dr. R. Fraser of the Liaison Office, ICSU and UNESCO. The United States was represented by Professor Marshall H. Stone, Chairman of the United States Delegation, Professors J. R. Kline, Einar Hille, Saunders MacLane and G. T. Whyburn.

It was the unanimous opinion of the United States Delegation that the meetings were a decided success.

The General Assembly voted to convene its next ordinary meeting in connection with the International Congress of Mathematicians scheduled to be held in the Netherlands in 1954.

The following officers were elected:

President: M. H. Stone (U.S.A.)
1st Vice-President: E. Borel (France)
2nd Vice-President: E. Kamke (Germany)
Secretary: E. Bompiani (Italy)
Members-at-Large of the Executive Committee:
W. V. D. Hodge (Great Britain)
S. Iyanaga (Japan)
B. Jessen (Denmark).

The Nachrichten der Oesterreichischen Mathematischen Gesellschaft has recently begun to publish international mathematical news. Of special interest to the editors are such items as permanent changes of position and temporary travels abroad. Since the International Mathematical Union is studying the possibility of using the Nachrichten as its official news bulletin, the journal may soon acquire a large international circulation and be of great value to all mathematicians. The members of the Society, and especially chairmen of departments of mathematics or their deputies, are cordially requested to inform the editors of all up-to-date mathematical news.

Items concerning the United States should be sent to Paul R. Halmos, Eckhart Hall, University of Chicago, Chicago 37, Illinois.

As a tribute in memory of the late Professor Abraham Wald, who was killed with Mrs. Wald in an airplane accident in India, December 13, 1950, a group of friends and colleagues are establishing a fund to help in defraying the expenses of a college education for his two children Betty, eight years old, and Bobby, now four. Trustees of the fund are Theodore W. Anderson, Howard Levene, and Mortimer Spiegelman. Contributions may be made payable to Wald Memorial Fund and may be sent to Howard Levene, Box 23 Fayerweather, Columbia University, New York 27, New York.

Associate Professor E. S. Kennedy of American University of Beirut, Lebanon, has been serving as Fulbright Lecturer at Tehran University, Iran.

Dr. Murray Gerstenhaber of Harvard University has been awarded a Frank B. Jewett Postdoctoral Fellowship of the Bell Telephone Laboratories for 1952–1953.

Professor M. H. Heins of Brown University has received a President's Fellowship for study in Europe in 1952–1953.

Dr. Henry Helson of Yale University has been awarded a Frank B. Jewett Postdoctoral Fellowship of the Bell Telephone Laboratories for 1952–1953.

Professor I. I. Hirschman of Washington University has been awarded a Guggenheim Fellowship and will spend the year 1952–1953 as a member of the Institute for Advanced Study.

Professor O. E. Neugebauer of Brown University has been awarded the John F. Lewis prize of the American Philosophical Society.

Mr. E. T. Onat of Brown University has been awarded a Frank B. Jewett Postdoctoral Fellowship of the Bell Telephone Laboratories for 1952–1953.

Professor Herbert Robbins of the University of North Carolina has been awarded a Guggenheim Fellowship and will spend the year 1952–1953 as a member of the Institute for Advanced Study.

Dr. A. L. Blakers of Lehigh University has been appointed to a professorship at the University of Western Australia.

Dr. A. W. Boldyreff of the Sandia Corporation has accepted a position as a research specialist at North American Aviation, Downey, California.

Mr. C. W. Bostick of the University of Illinois has accepted a position as analyst with the Armed Forces Security Agency.

Mr. H. N. Browne of Harvard University has accepted a position with the ENIAC Branch, Aberdeen Proving Ground, Maryland.

Dr. W. C. Carter of the Ballistic Research Laboratories has accepted a position as senior engineer in the computing department of the Raytheon Company, Waltham, Massachusetts.

Associate Professor Herman Chernoff of the University of Illinois has been appointed to an associate professorship at Stanford University.

Professor Arnold Dresden of Swarthmore College has retired.

Dr. Louis Garfin of the Insurance Department of the State of Oregon has accepted a position as associate actuary with the Pacific Mutual Life Insurance Company, Los Angeles, California.

Professor R. E. Gilman of Brown University is on leave of absence and has accepted a position with the Weapons Systems Evaluation Group, Department of Defense, Washington, D. C.

Dr. R. E. Graves has accepted a position as mathematician with the Goodyear Aircraft Corporation, Litchfield Park, Arizona.

Assistant Professor Simon Gruenzweig of Lincoln University has been appointed to a professorship at Philander Smith College.

Miss Emilie V. Haynsworth of the University of North Carolina has been appointed to an assistant professorship at Wilson College.

Mr. D. W. Hullinghorst of the U. S. Army Base, San Francisco, has accepted a position as hydrologist with Sanderson and Porter, Engineers, New Orleans, Louisiana.

Dr. W. G. Lister of Yale University has been appointed to an assistant professorship at Brown University.

Professor R. W. Marriott of Swarthmore College has retired.

Mr. Paul Meier of Philadelphia Tuberculosis and Health Association has accepted a position at the Forrestal Research Center, Princeton, New Jersey.

Assistant Professor Daniel Orloff of Southern Illinois University has accepted a position as mathematician with the Bell Aircraft Corporation, Buffalo, New York.

Assistant Professor I. H. Rose of the University of Massachusetts has been appointed to a visiting professorship at Mount Holyoke College.

Mr. M. A. Shader of Stanford University has accepted a position as aeronautical research scientist with the Ames Laboratory, National Advisory Committee for Aeronautics, Moffett Field, California.

Mr. D. J. Smith has accepted a position as engineer with Breeze Corporations, Newark, New Jersey.

Dr. R. H. Stark of the Los Alamos Scientific Laboratory has accepted a position as research associate with the Knolls Atomic Power Laboratory, Schenectady, New York.

Mr. Arthur Stein of the Ballistic Research Laboratories has accepted a position as ordnance engineer with the Ordnance Ammunition Center, Joliet, Illinois.

Assistant Professor J. W. Warwick of Wellesley College has accepted a position as scientific consultant with the Harvard College Observatory, High Rolls, New Mexico.

Miss Betty R. Weber of the University of South Carolina has accepted a position as engineer with Melpar, Incorporated, Alexandria, Virginia.

The following promotions are announced:

Dorothy L. Bernstein, University of Rochester, to an associate professorship.

W. C. Doyle, Rockhurst College, to a professorship.

W. T. Fishback, University of Vermont, to an assistant professorship.

I. I. Hirschman, Washington University, to an associate professorship.

D. B. Houghton, Franklin Institute, to Acting Chief of the Analysis Section.

A. W. Landers, Brooklyn College, to an associate professorship.

W. S. Massey, Brown University, to an associate professorship.

D. R. Ryan, Gonzaga University, to an associate professorship.

M. C. Sholander, Washington University, to an associate professorship.

Dr. Harold Weintraub, Tufts College, to an assistant professorship.

The following appointments to instructorships are announced: California Institute of Technology: Dr. H. A. Dye, Mr. F. B. Fuller; Fresno State College: Mr. D. J. Ewy; Princeton University: Dr. L. J. Snell.

Professor Guido Castelnuovo, president of the Accademia Nazionale dei Lincei, died on April 27, 1952.

Mr. H. R. Bassford of the Metropolitan Life Insurance Company died on March 12, 1952, at the age of sixty-three years. He had been a member of the Society for twenty-seven years.

Dr. L. L. Jackson died on March 28, 1952, at the age of eighty-one years. He had been a member of the Society for fifty years.

Mr. J. J. Newman of Columbia University died on September 10, 1951, at the age of twenty-three years.

Dr. C. D. Perrine died on June 17, 1952, at the age of eighty-three years.

Mr. R. E. Powers died on January 31, 1952, at the age of seventy-six years. He had been a member of the Society for forty-one years.

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SOME ASPECTS OF THE SEQUENTIAL DESIGN OF EXPERIMENTS

HERBERT ROBBINS

1. Introduction. Until recently, statistical theory has been restricted to the design and analysis of sampling experiments in which the size and composition of the samples are completely determined before the experimentation begins. The reasons for this are partly historical, dating back to the time when the statistician was consulted, if at all, only after the experiment was over, and partly intrinsic in the mathematical difficulty of working with anything but a fixed number of independent random variables. A major advance now appears to be in the making with the creation of a theory of the *sequential design* of experiments, in which the size and composition of the samples are not fixed in advance but are functions of the observations themselves.

The first important departure from fixed sample size came in the field of industrial quality control, with the double sampling inspection method of Dodge and Romig [1]. Here there is only one population to be sampled, and the question at issue is whether the proportion of defectives in a lot exceeds a given level. A preliminary sample of n_1 objects is drawn from the lot and the number x of defectives noted. If x is less than a fixed value a the lot is accepted without further sampling, if x is greater than a fixed value b $(a<b)$ the lot is rejected without further sampling, but if $a \leqq x \leqq b$ then a second sample, of size n_2, is drawn, and the decision to accept or reject the lot is made on the basis of the number of defectives in the total sample of n_1+n_2 objects. The total sample size n is thus a random variable with two values, n_1 and n_1+n_2, and the value of n is stochastically dependent on the observations. A logical extension of the idea of double sampling came during World War II with the development, chiefly by Wald, of sequential analysis [2], in which the observations are made one by one and the decision to terminate sampling and to accept or reject the lot (or, more generally, to accept or reject whatever statistical "null hypothesis" is being tested) can come at any stage. The total sample size n now becomes a random variable capable in principle of assuming infinitely many values, although in practice a finite upper limit on n is usually set. The advantage of sequential

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over fixed-size sampling lies in the fact that in some circumstances the judicious choice of a sequential plan can bring about a considerable reduction in the average sample size necessary to reduce the probability of erroneous decision to a desired low level. The theory of sequential analysis is still very incomplete, and much work remains to be done before optimum sequential methods become available for treating the standard problems of statistics.

The introduction of sequential methods of sampling freed statistics from the restriction to samples of fixed size. However, it is not only the sample *size* that is involved in the efficient design of an experiment. Most statistical problems met with in practice involve more than one population, and in dealing with such problems we must specify *which population* is to be sampled at each stage. An example will serve to clarify this point. Suppose we are dealing with two normally distributed populations with unknown means μ_1, μ_2 and variances σ_1^2, σ_2^2, and that we wish to estimate the value of the difference $\mu_1-\mu_2$. In order to concentrate on the point at issue we shall suppose that the total sample size, n, is fixed. There remains the question of how the n observations are to be divided between the two populations. If $\bar{x}_1$, $\bar{x}_2$ denote the means of samples of sizes n_1, n_2 from the two populations, then $\bar{x}_1-\bar{x}_2$ is an unbiased estimator of $\mu_1-\mu_2$, with variance $\sigma^2=(\sigma_1^2/n_1)+(\sigma_2^2/n_2)$. For fixed $n=n_1+n_2$, σ^2 is a minimum when $n_1/n_2=\sigma_1/\sigma_2$. If the latter ratio is known in advance, all is well. If this ratio is not known, but if the sampling can be done in two stages, then it would be reasonable to draw preliminary samples of some size m from each of the two populations and to use the values so obtained to estimate σ_1/σ_2; the remainder of the $n-2m$ observations could then be allocated to the two populations in accordance with the sample estimate of σ_1/σ_2. The question then becomes, what is the best choice for m? If m is small, no accurate estimate of σ_1/σ_2 can be made. If m is large, then the remaining $n-2m$ observations may be too few to permit full utilization of the approximate knowledge of σ_1/σ_2. (This kind of dilemma is characteristic of all sequential design problems.) More generally, we could consider schemes in which the observations are made one by one, with the decision as to which population each observation should come from being allowed to depend on all the previous observations; the total sample size n could be fixed or could be a random variable dependent on the observations.

Despite the total absence of theory, a notable pioneering venture in the spirit of sequential design was carried out in 1938 by Mahalanobis [3] to determine the acreage under jute in Bengal. Preliminary surveys were made on a small scale to estimate the values of certain parameters, a knowledge of which was essential to the efficient design of a subsequent large scale census. In a subsequent publication [4] Mahalanobis called attention to the desirability of revising the design of any experiment as data accumulates. The question, of course, is how best to do this.

We are indebted to Wald for the first significant contribution to the theory of sequential design. His book [5] states the problem in full generality and gives the outline of a general inductive method of solution. The probability problems involved are formidable, since dependent probabilities occur in all their complexity, and explicit recipes are not yet available for handling problems of practical interest. Nevertheless, enough is visible to justify a prediction that future results in the theory of sequential design will be of the greatest importance to mathematical statistics and to science as a whole.

In what follows we shall discuss a few simple problems in sequential design which are now under investigation and which are different from those usually met with in statistical literature. Optimum solutions to these problems are not known. Still, it is often better to have reasonably good solutions of the proper problems than optimum solutions of the wrong problems. In the present state of statistical theory this principle applies with particular force to problems in sequential design.

2. **A problem of two populations.** Let A and B denote two statistical populations (coins, urns, manufacturing processes, varieties of seed, treatments, etc.) specified respectively by univariate cumulative distribution functions $F(x)$ and $G(x)$ which are known only to belong to some class D. We shall suppose that the expectations

$$\alpha = \int_{-\infty}^{\infty} x\,dF(x), \qquad \beta = \int_{-\infty}^{\infty} x\,dG(x) \tag{1}$$

exist. *How should we draw a sample* $x_1, x_2, \cdots, x_n$ *from the two populations if our object is to achieve the greatest possible expected value of the sum* $S_n = x_1 + \cdots + x_n$?

For example, let A and B denote two coins of unknown bias, and suppose that we are allowed to make n tosses, with the promise of getting $1 for each head but nothing for tails. If $x_i = 1$ or 0 according as heads or tails occurs on the ith toss, then S_n denotes the total sum which we are to receive, and α and β $(0 \leq \alpha, \beta \leq 1)$ are the respective probabilities of obtaining heads on a single toss of coins A and B.

As a general intuitive principle, whenever we feel pretty sure from the results of previous observations that one of the two numbers α, β is the greater, we shall want to devote more of our future observations to that population. Note that there is no terminal decision to make; that is, we are not interested in estimating $\alpha-\beta$ or in testing the hypothesis that $\alpha=\beta$, etc., after the sample is drawn. The whole problem lies in deciding *how to draw the sample.* There certainly exist practical situations in which the present problem represents more nearly what one wants to solve than would any formulation in terms of testing hypotheses, estimating parameters, or making terminal decisions. In fact, the problem represents in a simplified way the general question of how we learn—or should learn—from past experience. A reasonably good solution of the problem must therefore be found if mathematical statistics is to provide a guide to what has been called by Neyman [6] inductive behavior.

To begin with we shall consider the special case already mentioned in which A and B are coins and the unknowns α and β are the respective probabilities of obtaining heads ($x_i=1$) on a single trial. Let us take as an example of a possible sampling rule the following.

Rule R_1. For the first toss choose A or B at random. Then, for $i=1, 2, \cdots$, if the ith toss results in heads, stick to the same coin for the $(i+1)$th toss, while if the ith toss results in tails, switch to the other coin for the $(i+1)$th toss.

What are the operating characteristics of the rule R_1? The successive tosses represent the evolution of a simple Markov chain with four states, (A, H), (A, T), (B, H), (B, T), and with transition probabilities which are easily written down; for example, the probability of a transition from (A, H) on the ith toss to (A, T) on the $(i+1)$th toss is $1-\alpha$. Let p_i denote the probability of obtaining heads on the ith toss. To avoid trivialities we shall suppose that α and β are not both 0 or both 1; then $|\alpha+\beta-1|<1$. It is easy to show that

$$p_{i+1} = (\alpha+\beta-1)p_i + (\alpha+\beta-2\alpha\beta), \tag{2}$$

from which it follows that

$$p_i = (\alpha+\beta-1)^{i-1}\left[p_1 - \frac{\alpha+\beta-2\alpha\beta}{2-(\alpha+\beta)}\right] + \frac{\alpha+\beta-2\alpha\beta}{2-(\alpha+\beta)}, \tag{3}$$

and hence that

$$\lim_{i\to\infty} p_i = \frac{\alpha+\beta-2\alpha\beta}{2-(\alpha+\beta)} = \gamma + \frac{\delta^2}{1-\gamma}, \tag{4}$$

where we have set

$$\gamma = \frac{\alpha + \beta}{2}, \qquad \delta = \frac{|\alpha - \beta|}{2}. \tag{5}$$

It follows that in using the rule R$_1$,

$$\lim_{n\to\infty} \mathrm{E}\left(\frac{S_n}{n}\right) = \lim_{n\to\infty} \left(\frac{p_1 + \cdots + p_n}{n}\right) = \gamma + \frac{\delta^2}{1-\gamma}. \tag{6}$$

Now, if we *knew* which of the two numbers α, β is the greater, then by using the corresponding coin exclusively we could achieve the result

$$\mathrm{E}\left(\frac{S_n}{n}\right) = \max(\alpha, \beta) = \gamma + \delta. \tag{7}$$

Hence it is natural to take the difference

$$L(A, B, \mathrm{R}_1) = (\gamma + \delta) - \left(\gamma + \frac{\delta^2}{1-\gamma}\right) = \delta\left[1 - \frac{\delta}{1-\gamma}\right] \geqq 0 \tag{8}$$

as a measure of the asymptotic *loss* per toss, by a person who uses R$_1$, *due to ignorance of the true state of affairs.* It is easy to show that $L(A, B, \mathrm{R}_1)$ has its maximum value, $M_1 = 3 - 2^{3/2} \cong .172$, when $\alpha = 0$ and $\beta = 2 - 2^{1/2} \cong .586$ or vice versa. Thus a person using R$_1$ will, for large n, lose on the average at most 17.2 cents per toss due to ignorance of which is the better coin. On the other hand, consider the rule R$_0$ which consists in choosing one of the two coins A, B at random and then sticking to it, come what may (or in tossing the two coins alternately). The corresponding quantity $L(A, B, \mathrm{R}_0)$ is easily seen to have the value $(\gamma+\delta) - \gamma = \delta$, which has its maximum, $M_0 = 1/2$, when $\alpha = 0$ and $\beta = 1$ or vice versa. Clearly, R$_1$ is considerably better than R$_0$.

The rule R$_0$ makes the choice of the coin for the ith toss independent of the results of previous tosses, while R$_1$ makes this choice depend on the result of the $(i-1)$th toss only. For the most general rule R the choice of the coin for the ith toss will depend on the results $x_1, \cdots, x_{i-1}$ of all the previous tosses. For any such rule R let

$$L_n(A, B, \mathrm{R}) = \max(\alpha, \beta) - \mathrm{E}\left(\frac{S_n}{n}\right) \tag{9}$$

where E denotes expectation computed on the basis of α, β, and R, and let

$$M_n(\mathrm{R}) = \max_{(\alpha,\beta)} [L_n(A, B, \mathrm{R})], \tag{10}$$

$$\phi(n) = \min_{(R)} [M_n(R)]. \tag{11}$$

It would be interesting to know the value of $\phi(n)$ and the explicit description of any "minimax" rule R for which the value $\phi(n)$ is attained.

A much simpler problem is: do there exist rules R such that

$$\lim_{n\to\infty} L_n(A, B, R) = 0 \qquad \text{for every } A, B? \tag{12}$$

We shall see in the next paragraph that the answer is yes, not only in the case of the coins but for any two populations.

Returning to the general case in which A and B are arbitrary statistical populations for which the values (1) exist, consider the sampling rule $\overline{R}$ defined as follows: let

$$\begin{aligned} 1 &= a_1 < a_2 < \cdots < a_n < \cdots, \\ 2 &= b_1 < b_2 < \cdots < b_n < \cdots \end{aligned} \tag{13}$$

be two fixed, disjoint, increasing sequences of positive integers of density 0; that is, such that the proportion of the integers $1, 2, \cdots, n$ which are either a's or b's tends to 0 as $n \to \infty$. We define inductively: if the integer i is one of the a's, take the ith observation, x_i, from population A, if i is one of the b's, take x_i from B, and if i is neither one of the a's nor one of the b's, take x_i from A or B according as the arithmetic mean of all previous observations from A exceeds or does not exceed the arithmetic mean of all previous observations from B. It can be shown to follow from the strong law of large numbers that with probability 1,

$$\lim_{n\to\infty} \frac{S_n}{n} = \max(\alpha, \beta). \tag{14}$$

This in turn can be shown to imply the relation

$$\lim_{n\to\infty} \mathrm{E}\left(\frac{S_n}{n}\right) = \max(\alpha, \beta), \tag{15}$$

so that

$$\lim_{n\to\infty} L_n(A, B, \overline{R}) = \max(\alpha, \beta) - \lim_{n\to\infty} \mathrm{E}\left(\frac{S_n}{n}\right) = 0 \tag{16}$$

for any A, B such that α, β exist.

3. **Some other problems of sequential design.** The problem of

§2 can be generalized in various ways. For one thing, we can let the total sample size n be a random variable, either independent of the observations or dependent on them. As an example of the latter case, suppose in the problem of the two coins that we have to pay a fixed amount c for the privilege of making each toss. We may then decide to stop tossing whenever it seems pretty certain that $\max(\alpha, \beta) < c$; this amounts to a special case of a problem of three populations. We can even consider the case of a continuum of populations. Suppose we can apply a certain treatment to some plant or animal at any intensity θ in some interval, and let $F(x, \theta)$ be the cumulative distribution function of the response x to the treatment of intensity θ. The expected value

$$\alpha(\theta) = \int_{-\infty}^{\infty} x dF(x, \theta), \tag{17}$$

the "regression" of x on θ, is assumed to be unknown. Let $\{\theta_i\}$ denote any sequence of θ values, chosen sequentially by the experimenter, and let $\{x_i\}$ denote the corresponding sequence of responses, so that each x_i has the distribution $\Pr[x_i \leqq x] = F(x, \theta_i)$. (I) Suppose $\alpha(\theta)$ has a unique maximum at some unknown point θ_0. How should the experimenter choose the sequence $\{\theta_i\}$ in order to maximize the expected value of the sum $S_n = x_1 + \cdots + x_n$ or, alternatively, in order to estimate the value of θ_0? (II) Suppose $\alpha(\theta)$ is an increasing function of θ which takes on a given value α_0 at some unknown point θ_0. How should the experimenter choose the sequence $\{\theta_i\}$ in order to estimate the value of θ_0? Problem I is the problem of the experimental determination of the maximum of a function when the observations are subject to a random error; Problem II is fundamental in sensitivity testing and bioassay.

It is clear that in both of these problems the choice of each θ_i should be made to depend on the responses $x_1, \cdots, x_{i-1}$ at the previous levels $\theta_1, \cdots, \theta_{i-1}$ of the treatment, so that we are dealing with problems of sequential design. The non-sequential study of Problem I was initiated by Hotelling [7] (see also [8]), but no sequential theory has yet been published. Problem II has been considered by Robbins and Monro [9]; their method is as follows. Let $\{a_n\}$ be a sequence of positive constants such that

$$\sum_1^{\infty} a_n^2 < \infty, \qquad \sum_1^{\infty} a_n = \infty, \tag{18}$$

let θ_1 be arbitrary, and set

$$\theta_{n+1} = \theta_n + a_n(\alpha_0 - x_n) \qquad (n = 1, 2, \cdots). \tag{19}$$

Then under certain mild restrictions on $F(x, \theta)$ it can be shown that

$$\lim_{n\to\infty} \theta_n = \theta_0 \qquad \text{in probability.} \tag{20}$$

In this and other problems, any sequential design with reasonably good properties is likely to find an appreciative audience. This will encourage the use of random sampling methods to find empirical approximations to the operating characteristics of sequential designs when a full mathematical solution is difficult. An empirical study of the rapidity of convergence in (20) has been made by Teichroew [10].

4. **The problem of optional stopping.** To fix the ideas, let x be normally distributed with unknown mean θ and unit variance. Suppose we wish to test the null hypothesis, H_0, that $\theta = 0$ against the alternative, H_1, that $\theta > 0$. The standard statistical test based on a sample of fixed size n is the following. Let $S_n = x_1 + \cdots + x_n$ and reject H_0 in favor of H_1 if and only if

$$S_n > \alpha n^{1/2}, \tag{21}$$

where α is some constant. The probability of rejecting H_0 when it is true will then be

$$\epsilon(\alpha) = 1 - \Phi(\alpha), \tag{22}$$

where we have set

$$\Phi(x) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{x} e^{-t^2/2} dt, \tag{23}$$

and by choosing α large we can make $\epsilon(\alpha)$ as small as we please. For example, if $\alpha = 3.09$ then $\epsilon(\alpha) \cong .001$.

Suppose now that H_0 is true but that an unscrupulous experimenter wishes to get an unwary statistician to reject it. If the sample size n has not been agreed upon in advance the experimenter could adopt the technique of stopping the sampling as soon as the inequality (21) is verified. The law of the iterated logarithm of probability theory implies that with probability 1 the inequality (21) will hold for infinitely many values of n if the sampling continues indefinitely, no matter how large the value of α. Hence the experimenter is "sure" to come eventually to a value of n for which (21) holds, and by stopping the experiment at this point he will cause the statistician to reject H_0 even though it is true. This fact immediately vitiates the use of (21) as a test of H_0 if there is any possibility that optional stopping may be involved.

The simplest way for the statistician to guard against the effect of optional stopping is to insist that the size of the sample be fixed in advance of the experimentation. Such a restriction would often be too rigid for practical use. The statistician might therefore content himself with setting limits $n_1 \leqq n \leqq n_2$ on the sample size which will be flexible enough to meet the contingencies of experimentation but narrow enough to eliminate the worst effects of optional stopping. To this end the statistician would like to know the value of the function

$$g(n_1, n_2, \alpha) = \Pr\left[S_n > \alpha n^{1/2} \text{ for some } n_1, \leqq n \leqq n_2\right], \tag{24}$$

where the x_i are independent and normal (0, 1). It is quite easy to establish the inequality

$$g(n_1, n_2, \alpha) < \frac{1 - \Phi(\alpha)}{1 - \Phi\left(\alpha \cdot \dfrac{\lambda^{1/2} - 1}{(\lambda - 1)^{1/2}}\right)}, \text{ where } \lambda = \frac{n_2}{n_1}, \tag{25}$$

which is useful when λ is not too large, and sharper inequalities can no doubt be devised.

The problem of optional stopping has received little attention in statistical theory. (See, however, [11], especially pp. 286–292.) One need not assume that the experimenter is consciously trying to deceive the statistician—the two are often the same person—to recognize the desirability of devising methods of statistical analysis which would be relatively insensitive to the effects of optional stopping.

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UNIVERSITY OF NORTH CAROLINA

PARTITIONING CONTINUOUS CURVES

R. H. BING

I. INTRODUCTION

1. **The continuous curve.** By a continuous curve we mean a compact, locally connected, metric continuum. For definitions of locally connected, continuum, and other terms used in this paper, see either [18] or [21]. Those who like to visualize topology concretely may wish to think of a continuous curve as a chunk out of Euclidean 3-space—one that is connected (all in one piece), one that is bounded (lies on the interior of a sphere), and one that is locally connected (nearby points belong to small connected subsets). A wad of paper, an irregularly shaped rock, or the earth itself may be considered as examples. However, our remarks about continuous curves will apply equally well to those in Euclidean spaces of all dimensions and to those in a Hilbert space.

When Jordan first introduced the term continuous curve, he defined it analytically to be the image (in the plane) of a straight line interval under a continuous transformation. It was not until over twenty years later that it was discovered that any compact locally connected metric continuum was the image of a straight line interval under a continuous transformation and conversely. This interesting and unusual discovery adds spice to the study of mathematics [24, p. 12]. Another interesting aspect of this discovery is that it was made independently by two mathematicians, Hahn and Mazurkiewicz. Since Peano had shown earlier that a square plus its interior is the image of a straight line interval, a continuous curve is sometimes called a Peano continuum.

In this discussion we shall be interested in the continuous curve itself and not the continuous transformation of an interval. Hence, we use the definition in the first paragraph rather than the analytic one. In this discussion we shall be interested in the structure of a continuous curve.

2. **Examples of continuous curves.** A straight line interval, a square plus its interior in the plane, and a cube plus its interior in 3-space are examples of continuous curves. In fact, any closed n-cell or n-simplex is an example. One can get a less familiar continuous

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curve by taking a continuum which is the sum of a finite number of dissimilar continuous curves.

If M is any compact point set whatever, it is possible to obtain a continuous curve by adding to M a countable collection of arcs We use this fact in constructing the following two examples of continuous curves which we shall use.

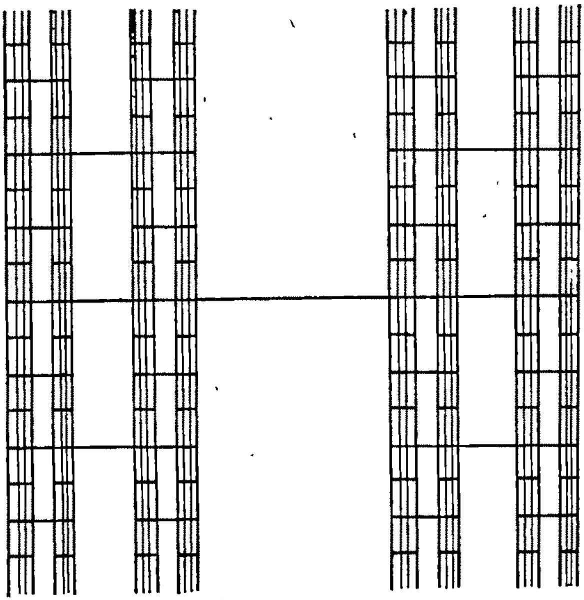

FIG. 1

EXAMPLE A. Let M be the cartesian product of a straight line interval and a Cantor set. Form a continuous curve by adding to M a decreasing sequence of horizontal straight line intervals as indicated in Figure 1. These intervals extend from $(0, 1/2)$ to $(1, 1/2)$, and $(0, 1/4)$ to $(1/3, 1/4)$, from $(2/3, 1/4)$ to $(1, 1/4)$, from $(0, 3/4)$ to $(1/3, 3/4)$, $\cdots$.

EXAMPLE B. It is shown in [5] that there is a bounded continuum S in Euclidean 3-space that is irreducible with respect to separating 3-space and has the property that if two of its subcontinua intersect, one is a subset of the other. The continuum S is 2-dimensional. It is called hereditarily indecomposable because no subcontinuum of it is the sum of two proper subcontinua of the subcontinuum. A continuous curve M may be formed by adding to S the sum of a decreasing sequence of mutually exclusive arcs. If K is a subset of M irreducible with respect to separating M, each component of K is indecomposable.

3. **Partitioning.** An interval can be partitioned. This fact is used in defining the integral of a continuous function over an interval. ($\int_a^b f(x)dx = \lim \sum f(\xi_i)\, \Delta x_i$ where ξ_i is a point of and Δx_i is the length of the ith element of the partitioning of $[a, b]$.)

In studying a square plus its interior it is sometimes convenient to partition it into small pieces. For example, in integrating over a square region, we may partition it into small rectangles (if we use rectangular coordinates) or into pieces of other shapes (in polar coordinates for instance).

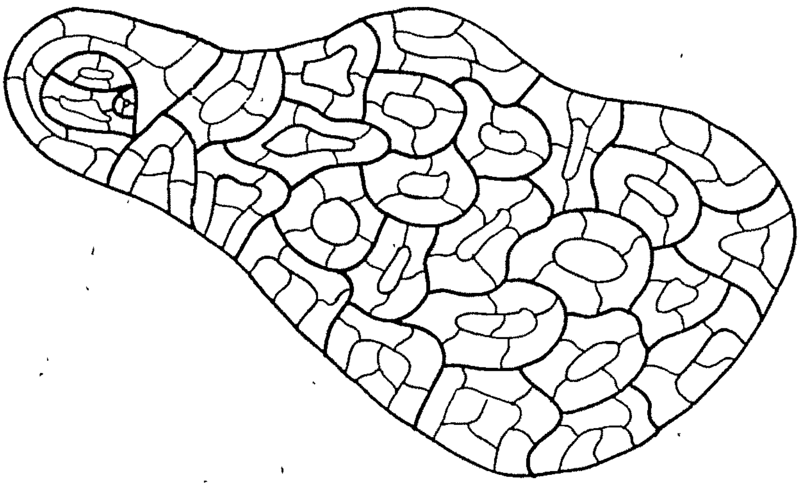

FIG. 2

A partitioning of a continuous curve M is a finite collection G of mutually exclusive connected open subsets of M whose sum is dense in M [6]. If the mesh of G is less than ϵ (each element of G is of diameter less than ϵ), G is called an ϵ-partitioning. The essential difference between a partitioning and a grating [13, p. 279] is that the elements of a partitioning are connected. It is much like a grille decomposition [15].

In defining the integral over a plane domain R ($\iint_R f(x, y)dS = \lim$ (mesh of partitioning $\rightarrow 0$) $\sum f(\xi_i, \eta_i)\Delta S_i$) we do not need to impose the conditions that the elements of the partitioning are either connected, mutually exclusive, or open. However, we shall be using partitionings in connections where these properties are important.

4. Decreasing sequence of partitionings. If G and H are two partitionings of the same continuous curve, G is called a refinement of H if each element of G is a subset of an element of H. In Figure 2, the partitioning represented by light lines is a refinement of the one represented by heavy lines. A sequence G_1, G_2, $\cdots$ of partitionings is called a decreasing sequence if, for each positive integer i, G_{i+1} is a refinement of G_i and the mesh of G_i approaches 0 as i increases without limit.

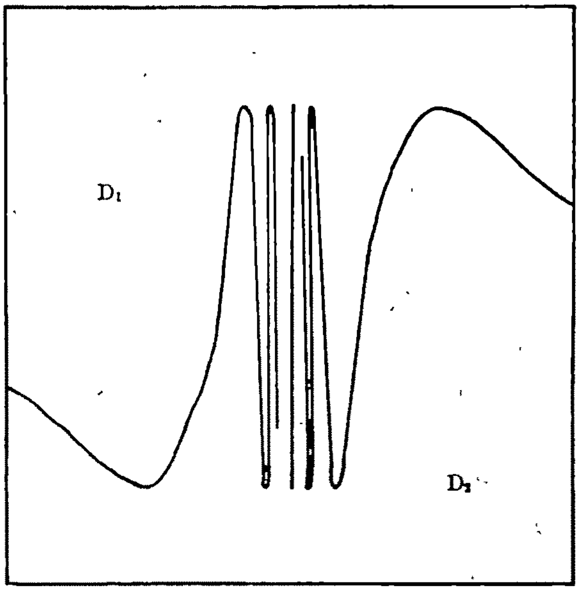

FIG. 3

While it is apparent that there is a decreasing sequence of partitionings of a square plus its interior, it is not so obvious that every continuous curve has such a sequence of partitionings. However, it is true that any continuous curve has such a sequence. This has given us new information about the continuous curve.

5. Purpose of paper. It is only within the last three years that it has been known that each continuous curve can be partitioned [6; 4; 15; 16]. It is the purpose of this paper to point out some recent developments, to show ways in which a continuous curve can be partitioned, and to point out analogies between the ways in which a square plus its interior can be partitioned and the ways in which an arbitrary continuous curve can be partitioned. Details of some proofs are included for completeness, but some readers may prefer to skip them.

II. BREAKING A CONTINUOUS CURVE INTO TWO PIECES

1. **The type of division needed.** If one pursues the proper attack, it is not difficult to show that any continuous curve can be partitioned. However, if one gets off on the wrong track, considerable difficulty may be encountered.

Suppose it is desired to partition a continuous curve into pieces of diameters less than a fixed positive number ϵ. A straightforward method of attack is to divide the continuous curve into two pieces, divide each of the two pieces, and continue this procedure until an ϵ-partitioning results. However, unless certain precautions are taken, this method does not succeed.

If $\epsilon=1$, M is a square plus its interior, and at the first stage it is partitioned into two pieces as shown in Figure 3, the above procedure will not lead to a partitioning. The common boundary of D_1 and D_2 is a part of the closure of the graph of $y=\sin 1/x$ and neither D_1 nor D_2 is the sum of a finite number of connected subsets each of diameter less than 1.

A set has property S if, for each positive number ϵ, the set is the sum of a finite number of connected subsets each of diameter less than ϵ. This notion was used by Sierpinski and when R. L. Moore used it later, he called it property S [19; 17; 18].

A precaution to take in partitioning a continuous curve is to divide it at each step so that the pieces have property S.

We shall be interested in the following result [6].

THEOREM 1. *If H and K are two mutually exclusive closed subsets of a continuous curve M, there are two mutually exclusive open subsets D_H and D_K of M containing H and K respectively such that* (a) *each of D_H and D_K has property* S *and* (b) $M=\bar{D}_H+\bar{D}_K$.

Before proving this theorem, we shall review the method of constructing a set with property S.

2. **Gradual growing process.** This is a method used to form sets with property S. Compare with [22; 21].

A set H is said to ϵ-grow into a set H' in a set M if M contains H', each point of H' belongs to a connected ϵ-subset (subset of diameter less than ϵ) of H' that intersects H, and for some positive number δ, H' contains all connected δ-subsets of M that intersect H.

Let M be a set with property S, H be a subset of M, and ϵ be a positive number. Suppose one wishes to obtain a subset of M with property S which contains H but no point whose distance is more than ϵ from H. Such a set can be obtained by the following gradual growing process. Let $\epsilon_1, \epsilon_2, \cdots$ be a sequence of positive numbers whose sum is no more than ϵ. Let H ϵ_1-grow into a set D_1 in M, D_1 ϵ_2-grow into a set D_2 in $M, \cdots, D_i$ ϵ_{i+1}-grow into a set D_{i+1} in $M, \cdots$. Then the sum D_0 of $D_1, D_2, \cdots$ is an open subset of M, has property S, contains H but no point at a distance more than ϵ from H.

3. **An attempt to divide.** One might try to prove Theorem 1 by letting each of the sets H and K grow gradually in M into mutually exclusive open subsets whose sum is dense in M. Some difficulty may be encountered in trying to keep the two subsets from intersecting and at the same time making their sum dense in M. This difficulty could be overcome if we could answer the following question in the affirmative.

QUESTION. Does there exist a positive integer n such that the following result holds for each continuous curve M, each positive number ϵ, and each pair of mutually exclusive closed subsets H and K of M? *If R is a finite subset of M such that each point of R belongs to an arc in M of diameter less than ϵ that intersects $H+K$, there are two collections A_H and A_K of arcs satisfying the following conditions*:

(a) *Each element of A_H intersects H but not K and each element of A_K intersects K but not H nor any element of A_H.*

(b) *Each element of R belongs to an element of A_H+A_K.*

(c) *Each element of A_H+A_K is of diameter less than $n\epsilon$.*

We cannot let $n=1$ and always get such collections A_H and A_K, for let M be the sum of the interval from $(-10, -1)$ to $(10, 1)$ and the interval from $(-10, 1)$ to $(10, -1)$, $H=\{(-10, 1)\}$, $K=\{(10,1)\}$, $R=\{(-10, -1), (10, -1)\}$, and $\epsilon=11$. Had the answer been in the affirmative for $n=1$, the attempts made in [15] to partition a continuous curve might have succeeded [16].

Since we do not know the answer to the above question, we shall get some other results before proving Theorem 1.

4. **A crude division.** Before proving Theorem 1, it is convenient to prove a weaker theorem.

THEOREM 1′. *Theorem 1 is true if condition* (a) *is replaced by the weaker conditions that D_H has property* S *and D_K has only a finite number of components.*

There are two essentially different methods of proving Theorem 1′. One method is due to Whyburn [22, Corollary, p. 137]. He considers an uncountable family of sets with property S such that of any two of the sets, one contains the closure of the other. He shows that one of the sets can be modified so that it still has property S and the complement of its closure has a finite number of components.

An essentially different method is to let H grow gradually and K in a wild fashion to form sets D_H and D_K satisfying the conditions of Theorem 1′ [6]. This method is similar to another attack we shall use later so we include the details.

Details of second method of proof of Theorem 1′. The open subsets D_H and D_K are formed by permitting H and K to grow so that H grows in a convergent fashion. The procedure is as follows.

We use $\rho(p, q)$ to denote the distance between p and q.

Suppose that, at the ith stage, H and K have been enlarged to sets H_i and K_i such that $\rho(H_i, K_i)>0$ and each point of M belongs to an arc in M of diameter less than $1/2^i$ that intersects H_i+K_i. Let R_i be a finite subset of M such that each point of M belongs to an arc in M of diameter less than $1/2^{i+1}$ that intersects R_i. For each point p of R_i let A_p be an arc in $M-\bar{H}_i$ from p to a point of K_i if there is one; if there is no such arc A_p, let B_p be an arc of diameter less than $1/2^i$ in $M-\bar{K}_i$ from p to a point of H_i.

Use H_i' to denote H_i plus the sum of the B_p's, K_i' to denote K_i plus the sum of the A_p's, and δ_i to denote $\rho(H_i', K_i')$. Let $H_{i+1} = \{p \mid p$ lies on an arc of diameter less than $\delta_i/3$ in M that intersects $H_i'\}$ and $K_{i+1} = \{p \mid p$ lies on an arc of diameter less than $\delta_i/3$ in M that intersects $K_i'\}$.

Since H_i has ϵ_i-grown into H_{i+1} for $\epsilon_i=7/(2^{i+1}\cdot 3)$ and the ϵ_i's have a finite sum, $\sum H_i = D_H$ has property S.

Since K_i has only a finite number of components and each component of K_{i+1} intersects K_i, $\sum K_i = D_K$ has only a finite number of components.

Since no point of M is at a distance of more than $1/2^i$ from H_i+K_i, $M=\bar{D}_H+\bar{D}_K$.

5. Proof of Theorem 1. By using Theorem 1′, it is shown in Theorem 3′ of the next section that for each positive number ϵ, there is an ϵ-partitioning of M. We use this result in establishing Theorem 1. For each integer i, let G_i be a $1/2^i$-partitioning of M.

The general idea is to let both H and K grow gradually into sets D_H and D_K. Suppose at the ith stage H and K have been enlarged to H_i and K_i so that $\rho(H_i, K_i)>0$ and H_i+K_i intersects each element of G_i. Let $\{p_j\}$ be a finite number of points such that each element of G_{i+1} contains an element of $\{p_j\}$ and each point of $\{p_j\}$ lies in an element of G_i.

If there is an arc in both $M-\bar{H}_i$ and an element of G_i from p_j to a point of K_i, let A_j be such an arc; if there is no such arc, let B_j be an arc in both $M-\overline{K}_i$ and an element of G_i which is irreducible from p_j to $\overline{H}_i$.

Let $\rho(H_i', K_i')=\delta_i$ where H_i' is the sum of H_i and all the B_j's while K_i' is the sum of K_i and all the A_j's.

Suppose $H_{i+1}=\{p \mid \text{there is an arc in } M \text{ of diameter less than } \delta_i/3 \text{ from } p \text{ to } H_i'\}$ and $K_{i+1}=\{p \mid \text{there is an arc in } M \text{ of diameter less than } \delta_i/3 \text{ from } p \text{ to } K_i'\}$.

The sets H_i and K_i have ϵ_i-grown into H_{i+1} and K_{i+1} where $\epsilon_i=7/(2^{i+1}\cdot 3)$. Hence, the sets $D_H=\sum H_i$ and $D_K=\sum K_i$ have property S. Since H_i+K_i intersects each element of G_i, $M=\overline{D}_H+\overline{D}_K$.

III. PARTITIONING

1. **Treating S-sets as continuous curves.** It is necessary that a set M have property S in order to be partitionable. (A set M is partitionable if for each positive number ϵ, there is an ϵ-partitioning of it.) The converse is true but it is not so obvious. However this converse follows from the facts that a continuous curve is partitionable and that any connected set with property S can be imbedded in a continuous curve so that any partitioning of the continuous curve produces a partitioning of it with no larger mesh.

If G is a partitioning of a continuous curve, the pieces of G are not continuous curves. However, if these pieces have property S, they can be satisfactorily dealt with.

Suppose M is a connected set with property S and $R(p, q)$ is the relative distance for M, that is, $R(p, q)$ is the greatest lower bound of the diameters of all connected subsets of M containing $p+q$. It may be shown that $R(p, q)$ preserves the topology of M and the diameter of any connected set is not increased under $R(p, q)$. Hence, any ϵ-partitioning of M under the metric $R(p, q)$ is an ϵ-partitioning of it under its original metric.

If we add only enough points to M to make it complete under $R(p, q)$, the resulting set M' is a continuous curve. Any ϵ-partitioning of M' induces an ϵ-partitioning of M. In fact we have the following result [6, Lemma 1; 21, pp. 154–158].

THEOREM 2. *If M is a connected set with property* S *with a relative distance metric, M' is the complete enclosure of M, and D is a connected open subset of M', then $M\cdot D$ is a connected open subset of M.*

2. **Crude partitioning.** Before showing that a continuous curve can be partitioned in various fashions, we show that it can be partitioned in some manner.

THEOREM 3'. *For each positive number* ϵ *and each continuous curve* M, *there is an* ϵ-*partitioning* G *of* M.

We use the following corollary which results from applying Theorems 1' and 2 to each component of M.

COROLLARY OF THEOREM 1'. *Suppose that the set* M *has property* S *and that* H *and* K *are two of its subsets such that* $\rho(H, K)>0$. *Then* M *contains two mutually exclusive open subsets* D_H *and* D_K *containing* H *and* K *respectively such that* D_H+D_K *is dense in* M, D_H *has property* S, *and* D_K *has only a finite number of components.*

The general idea in proving Theorem 3' is to use Theorem 1' and divide M into two pieces D_H and D_K so that D_H has property S, D_K has only a finite number of components and is of diameter less than ϵ. We apply the corollary of Theorem 1' to D_H and divide it into two pieces one of which has property S and the other of which has only a finite number of components each of diameter less than ϵ. The procedure is continued so as to get an ϵ-partitioning of M.

DETAILS OF PROOF OF THEOREM 3'. Let $(p_1, p_2, \cdots, p_n)$ be a finite subset of M such that no point of M is at a distance as much as $\epsilon/4$ from $\{p_i\}$. Let K_i be the set of all points of M that are at a distance from p_i of less than $\epsilon/4$ and H_i be the set of those that are at a distance from p_i of more than $\epsilon/3$.

Applying Theorem 1' to M, we find that there are mutually exclusive subsets D_{H_1} and D_{K_1} of M containing H_1 and K_1 respectively such that $\bar{D}_{H_1}+\bar{D}_{K_1}=M$, D_{H_1} has property S, and D_{K_1} has only a finite number of components. The components of D_{K_1} are of diameters less than ϵ and are elements of G. We divide the components of D_{H_1} further.

If D_{H_1} intersects both H_2 and K_2, we apply the corollary of Theorem 1' to D_{H_1} and find that there are open subsets D_{H_2} and D_{K_2} of D_{H_1} containing $H_2\cdot D_{H_1}$ and $K_2\cdot D_{H_1}$ respectively such that $D_{H_2}+D_{K_2}$ is dense in D_{H_1}, D_{H_2} has property S, and D_{K_2} has only a finite number of components. The components of D_{K_2} are elements of G. If K_2 does not intersect D_{H_1}, we do not add elements to G at this stage but let $D_{H_2}=D_{H_1}$.

The process is continued. For each positive integer i less than $n-1$ for which D_{H_i} intersects both H_{i+1} and K_{i+1} we apply the corollary to D_{H_i} and get mutually exclusive open subsets $D_{H_{i+1}}$, $D_{K_{i+1}}$ of D_{H_i} containing $H_{i+1}\cdot D_{H_i}$ and $K_{i+1}\cdot D_{H_i}$ respectively such that $D_{H_{i+1}}+D_{K_{i+1}}$ is dense in D_{H_i}, $D_{H_{i+1}}$ has property S, and $D_{K_{i+1}}$ has only a finite number of components. If D_{H_i} does not intersect K_{i+1}, we suppose $D_{K_{i+1}}=0$ and $D_{H_{i+1}}=D_{H_i}$.

The elements of G are the components of D_{K_1}, D_{K_2}, $\cdots$, $D_{K_{n-1}}$, and $D_{H_{n-1}}$.

3. **A more useful partitioning.** If G is an arbitrary partitioning of a continuous curve as described in Theorem 3′, it may not be the member of any decreasing sequence of partitionings because some element of it may not have property S. However, if we use Theorem 1 instead of Theorem 1′, we obtain the following extension of Theorem 3′.

THEOREM 3. *For each continuous curve M and each positive number ϵ there is an ϵ-partitioning G of M such that each element of G has property* S.

A partitioning whose pieces have property S is called an S-partitioning. If its mesh is less than ϵ, it is an ϵ-S-partitioning.

4. **Decreasing sequence of partitionings.** Theorem 2 may be applied to any set with property S to get the following corollary of Theorem 3.

COROLLARY OF THEOREM 3. *For each positive number ϵ and each set M with property* S, *there is an ϵ-S-partitioning of M.*

Using the corollary we find that if G is an S-partitioning of a continuous curve and ϵ is a positive number, each piece of G may be ϵ-S-partitioned. Hence, we have the following useful result [6].

THEOREM 4. *Each continuous curve has a decreasing sequence of partitionings.*

IV. CORE REFINEMENTS

In Figure 2, the partitioning represented by light lines is a core refinement of the one represented by heavy lines.

A partitioning is regular if each of its elements is the interior of its closure. This prevents pieces from having unnecessary boundary points.

If the partitioning H is a refinement of the partitioning G, the elements of H which have a boundary point in common with the boundary of an element of G are called border elements of H. Other elements are called interior elements.

The partitioning H is called a core refinement of the partitioning G if the following conditions are satisfied:

(a) Each border element of H is adjacent to an interior element.

(b) For each element g of G, the closure of the sum of the interior elements of H in g is connected.

(c) Both G and H are regular partitionings.

In [8] additional conditions were used in defining a core refinement for brick partitionings but we shall not need them here. We regard the interior of the closure of the sum of the interior elements of H in g as the core of g.

In some of the applications of partitionings, use is made of the following result.

THEOREM 5. *For each continuous curve M there is a decreasing sequence $G_1, G_2, \cdots$ of partitionings of M such that G_{i+1} is a core refinement of G_i.*

DETAILS OF PROOF. We prove this theorem by showing that if G is a regular S-partitioning of M and ϵ is a positive number, there is an ϵ-S-partitioning H which is a core refinement of G.

For each element g of G, let H' be a regular ϵ-S-partitioning of g. There is a dendron T in g that intersects each element of H'. Let H'' be a regular S-partitioning of g that refines H' and has a mesh less than the distance between T and the boundary of g.

Each element h'' of H'' with a boundary point on the boundary of g is a subset of a border element of H. If h' is the element of H' containing h'', the border element h of H containing h'' is the interior of the closure of the sum of all elements k of H'' such that there is an arc in h' from h'' to k that does not intersect the closure of any element of H'' whose closure intersects T. All elements of H'' not in such a border element of H are interior elements of H.

V. THE CONVEXIFICATION PROBLEM

1. **The method of solution.** A metric is a convex metric for a continuous curve M if for each pair of points of M there is a point of M halfway between them. In 1928 Menger raised [14] the question as to whether or not each continuous curve has a convex metric that preserves its topology. During the following twenty years a number of contributions [1; 3; 9; 10; 12] were made toward the solution of this question but it was not until the notion of partitioning was introduced that the final answer was given [6; 15].

THEOREM 6. *Each continuous curve has a convex metric.*

The fact that each continuous curve has a decreasing sequence $G_1, G_2, \cdots$ of partitionings is used in showing that each continuous curve has a convex metric. Each piece of each G_i is assigned a size and the distance between two points p, q is defined in terms of the sizes of the chains from p to q whose links are elements of the G_i's.

2. Definition of convex metric. Let $G_1, G_2, \cdots$ be a sequence of partitionings of M such that G_{i+1} is a core refinement of G_i and the mesh of G_{i+1} is less than one third the distance between any two nonadjacent elements of G_i.

The elements of G_1 are assigned a size of $1/2$ Each border element of G_{i+1} is assigned a size equal to one half the size of the element of G_i containing it. Each interior element of G_{i+1} is given a size equal to $1/(2^{i+1}\cdot N_{i+1})$ where N_{i+1} is the number of elements in G_{i+1}.

If K is a continuum which is equal to the closure of the sum of the elements of G_i, the ith size of K is the sum of the sizes of the elements of G_i in K. We define $E_i(p, q)$ as the minimum of the ith sizes of all continua containing $p+q$. It may be noted that if $E_i(p, q)$ is the ith size of K, then K is the sum of a chain $C_1, C_2, \cdots, C_n$ of continua such that C_1 contains p, C_n contains q, C_j intersects C_k if and only if $|j-k| \leqq 1$ and C_j is the closure of an element of G_i. The convex metric for M is $D(p, q) = \lim E_i(p, q)$.

We note that $D(p, p) = 0$ and $D(p, q) \geqq 0$. Also $D(p, q) = D(q, p)$ because $E_i(p, q) = E_i(q, p)$.

If p, q, r are three points, $E_i(p, q)+E_i(q, r) \geqq E_i(p, r)$ because if K_{pq} and K_{qr} are continua with ith sizes containing $p+q$ and $q+r$ respectively, the ith size of the continuum $K_{pq}+K_{qr}$ is no more than the sum of the ith sizes of K_{pq} and K_{qr}. Hence; $D(x, y)$ satisfies the triangle condition.

We now show that if p is a limit point of the set A, then $D(p, A) = 0$. Since $E_i(p, q)+1/2^{i+1} \geqq E_{i+1}(p, q)$, $D(p, q) \leqq E_i(p, q)+1/2^i$. However, there is a point q of A such that $p+q$ belongs to the closure of the same element of G_i. Then $D(p, q) \leqq 1/2^{i-1}$ and $D(p, a)$ (a element of A) is not bounded from 0.

We now show that if p is not a point of the closed set A, $D(p, A) > 0$. Suppose i is an integer such that no element of G_i is adjacent to both an element of G_i whose closure contains p and also to an element of G_i whose closure intersects A. If ϵ is the least of the sizes of the elements of G_i, we shall show that $D(p, A) \geqq \epsilon$.

If K is a continuum containing p, intersecting A, and having an $(i+k)$th size, K contains 2^k elements g of G_{i+k} such that g is a border element of G_{i+k} and lies in border elements of $G_{i+1}, G_{i+2}, \cdots, G_{i+k-1}$. The size of g is as much as $\epsilon/2^k$ and the $(i+k)$th size of K is more than ϵ. Hence, for each point a of A, $E_{i+k}(p, a) > \epsilon$ and $D(p, a) \geqq \epsilon$.

We show that the metric $D(x, y)$ is convex by showing that if p and q are two points, there is a point halfway between them. Let K be a continuum containing $p+q$ of ith size $E_i(p, q)$. Then K is the sum of two subcontinua K_p and K_q containing p and q respectively each of whose ith sizes differ from one half the ith size of K by less than $1/2^i$. There is a point r_i in $K_p \cdot K_q$ such that neither $E_i(p, r_i)$ nor $E_i(q, r_i)$ differs from $E_i(p, q)/2$ by more than $1/2^i$. If r is a limit point of $r_1, r_2, \cdots$, it is halfway between p and q under the metric $D(x, y)$.

Although a solution to the convexification problem has been given by means of partitionings, it would still be interesting to see the answer given by other methods.

3. **Unique segments.** If a circle is given a convex metric, for each point p there is a point q and two points r_1, r_2 such that both r_1 and r_2 are halfway between p and q. There is not a unique segment from p to q.

Although under a convex metric there is not a unique segment between each pair of points of a circle, there is a dense subset of a circle such that each pair of points of this dense subset is joined by a unique segment. In another paper I shall show that each continuous curve has such a convex metric. The result may be stated as follows.

THEOREM 7. *Each continuous curve has a dense subset R and a convex metric D such that each pair of points of R are the end points of a unique segment under D.*

For the surface of a sphere, there is a convex metric and a positive number ϵ such that each pair of points whose distance apart is less than ϵ are end points of a unique segment. It would be interesting to know a topological characterization of such sets.

A closed n-cell in a Euclidean space has a convex metric such that if two segments intersect in more than one point, the sum of these two segments is a segment. It would be interesting to obtain a topological characterization of continuous curves that can be given such a convex metric.

Each continuous curve with a convex metric that gives unique segments can be shrunk to each of its points in a very nice fashion. One might wonder if those continuous curves with such nice convex metrics could be characterized by the fashion in which they can be shrunk to points.

VI. MAPPING CONTINUOUS CURVES ONTO INTERVALS

1. **Introduction.** We have seen that if H and K are two mutually exclusive closed subsets of a continuous curve M, M can be divided into two mutually exclusive open subsets D_H and D_K containing H and K respectively such that each of the sets D_H, D_K has property S. We shall now see how uncountably many such divisions can be made.

We map M onto a straight line interval $[0, 1]$ so that H goes into 0, K goes into 1, and the inverse of each nondegenerate connected subset of $[0, 1]$ has property S [4]. Then except for a countable number of points x of $[0, 1]$, the collection of components of $T^{-1}[0, x)$ and $T^{-1}(x, 1]$ is an S-partitioning of M. We use $(a, b]$ to denote the closed interval $[a, b]$ minus the point a.

2. **Details of the transformation.** To prove the theorem regarding the mapping of M onto $[0, 1]$ we first prove the following lemma.

LEMMA FOR THEOREM 8. *Suppose C_1 and C_2 are mutually exclusive closed subsets of a continuous curve M, $M-C_i$ ($i=1, 2$) has property* S, *and ϵ is a positive number such that C_j ($j\neq i$) can ϵ-grow into $M-C_i$. Then for each positive number δ, M contains mutually exclusive open subsets D_1 and D_2 satisfying the following conditions:*

(a) *C_i can δ-grow into D_i and $(\epsilon+\delta)$-grow into $M-D_j$.*

(b) *$M-(D_1+D_2)$ can δ-grow into $M-(C_1+C_2)$.*

(c) *D_i and $M-D_i$ have property* S.

PROOF OF LEMMA. Let G be a γ-S-partitioning of $M-(C_1+C_2)$ where γ is less than either δ or $\rho(C_1, C_2)/3$.

Let A_i be a finite collection of arcs in $M-C_j$ ($i\neq j$) such that each of these arcs is of diameter less than ϵ, each intersects C_i, and each element of G intersects both an arc of A_1 and also an arc of A_2.

Let G' be a γ'-S-partitioning of $M-(C_1+C_2)$ which refines G, where γ' is less than the distance between C_i and any arc of A_j ($i\neq j$). Let R' be the closure of the sum of all elements g of G' such that g either lies in an element of G not adjacent to C_1+C_2 or g lies in an element of G adjacent to C_i and $\bar{g}$ intersects an arc in A_j ($i\neq j$). Then R is the sum of R' and the closure of all elements g' of G' of the following sort: g' lies in an element g of G which is adjacent to C_i but R' separates g' from C_i in $g+C_i$.

We define D_i to be the interior of the closure of the sum of C_i and all elements of G' which do not lie in R but lie in an element of G adjacent to C_i.

Each point of D_i-C_i belongs to an arc intersecting C_i and lying in both D_i and also in the closure of an element of G. This shows the first part of condition (a) is satisfied. An arc showing that the second part of this condition is satisfied may be obtained by going out an element of A_i and then continuing in the closure of an element of G.

To see that condition (b) is satisfied, note that each point of $M-(C_1+C_2)$ belongs to an arc that intersects R and lies in both $M-(C_1+C_2)$ and also in the closure of an element of G.

Condition (c) is satisfied because G' is an S-partitioning.

THEOREM 8. *If H and K are mutually exclusive closed subsets of the continuous curve M, there is a continuous transformation T of M onto the interval* $[0, 1]$ *such that* (a) $T(H)=0$ *and* $T(K)=1$, (b) *if W is a nondegenerate connected subset of* $[0, 1]$, $T^{-1}(W)$ *has property* S, (d) *if* 0 *belongs to W, each component of $T^{-1}(W)$ intersects H and if* 1 *belongs to W, each component of $T^{-1}(W)$ intersects K.*

We prove this result by repeated applications of the preceding lemma. To get a start we obtain $C_1=T^{-1}(0)$ and $C_2=T^{-1}(1)$. Let G be an ϵ-S-partitioning of M where $\epsilon=\rho(H, K)/3$. Then C_1 is the sum of the closures of all elements of G whose closures intersect H plus all points that this sum separates from K. Similarly, C_2 is the sum of the closures of all elements of G whose closures intersect K plus all points that this sum separates from H.

In our first application of the lemma we use $T^{-1}(0)$, $T^{-1}(1)$, $1/2$ for C_1, C_2, δ. The sets $M-(D_1+D_2)$, D_1, D_2 we obtain are called $T^{-1}(1/2)$, $T^{-1}[0, 1/2)$, $T^{-1}(1/2, 1]$.

In our next application of the lemma we let $T^{-1}(0)$, $T^{-1}[1/2, 1]$ $=T^{-1}(1/2)+T^{-1}(1/2, 1]$, $1/2$, $1/4$ be the C_1, C_2, ϵ, δ of the lemma and denote by $T^{-1}(1/4)$, $T^{-1}[0, 1/4)$, $T^{-1}(1/4, 1]$ the sets $M-(D_1+D_2)$, D_1, D_2 obtained.

In general, suppose that $T^{-1}[0, i/2^j]$ and $T^{-1}[(i+1)/2^j, 1]$ have been defined. We apply the lemma using $T^{-1}[0, i/2^j]$, $T^{-1}[(i+1)/2^j, 1]$, $1/2^j$, $1/2^{j+1}$ for C_1, C_2, ϵ, δ and denote by $T^{-1}[(2i+1)/2^{j+1}]$, $T^{-1}[0, (2i+1)/2^{j+1})$, $T^{-1}((2i+1)/2^{j+1}, 1]$ the sets $M-(D_1+D_2)$, D_1, D_2 obtained.

After $T^{-1}(x)$ has been defined for all numbers of $[0, 1]$ of the form $x=i/2^j$, the definition of T is continued in a continuous fashion—that is, $T^{-1}(x)$ is the intersection of all closed sets $T^{-1}[i/2^j, (i+1)/2^j]$ for which $i/2^j\leqq x\leqq(i+1)/2^j$.

In the next section we need to use the following information about the transformation T.

THEOREM 9. *If T is a continuous transformation mentioned in Theorem* 8, *then except for a countable number of values x* $(0\leqq x\leqq 1)$, *both* $T^{-1}[0, x)$ *and* $T^{-1}(x, 1]$ *are uniformly locally connected. Furthermore, if M is the closure of a uniformly locally connected domain D and B is a subset of the boundary of D, except for a countable number of values x, each point of* $T^{-1}(x)\cdot B$ *is a limit point of both* $T^{-1}[0, x)\cdot B$ *and* $T^{-1}(x, 1]\cdot B$.

A set D is uniformly locally connected if for each positive number ϵ there is a positive number δ such that each pair of points of D

whose distance apart is no more than δ belongs to a connected subset of D of diameter less than ϵ.

The proof of Theorem 9 is contained in the proof of Theorem 5 of [4].

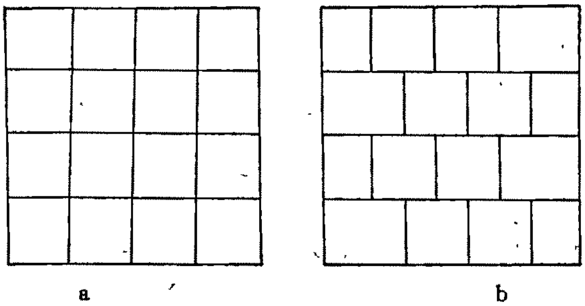

FIG. 4

VII. BRICK PARTITIONING

Figure 4 shows two ways in which a square plus its interior may be partitioned. The partitioning represented in Figure 4b is called a brick partitioning. A partitioning G is a brick partitioning if each of its elements is uniformly locally connected and equal to the interior of its closure while the interior of the sum of the closures of each pair of elements of G is uniformly locally connected. If G is a brick partitioning, not only can the elements of G be subdivided so as to form a partitioning G' which refines G but also adjacent elements of G and part of their common boundary can be added so as to form a partitioning G'' such that G refines G''.

In Figure 2, the left-most piece of the partitioning shown by heavy lines is not uniformly locally connected. Also, the interior of the sum of the closures of the two left-most pieces is not uniformly locally connected. If the three left-most pieces are replaced by the interior of the sum of their closures, a brick partitioning results.

Wilder showed [24, Theorem 3.4, p. 79] that if p is a point of a continuous curve M and D is a domain containing p, there is a uniformly locally connected domain E containing p and lying in D. However, to get a brick partitioning, we need to know that there is such a domain E such that $M-\bar{E}$ is uniformly locally connected. We get this result from Theorems 8 and 9.

It would be interesting to know if a modification of the methods used by Wilder would give this result. It appears that Moise had this in mind in [15].

Repeated applications of Theorems 8 and 9 show the following result [4].

THEOREM 10. *Each continuous curve has a decreasing sequence of brick partitionings.*

The first part of Theorem 9 is used in getting the elements of the partitionings to be uniformly locally connected and the second part is used to get the interior of the sum of the closures of two elements to be uniformly locally connected.

VIII. THE BOUNDARIES OF THE PIECES

1. **Locally connected boundaries.** In partitioning a square plus its interior as shown in Figure 4, the boundaries of the pieces are continuous curves and can be partitioned. If M is a finite complex and G is the collection of simplexes of M which are not part of the boundary of any other simplex, the elements of G have boundaries which are the sums of a finite number of continuous curves. These boundaries can be partitioned. (G may not be a brick partitioning.)

One might wonder if each continuous curve can be partitioned so that the boundaries of the pieces are locally connected. However, Example A (Figure 1) described in §I shows that this is not always possible. If the continuous curve is partitioned into pieces of diameters less than 1, each piece which is not a subset of a horizontal line has a boundary with uncountably many components.

However, in the partitioning of Example A, each component of the boundary of a piece of a partitioning is either a point or an interval and is therefore locally connected. One might wonder if each continuous curve can be partitioned so that the components of the boundaries of the pieces are locally connected. Example B of §I shows that this is not always possible.

It would be interesting to be able to characterize the types of continuous curves that can be partitioned into pieces with nice boundaries.

2. **Dimensionally regular partitionings.** Suppose that M is an n-dimensional continuous curve. It is known that for each positive number ϵ there is an open ϵ-covering H of M such that no point of M is covered by more than $n+1$ elements of H [11].

A partitioning G of the n-dimensional continuous curve M is called dimensionally regular if it is regular (each piece is the interior of its closure) and if the following inequality is satisfied when $j \leqq n+1$.

$$\text{Dimension (intersection of any } j \text{ boundaries)} \leqq n + 1 - j.$$

In another paper I shall prove the following result.

THEOREM 11. *There is a decreasing sequence of dimensionally regular partitionings for each n-dimensional continuous curve.*

It may be noted that the boundaries of the pieces will be of dimension less than n.

IX. APPLICATIONS OF PARTITIONINGS

The study of partitionings throws light on the structure of continuous curves and is of interest aside from its applications. In fact, I regard the result that any continuous curve is partitionable as more interesting and topologically important than any of its applications.

The first use made of the fact that any continuous curve is partitionable was to solve the convexification problem as mentioned in §V.

In studying a continuous curve M, one may consider a decreasing sequence of finite open coverings of M. A decreasing sequence of partitionings might be used instead. For example, in showing that each connected open subset of a continuous curve is arcwise connected, one may use chains whose links are connected open sets to define the arcs [18]. Chains whose links are pieces of partitionings could be used instead. This was done in the following application.

1. The Kline sphere characterization. A nondegenerate continuous curve M is topologically equivalent to the surface of a sphere if and only if it satisfies the following two conditions:

(a) No pair of points separates M.

(b) Each simple closed curve separates M.

Several mathematicians contributed to the proving of this result ([25] and other references in [7]). The final link was in showing that conditions (a) and (b) also imply that no arc separates M [7]. This was done by showing that if there is an arc which separates M, there is a simple closed curve that does not. The simple closed curve was constructed by means of chains whose links are open sets. By using the notion of brick partitioning, the above result was strengthened [4] by showing that condition (b) could be replaced by the following weaker condition:

(b′) Each simple closed curve whose complement has property S separates M.

2. Basis with connected intersections. A countable basis for a continuous curve is a countable collection D_1, D_2, $\cdots$ of open subsets such that if p is a point and E is an open set containing p there is an integer $i(p, E)$ such that D_i contains p and lies in E. If S is a square plus its interior, there is a countable basis for S whose elements are the intersections of S with interiors of circles with rational radii and rational centers. The intersection of any two elements of such a basis is connected. One might wonder if each continuous curve has such a basis.

E. E. Floyd noticed that if G is a brick partitioning of M and H is the collection of all open sets h such that for some point p, h is the interior of the sum of the closures of all elements of G whose closures contain p, then the intersection of any two elements of H is connected. Consequently, Floyd and I showed [8] that any continuous curve has a countable basis such that the intersections of the elements of this basis are connected and uniformly locally connected.

3. **Characterizing 3-space.** Suppose H and K are two sets such that for each positive number ϵ there are ϵ-partitionings G_H and G_K of H and K respectively and a 1-1 correspondence between the elements of G_H and the elements of G_K such that two elements of G_H have a common boundary point if and only if the corresponding elements of G_K have a common boundary point. One might wonder if this implies that H is homeomorphic to K. It does not because if H is a square plus its interior and K is the sum of H and a straight line interval that intersects H in just one point, there are such partitionings G_H and G_K. However, it is possible to use the notion of partitionings to characterize certain sets.

The two-manifold can be partitioned so that the boundary of the elements of the partitionings are simple closed curves. Some 3-manifolds can be partitioned so that the boundaries of the pieces are simple surfaces (topologically equivalent to the surface of a sphere). It is an open question as to whether or not all compact 3-manifolds without boundary have this property. The 3-manifold obtained by adding an ideal point at infinity to ordinary Euclidean 3-space is called a 3-sphere. In fact any set homeomorphic to it is called a 3-sphere—for example, the surface of a sphere in Euclidean 4-space. Using the notion of partitioning, we can get the following characterization of a 3-sphere [2].

A necessary and sufficient condition that a continuous curve M be a 3-sphere is that one of its decreasing sequences of regular partitionings $G_1, G_2, \cdots$ have the following properties:

(a) *The boundary of each element of G_i is a simple surface.*

(b) *If the boundaries of two elements of $\sum G_i$ intersect, this intersection is a 2-cell.*

(c) *The intersection of the boundaries of three elements of G_i is one-dimensional at each of its points.*

(d) *If* g *is an element of* G_{i-1} ($g=M$ *if* $i=1$) *the elements of* G_i *may be ordered* $g_1, g_2, \cdots, g_n$ *so that the boundary of* g_j ($j=1, 2, \cdots, n$) *intersects the boundary of* g *plus the boundary of* $g_1+g_2+\cdots+g_{j-1}$ *in a connected set.*

4. **Simultaneous partitionings.** In a forthcoming paper [20] Garth Thomas shows that if the continuous curve M_1 is imbedded in the continuous curve M_2, there is a partitioning G_2 of M_2 such that the intersections of M_1 and the elements of G_2 give a partitioning G_1 of M_1.

X. QUESTIONS

We have already raised certain questions in the preceding sections but we mention some more here.

1. If M is locally simply connected, can it be partitioned into simply connected pieces?

2. If M is a chain from H to K where each of the sets H and K contains only one point, is there an open transformation satisfying the conditions of Theorem 8? Whyburn shows such a transformation in [23] if M is 1-dimensional. If the dimension of M is finite, is there such a transformation such that $T^{-1}(x)$ is of dimension less than the dimension of M?

3. Is there a decreasing sequence of partitionings $G_1, G_2, \cdots$ of M and a boundary operation carrying subsets of G_i into subsets of G_{i+1} that gives a homology theory?

4. Suppose M_1 and M_2 are two continuous curves such that one can be given a convex metric with unique segments but the other cannot. Is there an essential difference between the types of partitionings that M_1 and M_2 can have?

Many other things could be said concerning partitioning, but there is much to learn. We summarize the results of this paper in one sentence: Any continuous curve can be partitioned in somewhat the same way that a square plus its interior can be partitioned.

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THE UNIVERSITY OF WISCONSIN

THE FIFTH SYMPOSIUM ON APPLIED MATHEMATICS

The Fifth Symposium on Applied Mathematics of the American Mathematical Society was held at the Carnegie Institute of Technology, Pittsburgh, Pennsylvania, on Monday and Tuesday, June 16–17, 1952. The subject of the Symposium was *Wave motion and vibration theory*. Attendance at each of the sessions was approximately 100. The following 119 persons, including 64 members of the Society, attended.

L. R. Alldredge, Julius Ashkin, S. B. Batdorf, E. G. Begle, R. A. Berg, Walker Bleakney, J. L. Bogdanoff, J. I. Bohnert, G. F. Carrier, Subrahmanyan Chandrasekhar, C. C. Chang, Abraham Charnes, Peter Chiarulli, Norman Davids, C. R. DePrima, F. C. DeSua, G. P. Dinneen, R. C. DiPrima, W. S. Dorn, R. J. Duffin, Nat Edmonson, J. J. Eisenhuth, T. A. Elkins, Sabri Ergun, R. L. Evans, J. B. Fanucci, H. E. Fettis, E. T. Frankel, J. B. Freier, Bernard Friedman, P. R. Garabedian, A. Garren, Edward Gerjuoy, R. D. Glanz, H. J. Greenberg, Mayo Greenberg, P. N. Gustafson, George Habetler, G. H. Handelman, H. W. Handsfield, Duncan Harkin, C. D. Harris, A. E. Heins, G. S. Heller, R. G. Helsel, D. S. Hoffman, J. P. Hoyt, W. S. Jardetzky, Fritz John, W. P. Jones, Edna Keitzer, T. B. Khammash, Martin Krakowski, J. M. Lande, E. H. Lee, Solomon Lefschetz, G. F. Leger, C. E. Lemke, Martin Lessen, Harold Levine, H. G. Lew, J. A. Lewis, C. C. Lin, B. W. McCormick, N. W. McLachlan, S. S. McNeary, Wilhelm Magnus, L. E. Malvern, M. H. Martin, R. C. Meacham, C. T. Molloy, E. W. Montroll, T. W. Moore, G. W. Morgan, David Moskovitz, C. E. Mullan, D. J. Newman, Simon Ostrach, N. A. Patton, W. H. Pell, R. C. Prim, Eric Reissner, L. P. Rosenberry, E. A. Saibel, S. A. Schelkunoff, M. E. Scherberg, Albert Schild, Morris Schulkin, R. J. Seeger, Hyman Serbin, D. H. Shaffer, R. Skinner, M. F. Smith, Marvin Stern, Eli Sternberg, Andrew Sterrett, J. J. Stoker, T. T. Tanimoto, Olga Taussky, J. S. Taylor, M. O. Taylor, T. Theodorsen, W. B. Thompson, John Todd, L. R. Turner, H. C. Volkin, E. G. Volterra, J. C. Warner, W. H. Warner, B. R. Weber, Herschel Weil, E. B. Weinberger, Alexander Weinstein, H. J. Weiss, Paul Weiss, Bernard Weitzer, E. A. Whitman, W. R. K. Wu, Arthur Zeichner.

Dormitories, lounges, and dining facilities were made available to those attending the Symposium by the host institution. On Monday afternoon, Symposium members were guests of the Department of Mathematics at a tea, held in the Exhibition Room of the Fine Arts Building, and on Monday evening at a social gathering, held in the Faculty Club.

Dr. J. C. Warner, President of the Carnegie Institute of Technology, extended greetings from the Institute to Symposium members at the first session on Monday morning.

The program of the Symposium consisted of 15 invited addresses. The papers were presented in the following order (in those cases where two authors are named, the paper was presented by the first-named author).

1. C. C. Lin: *Hydrodynamic stability.*

2. Subrahmanyan Chandrasekhar: *The stability of fluid motion in the presence of the magnetic field.*

3. P. R. Garabedian: *On free surface flows.*

4. G. F. Carrier and W. H. Munk: *On the diffusion of tides into permeable rock.*

5. J. J. Stoker: *Water waves.*

6. Walker Bleakney: *Review of significant observations on the Mach reflection of shock waves.*

7. N. W. McLachlan: *On the solution of non-linear differential equation to hydraulics.*

8. E. W. Montroll and Mayo Greenberg: *Scattering of plane waves by a soft obstacle.*

9. Harold Levine: *Variational methods in boundary value problems.*

10. Wilhelm Magnus: *Infinite matrices associated with a diffraction problem.*

11. A. E. Heins and H. Feshbach: *On the coupling of two half planes.*

12. E. H. Lee: *Wave propagation in helical compression springs.*

13. Alexander Weinstein and J. B. Diaz: *On the wave equation and the equation of Euler-Poisson.*

14. Solomon Lefschetz: *On the Lienard equation.*

15. R. J. Duffin and Albert Schild: *On some vibration problems.*

The presiding officers at the sessions were, respectively, Professor Peter Chiarulli, Dr. T. Theodorsen, Dr. S. A. Schelkunoff, and Professor E. A. Saibel.

At the beginning of the last session, a resolution of thanks to the Carnegie Institute of Technology, the Department of Mathematics, and the Committee on Arrangements was adopted unanimously.

The papers presented at the Symposium will be published as Volume V of the Proceedings of the Symposia in Applied Mathematics by the McGraw-Hill Book Company.

A. E. HEINS,
Chairman of the Committee on Arrangements

E. G. BEGLE,
Secretary of the Society

THE JUNE MEETING IN EUGENE

The four hundred eight-second meeting of the American Mathematical Society was held at the University of Oregon, Eugene, Oregon, on Saturday, June 21, 1952. A total of 173 persons registered for the meeting, including the following 86 members of the Society:

C. B. Allendoerfer, N. C. Ankeny, T. M. Apostol, B. H. Arnold, M. G. Arsove, G. A. Baker, R. W. Ball, C. R. Ballantine, J. P. Ballantine, R. A. Beaumont, R. F. Bell, J. S. Biggerstaff, Z. W. Birnbaum, Gertrude Blanch, R. N. Bradt, J. V. Breakwell, J. L. Brenner, B. W. Brewer, F. H. Brownell, D. G. Chapman, Harold Chatland, Paul Civin, B. H. Colvin, K. L. Cooke, C. M. Cramlet, E. L. Crow, G. B. Dantzig, D. B. Dekker, Douglas Derry, W. J. Dixon, Melvin Dresher, L. A. Dye, E. A. Fay, G. E. Forsythe, J. W. Gaddum, R. S. Gardner, K. S. Ghent, M. A. Girshick, E. G. Goman, C. H. Gordon, F. L. Griffin, Margaret Gurney, S. G. Hacker, M. E. Haller, A. J. Hoffman, V. E. Hoggatt, Mark Kac, J. M. Kingston, M. S. Knebelman, L. M. LeCam, D. H. Lehmer, R. B. Leipnik, J. C. R. Li, A. E. Livingston, Eugene Lukacs, M. W. Maxfield, A. S. Merrill, W. E. Milne, T. S. Motzkin, A. F. Moursund, W. L. Nicholson, O. M. Nikodým, Ivan Niven, A. R. Noble, T. G. Ostrom, D. B. Owen, T. S. Peterson, J. H. Raymond, P. V. Reichelderfer, R. A. Rosenbaum, Herman Rubin, W. G. Scobert, R. G. Selfridge, W. L. Shepherd, Judson C. Smith, Andrew Sobczyk, R. D. Stalley, W. M. Stone, Gabor Szegö, F. H. Tingey, G. E. Uhrich, J. L. Ullman, J. G. Wendel, R. F. Williams, R. J. Wisner, F. H. Young.

By invitation of the Committee to Select Hour Speakers for Far Western Sectional Meetings, Dr. G. E. Forsythe, National Bureau of Standards, Los Angeles, California, delivered an address entitled *Solving linear equations is not trivial*. Dr. Forsythe was introduced by Professor W. E. Milne. There were sessions for contributed papers in the morning and afternoon, presided over by Professors Gabor Szegö, R. A. Beaumont, and R. A. Rosenbaum.

Following are abstracts of papers presented at the meeting. Papers with abstract numbers followed by "*t*" were presented by title. Paper number 505 was presented by Professor Apostol, number 518 by Dr. Wendel, number 523 by Dr. M. W. Maxfield, number 531 by Professor Civin, and number 556 by Professor Reichelderfer. Dr. J. E. Maxfield was introduced by Dr. M. W. Maxfield and Professor Hostetter by Professor W. T. Puckett.

ALGEBRA AND THEORY OF NUMBERS

505. D. R. Anderson and T. M. Apostol: *The evaluation of Ramanujan's sum and generalisations.*

Ramanujan obtained an expression for the sum $R(n; k)$ of the nth powers of the primitive kth roots of unity in terms of the Möbius function $\mu(n)$, namely, $R(n; k) = \sum_{d\mid(n,k)} d\mu(k/d)$, where d runs over the divisors of the g.c.d. (n, k). The authors reduce this sum to a closed form, $R(n;k)=c\mu(b)\phi(a)/\phi(c)$, where $\phi(n)$ is Euler's totient, $a=(n, k)$, $b=k/a$, and $c=(a, b)$. Similar formulae are obtained for sums of the form $\sum_{d|(n,k)} f(d)g(k/d)\mu(k/d)$, where $g(n)$ is multiplicative and $f(n)$ is completely multiplicative. Also, a connection is established between these generalized sums and certain exponential sums involving nth powers of kth roots of unity. (Received May 5, 1952.)

506. N. C. Ankeny: *An application of algebraic geometry to number theory.*

Let $f(X)$ be a polynomial whose coefficients are rational integers. Define $N(f(X), p)$ as the number of different values $f(i)$ (mod p) for $i=0, 1, \cdots, p-1$ with p a prime number. The author then finds necessary and sufficient conditions on $f(X)$ such that $\lim_{p\to\infty} N(f(X), p)/p$ exists, and further shows that if the limit exists it depends only upon the degree of $f(X)$. The method is based on the ideas of A. Weil on algebraic curves over finite fields. Some applications are given. (Received May 5, 1952.)

507t. H. W. Becker: *Eels vectors.*

W. C. Eels, Amer. Math. Monthly vol. 21 (1914) pp. 269–273, gave an extensive table of solutions of the Diophantine system: $(t)\ x^2+y^2+z^2=t^2$, $(u)\ x^2+y^2=u^2$. Taking x and t odd, from the solution (11) of (t), R. D. Carmichael, *Diophantine analysis*, (1915) p. 38, is derived the Eels vector of the first kind: $t, u=(M^2+N^2+P^2+Q^2)^2 \pm 4(MP+NQ)^2=T^2\pm Z^2$, $x=(M^2-N^2-P^2+Q^2)^2-4(MN-PQ)^2=X^2-Y^2$, $y=4(M^2-N^2-P^2+Q^2)(MN-PQ)=2XY$, $z=4(M^2+N^2+P^2+Q^2)(MP+NQ)=2TZ$, in which both component Pythagorean triangles are primitive. Another primitive solution, in which however the component triangles are not necessarily primitive, has $m=p=rs$, $q=r^2-s^2-n$ in (11). The case $q=0$ gives $t, u=r^4\pm s^4$, $x=t-z=(r^2-s^2)^2$, $y=2rs(r^2-s^2)$, $z=2r^2s^2$. E.V. of the second kind, (t), $(v)\ y^2+z^2=v^2$ appear on solution of $(m^2+q^2)(n^2+p^2)=\Box$ in (11). There are no E.V. of the second kind with three of m, n, p, q odd: this is Spunar's theorum, Amer. Math. Monthly vol. 24 (1917) p. 393, proven under the impression that it was actually a proof of the impossibility of Martin vectors meeting the conditions (t), (u), (v), and $(w)\ x^2+z^2=w^2$. E.V. of both kinds (on reduction) are corollaries of (9), Carmichael, ibid.: $t, x=(M^2+Q^2R^2)^2\pm R^2(N^2+P^2)^2$, $y=4R(MN-PQR)(MP+NQR)$, $z=2R[(P^2-N^2)(M^2-Q^2R^2)+4MNPQR]$, $v=2R(P^2+N^2)(M^2+Q^2R^2)$. If further $x\mp y=u\mp z$, this is Frenicle's problem, Dickson's *History*, vol. II, pp. 184–186. (Received May 8, 1952.)

508t. H. W. Becker: *Williams vectors.*

In his *Algebra*, 1832, p. 419, J. D. Williams challenged his contemporaries with the Diophantine simultaneous equations (t), (u), (v) above, and gave the probably simplest solution $u, v, x, y, z, t=185, 680, 153, 104, 672, 697$. This was erroneously stated to be erroneous by Zerr, Amer. Math. Monthly vol. 15 (1908) p. 17, Dickson's *History*, vol. II, p. 492 (despite Cunliffe, 452, and Ward, 506). The system reduces to $\mu\nu=(i^2+j^2)(k^2+l^2)=(f^2k^2+e^2g^2)(e^2k^2+f^2g^2)=t^2$, $2(ijkl)^{1/2}=2efgh=y$, for which 15 types of solution are considered, $\xi\neq\Box\neq 1$: (0.1)$\mu=\Box$, $\nu=\Box$; (0.2) $\mu=\xi\Box$, $\nu=\xi\Box$; (0.3) $\nu=\mu\Box$; (0.4) $\mu=\nu$; and (I.1), etc., according as 1, 2, or 4 of i, j, k, l are squares. Many primitive parametric and numerical solutions are tabulated, including 16 with $t<10^4$, such as: (0.1) 975, 952, 495, 840, 448, 1073; (0.2) 1073, 561, 952, 495, 264, 1105; (0.3) 925, 3444, 533, 756, 3360, 3485; (I.3) 2175, 2431, 1092, 1881, 1540, 2665; (II.2) 533, 765, 520, 117, 756, 925; (II.3) 1073, 520, 975, 448, 264, 1105; (IV.3) Williams' example. Valuable preliminaries to or paraphrases of the problem are in Dickson, ibid., pp. 406–407, 428, 505–507, 630–634, 644–646, 667–668, Carmichael, ibid., chap. IV. The system has at least 5 different geometric interpretations, as the obvious parallelopipedon of volume xyz, and Hero triangle with sides u^2, v^2, x^2+z^2 and area $xyzt$, Turriere, L'Enseignment Mathématiques vol. 18 (1916) Arithmogeometrie. (Received May 8, 1952.)

509*t*. H. W. Becker: *Petrus and Carmichael vectors.*

Williams' problem was revived by R. D. Carmichael, *Diophantine analysis*, p. 23, ex. 8. His triadic solution (unpublished) takes off from 24–25 and ex. 6, ibid. (Hillyer), and is more general than the included Euler triad, Dickson's *History*, vol. II, p. 474. All solutions of $y/4=(abcd(a^2-b^2)(c^2-d^2))^{1/2}=(a, b, c, d)$, $t=(a^2+b^2)(c^2+d^2)$, u, $x=4abcd\pm(a^2-b^2)(c^2-d^2)$, v, $z=2(ac\mp bd)(ad\pm bc)$ are referred to as Carmichael vectors; they are type (.1) W.V. The first solution is (5, 2, 6, 1). Many explicit parametric solutions are developed from cognate relations in Dickson, ibid., pp. 172–176, 449, 459–474, 493, 511, 527, 661. There is a triadization transform, $T[(a, b, c, d), (a, b, c^*, d^*)]=(c, d, c^*, d^*)$, the latter often a maverick numerical solution yet to be generalized. There is a congruent number transform, $Q(a, b, c, d)=((a^2+b^2)^2, 4ab(a^2-b^2), (c^2+d^2)^2, 4cd(c^2-d^2))$, where Y^2 has 7 square factors, the maximum possible. Petrus (1674), Dickson, ibid., pp. 446–450, 2, 4, 6, 8, demanded 3 numbers such that their 6 sums and differences are squares. The solution is a subclass of C.V. Petrus gave a transform P by which every P.V. $=P$(W.V.). Remarkably, $P^2=Q$. The inverse Petrus transform: $((T\pm U)/2)^{1/2}=t$, y, $((T\pm X)/2)^{1/2}=u$, v, $((U\pm X)/2)^{1/2}=x$, z, correlates known, or yields new, W.V.; e.g., Euler's P.V., Dickson, ibid., p. 450, with a misprint which should read $q=-(f^4-g^4)^2-16f^4g^4$. (Received May 8, 1952.)

510*t*. H. W. Becker: *Pythagorean tetrahedrons, hexahedrons, and graphs.* Preliminary report.

Dickson's *History*, vol. II, pp. 221–224, reports tetrahedrons with rational faces and volume, yet none whose four faces are Pythagorean triangles. But the six components of a W.V. can be rearranged to form such a Pythagorean tetrahedron, of integer volume $xyz/6$, with two right angles at each of two vertices. From this point of view, each W.V. or P.T. depends on $(i^2-\phi^2)(k^2-\lambda^2)=(\zeta^2\theta^2-\epsilon^2\eta^2)$. $(\epsilon^2\theta^2-\zeta^2\eta^2)=x^2$, $2(i\phi k\lambda)^{1/2}=2\epsilon\zeta\eta\theta=z$ in two ways. So, automatically appear many explicit parametric solutions of these forms, for general or square values of the Greek letters, numerical solutions of which were supplied by Euler and Gerardin, Dickson, ibid., pp. 448, 458, 505, 661–662. Every Petrus vector is of types (− .1), (−IV). Any two P.T. with a common face make a Pythagorean hexahedron. Every P.T. u, v, x, y, z, t transforms to four others: ty, vz, vy, xy, xz, uv; uz, ty, xz, yz, uy, uv; uv, tz, ty, xz, vz, tv; tx, uv, ux, xz, ty, tu, each having three faces similar to three faces of the P.T. transformed. Thus from the first C.V. is derived the first P.H., whose base triangle is 1105, 1073, 264, with edges 448, 520, 975, and 495, 561, 952 at the apices. But the regular graph of ten branches, the pentagon-pentacle, all of whose ten triangles are Pythagorean, is impossible. In general, what graphs admit of Pythagorean triangulation? (Received May 8, 1952.)

511*t*. Leonard Carlitz: *Congruences connected with three-line latin rectangles.*

Let $K_n=K(3, n)$, the number of reduced three-line latin rectangles. Riordan

(Amer. Math. Monthly vol. 59 (1952) pp. 159–162) has proved the congruences $k_{n+p} \equiv 2k_n$, $K_{n+p} \equiv 2K_n \pmod{p}$, where p is a prime >2 and k_n is related to K_n by means of a certain recurrence. In the present paper it is proved that for arbitrary m, $k_{n+m} \equiv 2^m k_n$, $K_{n+m} \equiv 2^m K_n \pmod{m}$. Indeed if we put $\Delta f(n) = f(n+m) - 2^m f(n)$, $\Delta^r f(n) = \Delta\Delta^{r-1} f(n)$, then we have $\Delta^r k_n \equiv \Delta^r K_n \equiv 0 \pmod{m^r}$ for $r \geqq 1$. (Received April 17, 1952.)

512*t*. Leonard Carlitz: *Congruences for the ménage polynomials.*

Put $U_n = U_n(t) = \sum_0^n u_{n,r} t^r$, where $u_{n,r}$ denotes the generalized ménage number. Riordan (Duke Math. J. vol. 19 (1952) pp. 27–30) has proved the congruence $U_{p^2+n} \equiv (t^{p^2}-1) U_n \pmod{p}$, where p is a prime >2. In the present paper it is shown that $U_{m^2+n} \equiv (t-1)^{m^2} U_n \pmod{m}$, where m is an arbitrary integer. (Received April 17, 1952.)

513*t*. Leonard Carlitz: *Some congruences for Bernoulli numbers of higher order.*

S. Wachs (Bull. Sci. Math. (2) vol. 71 (1947) pp. 219–232) proved $B_{p+1}^{(p+1)} \equiv 0 \pmod{p^2}$ for p prime $\geqq 3$. The writer (Pacific Journal of Mathematics vol. 2, no. 2 (1952)) improved this to $B_{p+1}^{(p+1)} \equiv 0 \pmod{p^3}$ for $p > 3$. In the present paper the congruence $B_{p+1}^{(p+1)} \equiv p^{3/16} \pmod{p^4}$ for $p > 3$ is obtained. The writer had also proved $B_p^{(p)} \equiv p^2/2 \pmod{p^3}$. This is improved to $B_p^{(p)} \equiv -p^2(p-1)!/2 \pmod{p^5}$ for $p > 3$. (Received April 21, 1952.)

514*t*. Leonard Carlitz: *Some special equations in a finite field.*

In the first part of the paper asymptotic formulas are obtained for the number of solutions of equations of the type $f_1(\zeta_1) + \cdots + f_r(\zeta_r) = \alpha$, where the f_i are polynomials with coefficients in $GF(q)$; use is made of some theorems of Mordell. In the second part exact formulas are obtained for the number of solutions of equations of the form $Q(\zeta_1, \cdots, \zeta_r) = f(\eta_1, \cdots, \eta_s)$, where Q denotes a quadratic form while f is an arbitrary polynomial satisfying certain restrictions. In the latter problem the ζ_i are also permitted to be polynomials in $GF[q, x]$ of degree $<m$. (Received March 18, 1952.)

515*t*. Leonard Carlitz: *Weighted quadratic partitions over a finite field.*

This note is concerned with the evaluation of sums of the type $S = \sum e(2\lambda_1\zeta_1 + \cdots + 2\lambda_r\zeta_r)$ extended over all $\zeta_i \in GF(q)$ such that $Q(\zeta_1, \cdots, \zeta_r) = \alpha$, where α, $\lambda_i \in GF(q)$ and $e(\alpha) = e^{2\pi i t(\alpha)/p}$, $t(\alpha) = \alpha + \alpha^p + \cdots + \alpha^{p^{n-1}}$, $q = p^n$, $p > 2$. The evaluation of S depends upon the Kloosterman sums $\sum_\zeta e(\alpha\zeta + \beta\zeta^{-1})$, and $\sum_\zeta \psi(\zeta) e(\alpha\zeta + \beta\zeta^{-1})$, where $\psi(\zeta) = +1$ or -1 according as ζ is a square or a non-square of the field. (Received March 18, 1952.)

516*t*. C. W. Curtis: *A note on noncommutative polynomials.*

An integral domain R (i.e. an associative ring without divisors of zero) is said to have property (M) in case any two nonzero elements of R have a nonzero common right multiple. Property (M) is necessary and sufficient that R can be imbedded in a right quotient division ring. A ring S containing R is an *extension of type O* of R if S contains an element x not in R such that each element of S can be expressed uniquely in the form $\sum_{i=0}^{n} x^i r_i$, $r_i \in R$, where $rx = xT(r) + D(r)$, $T(r)$, $D(r) \in R$. In this note it is proved that if S is a ring containing an integral domain R with property (M), such that S is the join of a well ordered sequence of subrings beginning with R, where each subring is an extension of type O of the preceding, then S is an integral domain with property (M). In the case of a single extension of type O of a division ring, this result is due to Ore (Ann. of Math. vol. 34 (1933) pp. 480–508). A consequence of the result is that the general enveloping algebra of a solvable Lie algebra over a field of characteristic zero has property (M). (Received April 28, 1952.)

517t. H. A. Dye: *On group representations of finite type.*

It is proved that a connected locally compact group G having a faithful strongly continuous unitary representation in a finite W^*-algebra M is the direct product of a vector group and a compact group. The group G will be separable if M is countably decomposable. Applied to connected Lie groups, this result subsumes a theorem of Segal and von Neumann on unitary representations of semi-simple groups (Ann. of Math. vol. 52 (1951) pp. 509–517) and yields the following analogue in the solvable case: the kernel of a strongly continuous unitary representation of a connected solvable group in a finite ring contains the commutator subgroup. Another corollary concerns the dimension-type of group representations: factors of type II_1 do not arise among the W^*-algebras enveloping strongly continuous unitary representations of connected locally compact groups. Similar results have been conjectured and proved independently by R. V. Kadison and I. M. Singer. (Received May 5, 1952.)

518. Melvin Hausner and J. G. Wendel: *Ordered vector spaces.*

An *ordered vector space* is a vector space V over the reals, simply ordered under a relation $>$ and satisfying the usual axioms. Examples are furnished by spaces V_T defined as follows: T is a simply ordered set, V_T is the linear space of those real functions f which take nonzero values on well-ordered subsets of T, with $f > 0$ meaning that $f(t_0) > 0$, where t_0 is the least $t \in T$ at which $f(t) \neq 0$. A subspace of V_T is called *large* if it contains the characteristic functions of the points of T, and the truncations of its elements, g being a truncation of f in case for some t_0, $g(t) = f(t)$ or 0 according as $t < t_0$ or $t \geqq t_0$. *Theorem: The ordered vector space V is isomorphic to a large subspace of a uniquely determined V_T.* The chief tool in the proof is a notion of equivalence in the set of positive elements of V: $x \sim y$ means that $ax < y < bx$ for some real a, $b > 0$. The equivalence classes $[x]$ are simply ordered by the rule $[x] < [y]$ if and only if $x > cy$, all $c > 0$. T is the set of all equivalence classes, with this ordering. The construction of the isomorphism of V into V_T is begun by mapping an arbitrary system of representatives of the equivalence classes onto the characteristic functions of the respective points of T, and is completed by transfinite induction. (Received June 15, 1952.)

519. D. H. Lehmer: *Certain character matrices.*

Let $\chi(n)$ be a real nonprincipal character modulo the prime p (Legendre symbol). The paper is concerned with the square matrices M_α ($\alpha = 0, 1, \cdots, p-1$) of order $p-1$ whose general element $a_{ij} = \chi(\alpha + i + j)$. For each M_α it is possible to give explicit formulas for its determinant, its inverse, its characteristic roots, two of its latent vectors, and even the general element of its kth power, the latter depending in a peculiar way on the Fibonacci numbers in case $\alpha \neq 0$. Such matrices are useful in testing the correctness and efficacy of different matrix methods. (Received May 9, 1952.)

520*t*. O. I. Litoff: *On the equality of three subgroups of the general linear group.* Preliminary report.

Let $GL_n(R)$ be the group of nonsingular $n\times n$ matrices with elements in R. Let $U_n(R)$ be the subgroup of all elements of determinant 1. Let T be the group generated by the transvections (J. Dieudonné, Bull. Soc. Math. France vol. 71 (1943) pp. 27–45). It is known that if R is a division ring (Dieudonné, loc. cit.) or the integers mod p^r (J. Brenner, Ann. of Math. vol. 39 (1938) pp. 472–493), then $T=U$ and (except when $n=2$ and 2 is not invertible) $T=C$ (commutator subgroup). This paper extends these results to the case in which R is a commutative ring with unit which is either Euclidean or a valuation ring. Recently Hua and Reiner (Trans. Amer. Math. Soc. vol. 71 (1951) pp. 331–348) have given a different proof that $U=C$ in the case of the integers. (Received May 8, 1952.)

521*t*. O. I. Litoff: *On the normal subgroups of the general linear group.* Preliminary report.

This paper generalizes a classical theorem on the normal subgroups of the general linear group over a field (for example, J. Dieudonné, Bull. Soc. Math. France vol. 71 (1943) pp. 27–45) and a result of Brenner (Ann. of Math. vol. 39 (1938) pp. 472–493) for the integers mod p^r. Let R be a commutative valuation ring with unit which may have divisors of zero. (If $n=2$, it is further assumed that 2 is invertible and that the residue class field of R is not the field of three elements.) For any ideal I in R let N_I^* be the (normal) subgroup of all A in $GL_n(R)$ such that $A\in$center (mod I) and N_I be the (normal) subgroup of all A in $U_n(R)$ such that $A=1$ (mod I). For any normal subgroup N, let J be the smallest ideal such that $N\subset N_J^*$. Then $N_J\subset N\subset N_J^*$. (Received May 8, 1952.)

522. J. E. Maxfield: *Normal k-tuples.*

The definition of a normal k-tuple to the scale r is analogous to the definition of a normal number to the scale r. By use of a theorem stating that a k-tuple is normal if and only if a certain number derived from the k-tuple is normal and by use of the concept of k-dimensional uniform distribution (mod 1), most of the known results on normal numbers are extended to k-tuples. Further, it is shown that if A is a regular k by k matrix with rational elements and β is a normal k-tuple, then $\gamma=\beta A$ is a normal k-tuple. Let P be any permutation of the digits $0, 1, \cdots, r-1$, and let $P\alpha$ be the number obtained from α by performing the permutation on all the digits of α. If $\beta=(\alpha_1, \alpha_2, \cdots, \alpha_k)$ is normal to the scale r, then so are: (1) $(\alpha_1, \cdots, \alpha_{i-1}, P\alpha_i, \alpha_{i+1}, \cdots, \alpha_k)$ and (2) the k-tuple obtained from β by the application of P to the k-digits $b_{m}, b_{2m}, b_{3m}, \cdots$, where the mth k-digit is the k-tuple composed of the mth digits of the α_i, to the scale r. (Received May 2, 1952.)

523. J. E. Maxfield and Margaret W. Maxfield: *The existence of primitive roots* (mod p^r) *that are less than p.*

It is proved that for any odd prime p there is a primitive root (mod p^r) between 0 and p. This result is a corollary to the theorem of the paper, which applies to any divisor $d>1$ of $\phi(p)$ as exponent rather than to $\phi(p)$ alone. With the problem broken into cases according to whether d is not divisible by 2, divisible by 2 once, or divisible by 2 more than once, it is shown that an integer b can be found whose period (mod p) is d but whose period (mod p^2) is pd. An inductive argument extends the result to the modulus p^r. (Received April 28, 1952.)

524*t*. M. J. Norris: *Cofinally concentrated directed systems.*

A directed system is said to be cofinally concentrated if the cardinal of a set of elements not after a given arbitrary element is always less than the cardinal of the system. The main result is that every cofinally concentrated directed system has a well-ordered cofinal subset. (Received May 5, 1952.)

525*t*. W. L. Parker and B. A. Bernstein: *On uniquely solvable Boolean equations.*

Whitehead obtained a criterion for the unique solvability of a Boolean equation. The criterion is convenient when the equation is expressed in terms of logical addition, multiplication, and negation. The authors of the present paper obtain additional convenient criteria for the unique solvability of a Boolean equation expressed in terms of the logical operations, as well as suitable criteria for the unique solvability of any Boolean equation or system of Boolean equations expressed in terms of the ring operations of symmetric difference and multiplication. For each type of uniquely solvable equation or system of equations the solution is given in a form appropriate to the type. In the discussion of the Boolean ring equation a *normal ring form* of the equation is used and a simple relation between this form and the normal logical form is obtained. Systems of Boolean ring equations, linear and nonlinear, are studied with the help of "*complexes*"; systems of n linear equations in n unknowns are also studied with the help of ring determinants. In an incidental note a criterion is obtained for determining if the roots of one of two Boolean equations are also roots of the other. (Received April 28, 1952.)

526*t*. Dov Tamari: *On one-sided linear homogeneous equations in general rings.*

A (linear) equation with the unknowns on the right of the coefficients is a *right* equation, and a system of right equations is a *right* system $\mathfrak{S}_r$. *Solvable* means *non-trivially* solvable in the ring of coefficients R, and similarly *solution* means a nontrivial solution. R may contain divisors of zero and is said to have the property P_r if for any two $a_i \in R$, $a_i \neq 0$, there exists $b_i \in R$ such that $a_1b_1 = a_2b_2 \neq 0$: existence of C.R.M. $\neq 0$. One can extend to such rings the theorem, classical in fields: A system of n right linear homogeneous equations in $n+1$ or more unknowns is solvable. The straightforward proof, essentially different from the usual elimination process which can not be applied, is simpler than the classical one for fields. It is based on *shortest solutions*, i.e. solutions having the least number >0 of nonzero components. If R is not P_r but P_l, elimination can be applied to $\mathfrak{S}_r$. But as (generally) $\mathfrak{S}_r$ has no solution, this does only transform the *Diophantian problem* and is more useful to decide the eventual unsolvability of a specified system. (Received May 8, 1952.)

527*t*. Dov Tamari: *On the embedding of general Lie rings in quotient fields.*

Let $R[X]$ be a *locally finite* ring (not necessarily an algebra in the narrow sense) of Lie polynomials, i.e.: R any ring; X a finite or infinite set of symbols; $ax = xa$ ($a \in R$, $x \in X$); $[x_i, x_j] = x_ix_j - x_jx_i = \sum \left[{}^{ij}_{k}\right] x_k$ ($\left[{}^{ij}_{k}\right] \in R$), and any *finite* subset $Y \subset X$ contained in a *finite* subset $\bar{Y}$ such that $R[\bar{Y}]$ is a *Lie* (*sub*)*ring*. For any *linear order* $\mathfrak{O}$ of X every element of $R[X]$ has at least one $\mathfrak{O}$-*standard form* $\sum a_{\dots n_i \dots n_j \dots} \cdots x_i^{n_i} \cdots x_j^{n_j} \cdots$ ($x_i < x_j$ in $\mathfrak{O}$), and $R[X]$ is represented *on* the set $R(X)$ of ordinary (commutative) polynomials. The construction of *formally nontrivial* C.R.(L.)Ms. in $R[X]$ reduces to solving a system of right (left) linear homogenous equations in R (see preceding abstract). For sufficiently high degrees of the unknown standard forms the number of unknowns exceeds that of equations. If the fundamental theorem on systems of right (left) linear homogeneous equations is valid in R, any two $P, Q \in R[X]$ of degree $\leq m$ with $x_i \in Y$ have at least C.R.(L.)Ms. of any degree $r \geqq c_\nu m$ where $c_\nu = 1/(1-2^{-1/\nu})$ and $\nu =$ card $\overline{Y}$ a natural number. If the representation by standard forms is *unique*, $R[X]$ is a *proper* Lie ring, and *formally nontrivial* is always *nontrivial*; with R also $R[X]$ is regular (r., l.). This shows the embeddability of the Birkhoff-Witt algebras in fields of quotients by the well known construction of Ore. (Received May 8, 1952.)

ANALYSIS

528. M. G. Arsove: *Functions of potential type.*

A function w representable as the difference of two subharmonic functions is called *δ-subharmonic*. With the introduction of the characteristic function $T_r(w)$ (*Functions representable as differences of subharmonic functions*, Proceedings of the International Congress of Mathematicians, Cambridge, 1950) the concept of *order* and the theory of functions of finite order (familiar for entire meromorphic functions) extend to entire δ-subharmonic functions. A complete entire δ-subharmonic function is said to be of *potential type* provided it has order zero and its mass distribution has finite total variation; we have as a criterion that $\lim_{r\to\infty} [T_r(w)/\log r]$ exist finitely. Let E be a bounded nonpolar Borel set and q the equilibrium distribution of the unit mass on E. Under the norm $\|w\| = \left|\int w\,dq\right| + \lim_{r\to\infty} [T_r(w)/\log r]$ functions of potential type form a Banach space, and thereby play the fundamental role of completing the space of potentials. Moreover, a number of further problems that arise naturally in potential theory can be solved only in terms of functions of potential type. (Received May 5, 1952.)

529t. R. P. Boas: *Entire functions with negative zeros.*

A theorem of Valiron [Ann. Fac. Sci. Univ. Toulouse (3) vol. 5 (1914) pp. 117–257] states that for an entire function $f(z)$ of order less than 1 with real negative zeros the conditions (1) $\log f(r) \sim \pi \csc \pi\rho r^\rho$, $r \to \infty$, and (2) $n(r) \sim r^\rho$ are equivalent; here $n(r)$ is the number of zeros of absolute value not exceeding r. For recent simplified proofs see Bowen [Quart. J. Math. Oxford Ser. vol. 19 (1948) pp. 90–100], M. Heins [Ann. of Math. (2) vol. 49 (1948) pp. 200–213]. Paley and Wiener [*Fourier transforms in the complex domain*, New York, 1934, p. 70] proved a theorem for functions of order 1 which is equivalent to saying that when $\rho = 1/2$ and $f(0) = 1$, (1) and (2) are equivalent to $\int_0^\infty x^{-3/2} \log |f(-x)|\, dx = -\pi^2$. The author shows that for $0 < \rho < 1$ and $0 < \sigma < 1$, (1) and (2) are equivalent to $\int_0^r x^{-1-\sigma} \{\log |f(-x)| - \pi \cot \pi\sigma n(x)\} dx \sim \pi(\rho-\sigma)^{-1} \cdot (\cot \pi\rho - \cot \pi\sigma) r^{\rho-\sigma}$. (Received May 2, 1952.)

530t. R. C. Buck: *On a theorem of Pólya.*

A new and very simple proof is obtained for the following well known theorem; let $f(z)$ be entire, of order one and zero type, and let $f(n) = O(n^r)$ for $n = 0, \pm 1, \pm 2, \cdots$; then $f(z)$ is a polynomial of degree r at most. The proof depends on the use of the function $G(z) = \sum_0^\infty \Delta^n f(0) z^n$. After a preliminary reduction, this is shown to be entire and bounded in the half planes $x \geqq -1/2$ and $x \leqq -1/2$, so that $G(z)$ is constant, and from this $f(z)$ is a polynomial. (Received May 5, 1952).

531. Paul Civin and H. E. Chrestenson: *The multiplicity of a class of perfect sets.*

Let the perfect set P be formed in the following way: Let $[0, 2\pi] = p_1^0$. At the mth stage of the construction remove the open interval d_i^{m+1} from p_i^m and call the remaining intervals p_{2i-1}^{m+1}, p_{2i}^{m+1}, $i = 1, \cdots, 2^m$. Let $\epsilon_m = \sup_{i=1,\cdots,2^{m-1}} (d_i^m/p_i^{m-1})$, and $\theta_m = \sup_{i=1,\cdots,2^m} (p_{2i-1}^{m+1}/p_{2i}^{m+1}, p_{2i}^{m+1}/p_{2i-1}^{m+1})$. N. K. Bary [Fund. Math. vol. 9 (1928) pp. 84–100] showed that the hypotheses (1) $\epsilon_m = o(1)$ and (2) $\theta_m = O(1)$ implied that P was a set of multiplicity for trigonometrical series. She also conjectured that (2) was superfluous. S. Verblunsky [Acta Math. vol. 65 (1935) pp. 291–305] introduced a lemma to establish the Bary conjecture. The authors indicate the falsity of the Verblunsky lemma and prove that the multiplicity follows in the intermediate case in which (2) is replaced by $\theta_m = o(1/\sup_{m \leq n} \epsilon_n)$. (Received May 2, 1952.)

532. K. L. Cooke: *The asymptotic behavior of the solutions of linear and nonlinear differential-difference equations.*

Consider the equations (1) $u'(t+1) = a(t)u(t) + b(t)u(t+1)$ and (2) $u'(t+1) = a(t)u(t) + b(t)u(t+1) + D(u(t), u(t+1))$, where $D = \sum_{i+j \geq 2} b_{ij}(t)u(t)^i u(t+1)^j$. $a(t)$, $b(t)$, and the $b_{ij}(t)$ are assumed to be real and continuous for $t > t_0$. It is also supposed that $|b_{ij}(t)| \leq b_{ij}$ and that the series $\sum_{i+j \geq 2} b_{ij} x^i y^j$ converges for all sufficiently small $|x|$ and $|y|$. The boundary condition is $u(t) = g(t)$ for $t_0 \leq t \leq t_0 + 1$, where $g(t)$ is a given continuous function. Bellman (Ann. of Math. vol. 50 (1949) pp. 347–355) has studied (2) in case $a(t) \to a_0$ and $b(t) \to b_0$ as $t \to +\infty$. Using transform methods, the author shows that if $a(t) \sim a_0 + (a_1/t) + \cdots$ and $b(t) \sim b_0 + (b_1/t) + \cdots$ as $t \to +\infty$, the linear equation (1), subject to the stated boundary condition, has a unique solution for $t > t_0$. If S denotes the root of largest real part of the equation (3) $se^s - b_0 e^s - a_0 = 0$, and if δ is the residue of $(a_1 e^{-s} + b_1)/(s - b_0 - a_0 e^{-s})$ at S, this solution is $O(t^{\mathrm{Re}(\delta)} e^{\mathrm{Re}(S)t})$ as $t \to +\infty$. If all the roots of (3) have negative real parts and if $|g(t)|$ is sufficiently small over the initial interval, the same conclusions are valid for (2). (Received April 29, 1952.)

533. H. A. Dye: *The unitary structure in finite rings of operators.*

A W^*-algebra ($=$ weakly-closed self-adjoint algebra of operators on Hilbert space) is called *non-atomic* if it contains no minimal projections. By means of a theorem of Liapounov on the range of non-atomic vector-valued measures, it is shown that (i) the weak closure of the set of projections in any non-atomic W^*-algebra fills out the positive part of the unit sphere, and (ii) the weak closure of the set of unitary operators in a non-atomic W^*-algebra containing no purely infinite projections fills out the entire unit sphere. This fact is then used to characterize the algebraic and spatial types of a finite non-atomic W^*-algebra M in terms of its unitary structure. Here, by definition, the *unitary structure* in M is the group M_U of unitary operators in M viewed as a uniform space in the uniform structure induced by the weak topology. Two such structures M_U and N_U are called isomorphic if there exists a biunivalent mapping ϕ of M_U on N_U which is a group isomorphism, a unimorphism, and which satisfies $\phi(iI) = iI$, where $i = (-1)^{1/2}$. Each $*$-isomorphism between two non-atomic finite W^*-algebras M and N induces an isomorphism between their unitary structures, and conversely, an isomorphism between the unitary structures in M and N has a unique extension to a $*$-isomorphism of M on N. A corresponding notion of strong isomorphism is defined between unitary structures which serves to characterize the corresponding rings up to unitary equivalence. (Received May 2, 1952.)

534*t*. Abolghassem Ghaffari: *On the existence of limit cycles of a certain nonlinear differential equation.*

The author considers the nonlinear differential equation (1) $dy/dx = [y(x^2+y^2-2x-3)(x^2+y^2-2x-8)+x][x(x^2+y^2-2x-3)(x^2+y^2-2x-8)-y]^{-1}$ and discusses the behavior of its characteristics in the large. It is shown that the only critical point of (1) is the *origin* which is an *unstable focus*. In polar coordinates ρ, θ, the equation (1) transforms into (2) $d\rho/d\theta = \rho(\rho^2-2\rho\cos\theta-3)(\rho^2-2\rho\cos\theta-8)$, which is discussed incompletely by H. Poincaré (*Oeuvres*, vol. I, p. 83). Using as Poincaré's topographical system the circles $\rho =$ constant, one finds that the curves of contacts are the origin and the circles $\rho^2-2\rho\cos\theta-3=0$ and $\rho^2-2\rho\cos\theta-8=0$, so that the regions $0<\rho<1$, $\rho>4$ are *acyclic* and the annular region $1<\rho<4$ is doubtful. As the circles $\rho_1=1$, $\rho_4=4$ are of the same sign (positive), there exists an even number of limit cycles in the region $1<\rho<4$. The doubtful region can be divided into two smaller doubtful regions $\rho_1<\rho<C$ and $C<\rho<\rho_4$ by the auxiliary circle without contacts $C: \rho^2-2\rho\cos\theta-11/2=0$. It is found that the sign of C is negative and thus different from those of ρ_1 and ρ_4; therefore each of the two smaller doubtful regions is a *monocyclic* region. Then the equation (1) has, besides the equator, two limit cycles and the characteristics are composed of three families of curves spiralling away from the origin and the limit cycles. (Received May 2, 1952.)

535. I. M. Hostetter: *Boundary value solutions from the general solution of certain partial differential equations.* Preliminary report.

Let R be the position vector of a point R of space and $f(M\cdot R)$ an arbitrary function. Direct substitution of $u=f(M\cdot R)$ in the homogeneous space linear differential equation $\Phi:\nabla\nabla u=0$ shows that for f to be a solution M must satisfy the condition $M\cdot\Phi\cdot M=0$. If Φ is nonsingular and definite then M must be of the form $P+iQ$ where P and Q satisfy the conditions $P\cdot\Phi\cdot P=Q\cdot\Phi\cdot Q$, $P\cdot\Phi\cdot Q=0$. The totality of all solutions $f(P\cdot R+iQ\cdot R)$ where P and Q satisfy these conditions is the general solution of $\Phi:\nabla\nabla u=0$. If u is defined on a boundary σ, let Q, or P, depending upon the nature of the solution sought, represent the points of the boundary. If, say, Q_j are a set of n such points, then $u=a\sum_{j=1}^{n} f(P_j\cdot R+iQ_j\cdot R)\Delta\sigma$, and consequently $u=a\int_\sigma f(P\cdot R+iQ\cdot R)d\sigma$, is a solution. If f is properly chosen, the boundary conditions will be satisfied. The method is extended to obtain the solution of boundary value problems of special types of nonhomogeneous equations and leads directly and simply to the usual solutions of the standard boundary value problems. (Received May 8, 1952.)

536*t*. M. S. Klamkin: *Generalization of Clairaut's differential equation and the analogous difference equation.*

Clairaut's equation is generalized to $F(s_0, s_1, \cdots, s_{n-1})=0$ where $s_r = \sum_{s=0}^{n-r-1}(-1)^s x^s y^{(r+s)}/s!$, $y^{(r)}=d^ry/dx^r$. The solution is obtained by differentiating $F=0$, which yields $d^ny/dx^n=0$. Thus $y=\sum_{r=0}^{n-1} a_r x^r/r!$ where $F(a_0, a_1, \cdots, a_{n-1})=0$. The analogue of the generalized differential equation as a difference equation is $F(s_0, s_1, \cdots, s_{n-1})=0$ where $s_r=\sum_{s=0}^{n-r-1}(-1)^s C_{x+s-1,s}\,\Delta^{r+s}u_x$. The solution is obtained by considering $\Delta F=0$, which yields $\Delta^n u_x=0$. Thus $u_x=\sum_{r=0}^{n-1} a_r C_{x,r}$ where $F(a_0, a_1, \cdots, a_{n-1})=0$. (Received May 7, 1952.)

537*t*. V. L. Klee: *Convex sets in linear spaces.* III

(The first two papers of this series appeared in Duke Math. J. vol. 18 (1951) pp. 443–466 and 875–883.) Let L be a linear system and T the collection of all sets $X \subset L$ such that for each finite-dimensional linear variety $V \subset L$, $X \cap V$ is open in the natural "Euclidean" topology of V. It is proved that the linear space (L, T) (L topologized by T as its family of open sets) is locally convex if and only if $\dim L \leqq \aleph_0$; and for $\dim L > \aleph_0$, L actually contains a closed convex proper subset which intersects every nonempty open convex subset. This answers negatively a question raised earlier [loc. cit., p. 447, (Q_1)]. Supplementing an earlier result [loc. cit., p. 462, (12.4)], the following are proved: $\dim L < \aleph_0$ if and only if every p^+-functional on L majorizes at least one linear functional; $\dim L \leqq \aleph_0$ if and only if every lower semicontinuous p^+-functional on L majorizes at least one linear functional. (Received May 8, 1952.)

538. R. B. Leipnik: *New existence proof for semi-linear elliptic equations.*

Axiomatic inversion theory (cf. *Axiomatic Perron inversion*, Proceedings of the International Congress of Mathematicians, Cambridge, 1950, Vol. I, p. 462) is applied to semi-linear elliptic partial derivative equations to obtain existence theorems. The results are in some respects stronger than those based on the theory of completely continuous operators in Banach space. (Received May 7, 1952.)

539t. Norman Levinson: *The treatment of the irregular singular point for differential equations by the Phragmén-Lindelöf theorems.*

The system of differential equations $dw/dz = z^r A(z)\, w$, where w is a vector, z a complex variable, r an integer $\geqq 0$, and $A(z) = A_0 + A_1 z^{-1} + A_2 z^{-2} + \cdots$ with $A_0, A_1, \cdots$ constant matrices, can have $z = \infty$ as an irregular singular point. Formal solutions are known to exist. Recasting the system into a system of integral equations, by a well known procedure, the existence along any radial direction (or similar curve) of actual solutions for which the formal solutions are asymptotic is well known. The proof that the actual solutions are represented asymptotically by the formal solutions in certain sectors of the complex plane, due in the general case to Trijitzinsky, is more complicated. It can be shown that by use of familiar Phragmén-Lindelöf theorems (Problems 331 and 340 of Vol. 1, part III, of *Aufgaben und Lehrsätze* by Pólya and Szegö) the result along radial directions (or similar curves) implies the result in the appropriate sectors of the complex plane, thereby simplifying the proof considerably. (Received March 13, 1952.)

540. A. E. Livingston: *Some Hausdorff means which exhibit the Gibbs' phenomenon.*

The regular Hausdorff means of order n, with kernel $g(x)$, for the sequence $d_n(x) = \sum_{k=1}^{n} \sin kx/k$ are given by $h_{n,g}(x) = \sum_{k=1}^{n} C_{n,k} d_k(x) \int_0^1 t^k (1-t)^{n-k} dg(t)$, with $g(0+) = g(0)$, $g(1) - g(0) = 1$, and $g(x)$ of bounded variation on the interval $0 \leqq x \leqq 1$. Otto Szász (Trans. Amer. Math. Soc. vol. 69 (1950) pp. 440–456) has shown that if $x_n \to 0+$ and $nx_n \to a \leqq \infty$ as $n \to \infty$, then $h_{n,g}(x_n) \to \int_0^1 \mathrm{Si}\,(ax) dg(x)$ $(\mathrm{Si}\,(x) = \int_0^x (\sin t/t dt)$. Define $F(g) = \max_{a>0} (2/\pi) \int_0^1 \mathrm{Si}\,(ax) dg(x)$. If $F(g) > 1$, then the sequence $h_{n,g}(x)$ exhibits the Gibbs' phenomenon. The author proves that if the N points of jump of the step-function kernel $s_N(x)$ are linearly independent over the rationals, then $F(s_N) > 1$. For $N = 2$, he shows that every $s_2(x)$ satisfies $F(s_2) > 1$. (Received February 18, 1952.)

541*t*. E. R. Lorch: *A curvature study of convex bodies in Banach spaces.*

Let $\mathfrak{B}$ be a real Banach space with norm $\|x\|$. Let $r>1$. Write $G(x)=r^{-1}\|x\|^r$. Suppose that $\phi(\alpha, \beta)=G(x+\alpha y+\beta z)$ is twice continuously differentiable. Write the derivatives at $\alpha=\beta=0$ as $G_y(x)$, $G_{yz}(x)$, etc. Suppose that for $x\neq 0$, $y\neq 0$, $G_{yy}(x)>0$ (implying that the curvature of the unit sphere in $\mathfrak{B}$ is positive). It is shown that $G_y(x)$ is a bounded linear functional $f(y)$ in $\mathfrak{B}^*$ with $\|G_y(x)\|=\|x\|^{r-1}$. Further, $G_{yz}(x)$ is bilinear in y, z and is a bounded linear functional $g(z)$ for each y. If $x\neq 0$ is fixed, the mapping $y\to G_{yz}(x)$ is 1-1 bounded linear of $\mathfrak{B}$ into $\mathfrak{B}^*$. If $f(y)\in\mathfrak{B}^*$ assumes its l.u.b. at x, then $f(y)=\lambda G_y(x)$, $\lambda>0$. If there is an $L>0$ such that $L\|x\|^{r-2}\|y\|^2\leq G_{yy}(x)$, then $\mathfrak{B}$ is uniformly convex and the mapping $G_y(x)\to x$ is 1-1 strongly continuous (nonlinear) from $\mathfrak{B}^*$ onto $\mathfrak{B}$. Also the mapping $y\to G_{yz}(x)$ for fixed $x\neq 0$ is bicontinuous linear of $\mathfrak{B}$ onto $\mathfrak{B}^*$. If for $L>0$, $K>0$, $L\|x\|^{r-2}\|y\|^2\leq G_{yy}(x)\leq K\|x\|^{r-2}\|y\|^2$ (implying curvature of unit sphere is bounded away from 0, ∞), $\mathfrak{B}$ is isomorphic to a Hilbert space. The mapping $x\to G_y(x)$ is a strong homeomorphism of $\mathfrak{B}$ onto $\mathfrak{B}^*$ which maps the unit sphere of $\mathfrak{B}$ onto that of $\mathfrak{B}^*$. (Received May 6, 1952.)

542. O. M. Nikodým: *On accessibility in convex sets in abstract linear spaces.*

Let E be a convex set containing at least two points, in abstract linear infinite-dimensional real space L where no topology is supposed. By lin E we understand, with V. L. Klee, Jr., the set of all points $y\in L$ such that there exists $x\in E$ where the half-open rectilinear segment $[x, y)$ belongs to E. Klee has given an example of E where $\text{lin}^2 E\neq\text{lin } E$. The question whether the relation $\text{lin}^3 E\neq\text{lin}^2 E$ is possible has not been answered yet. In this respect the following theorems hold true: I. For every ordinal α where $1\leq\alpha<\Omega$ there exists E with $\text{lin}^\alpha E\neq\text{lin}^{\alpha+1} E$. II. If the Hamel basis of L is countable, then for every E there exists $\alpha<\Omega$ with $\text{lin}^\alpha E=\text{lin}^{\alpha+1} E$. III. If the Hamel basis is not countable, there exists E with $\text{lin}^\alpha E\neq\text{lin}^{\alpha+1} E$ for every $\alpha<\Omega$. IV. For every E we have $\text{lin}^\Omega E=\text{lin}^{\Omega+1} E$. V. The linear closure of E coincides with $\text{lin}^\Omega E$ and is a convex set. This paper is part of the work under a cooperative contract between A.E.C. and Kenyon College. (Received May 5, 1952.)

543*t*. H. S. Shapiro: *Maximum-minimum duality in extremal problems.* Preliminary report.

It is known that certain maximum problems in the theory of functions can be associated with minimum problems, and solved by exploiting this interrelation. For example, Macintyre and Rogosinski, Acta Math. vol. 82, state the result: "Let $K(z)$ be regular in $|z|<1$ except for a finite number of poles, and $\limsup_{r\to 1-0}\|K(re^{i\theta})\|_{p'}<\infty$. Let $I(f)=\left|(1/2\pi i)\int_{|z|=1}f(z)K(z)dz\right|$ and $J(g)=\|K+g\|_{p'}$. Then max $I(f)$ over all $f\in H_p$ with $\|f\|_p\leq 1$ equals min $J(g)$ over all $g\in H_{p'}$. The max and min are attained by f_1 and g_1 such that (1) $\left|(1/2\pi i)\int_0^{2\pi}f_1(e^{i\theta})K_1(e^{i\theta})d\theta\right|=\|f_1(e^{i\theta})\|_p\|K_1(e^{i\theta})\|_{p'}$, where $K_1=K+g_1$." From (1) much further information is then deduced, leading in many special cases to the determination of f_1, K_1, and max $I(f)$. (In the above, $\|F\|_p=\|F(\theta)\|_p$ denotes $((1/2\pi)\int_0^{2\pi}|F(\theta)|^p d\theta)^{1/p}$; H_p is the class of $f(z)$ regular in $|z|<1$ for which $\limsup_{r\to 1-0}\|f(re^{i\theta})\|_p<\infty$; $1\leq p\leq\infty$, and $1/p+1/p'=1$.) The author gives a simple proof of the above theorem, using the Hahn-Banach theorem and F. Riesz' theorems on the representation of linear functionals over the space L_p. The generality of the author's method permits him to extend the above result in several ways, for example: more general kernels K, more general domains than the unit circle, and replacement of the class H_p by a corresponding class of polynomials. The underlying reason for maximum-minimum duality is seen to be the fact that the norm of a linear functional relative to a subspace $\mathfrak{X}_1$ of a Banach space $\mathfrak{X}$ equals the distance in the conjugate space $\mathfrak{X}^*$ of T from the annihilator of $\mathfrak{X}_1$. (Received May 21, 1952.)

544. Andrew Sobczyk: *Generalized measures and functionals of testing functions.*

A continuous linear functional $f(\phi)$ on the space (m) of bounded sequences is of the form $f(\phi)=\int \phi(t)dF$, where F is a finitely additive set function on all subsets of the set T of positive integers. If $\phi_X(t)$ is the characteristic function of $X\subset T$, then $F(X)=f(\phi_X)$. For a nonlinear functional $g(\phi)$ on (m), the equation $G_a(X)=g(a\phi_X)$ associates with $g(\phi)$ a one-parameter family of set functions; in this paper set functions are studied to advantage from the point of view of functionals on (m). Characterizations are obtained of convex, quasi-convex, monotone, subadditive, finitely and countably-many valued set functions, in terms of the associated functionals. Properties are determined of the class of two-valued set functions modulo the class of two-valued additive set functions. Similarly nonlinear functionals on the linear topological space $(\mathfrak{D})$ of the Schwartzian theory of distributions, and how they correspond to nonadditive set functions and multiple-layers, are considered. Other spaces of testing functions than $(\mathfrak{D})$ and (m), appropriate for other families of sets than Borel fields and the family of all subsets of T, also are introduced, and the corresponding functionals and generalized measures are studied. (Received May 7, 1952.)

545. G. E. Uhrich: *A generalized hyperbolic differential equation.* Preliminary report.

An examination is made of the partial differential equation $D_0u=f(x_1, \cdots, x_n; u, D_1u, \cdots, D_mu)$ in which D_pu $(p=1, \cdots, m)$ is a partial derivative of the dependent variable u with respect to the independent variables $x_1, \cdots, x_n$, the order of D_0u exceeds that of each of the other derivatives involved, and further its order with respect to x_k $(k=1, \cdots, n)$ is not exceeded by the order with respect to x_k of any of the other derivatives. When a certain set of functions called "initial determinations" are given, and when these functions together with the function f of the equation satisfy a set of specified conditions, the existence of a unique solution $u=u(x_1, \cdots, x_n)$ of the differential equation is insured. (Received May 8, 1952.)

546. J. L. Ullman: *A notion of capacity for distributions.*

In an investigation of the relationship of Hankel determinants whose elements are the coefficients of a Taylor series to the singularities of the corresponding function, it was shown that the following notion of average capacity played a significant role. Let $\mu(t)$ be a nondecreasing function, constant outside the interval (0, 1), let E be the set of t for which $\mu(t+\epsilon)-\mu(t-\epsilon)>0$ for all $\epsilon>0$, and let E_ρ be the subset of E in the closed interval $(\rho, 1)$. The average capacity of E_ρ, denoted by $\bar{\tau}(E_\rho, \mu)$, is defined as $\limsup_{k\to\infty} |\overline{M}_k|^{1/k(k-1)}$, where $\overline{M}_k=\int_\rho^1 \cdots \int_\rho^1 V(t_1, \cdots, t_k)d\mu(t_1) \cdots d\mu(t_k)$, $V(t_1, \cdots, t_k)=\prod_1^k \prod_{1, i\neq j}^k (t_i-t_j)$. Let λ be the lower bound of ρ for which $\bar{\tau}(E_\rho, \mu)=0$, and let $f(z)=\int_0^1\mu(t)/(t-z)$. The problem is to give a function-theoretic characterization of the behavior of $f(z)$ in a neighborhood of E_λ. The following represents a partial solution, namely, it is shown that $\lim_{y\to 0} f(x+iy)-f(x-iy)=0$ for almost all x in the interval $(\lambda, 1)$. This is proved by first showing that $\mu(t)$ has derivative zero except for a set of measure zero in the interval $(\lambda, 1)$, and an important step is the generalization of a relationship due to Edrei (Compositio Math. (1939), Proposition 7, p. 65). (Received May 9, 1952.)

547t. Albert Wilansky: *The basis in summability space.*

Let Φ be the set $\{1, \delta^1, \delta^2, \delta^3, \cdots\}$, where $1 = \{1, 1, 1, \cdots\}$, $\delta^k = \{0, 0, \cdots, 0, 1, 0, \cdots\}$ (1 in the kth place). S. Mazur (Studia Mathematica vol. 2) proved that Φ is fundamental in the field of a summability matrix if and only if it is of type M. A matrix A is said to have the PMI property if for every equipotent B, $\sum b_k x_k$ converges for every x summable-B, where $b_k = \lim_n a_{nk}$. The $(C, 1)$ matrix has the PMI property, the Norlund matrix ($p_0 = p_1 = 1$, $p_k = 0$ for $k > 1$) has not. Theorem. *Φ is a basis for the field of A if and only if A has the* PMI *property.* Principal tool: a known lemma on biorthogonal sets and the result: A has the PMI property if and only if $\left|\sum_{k=1}^{m} a_{nk} x_k\right| < M$, M depending on x, for all m, n and each x which is summable A. (Received April 23, 1952.)

APPLIED MATHEMATICS

548. Gertrude Blanch: *Note on the numerical solution of differential equations involving linear differential operators.*

This note is concerned with the numerical solution of equations of the type $Ty + F(y, x) = 0$, where T is a linear differential operator, and initial conditions are given at the origin of the system. It is shown that when the solutions of the homogeneous equation are known, then the computations can be so arranged that the contribution to the integral from the upper limit, at each step of the numerical process, when y is still unknown at the upper limit, drops out. The method involves, essentially, using the integral representation of the system. Thus it is not necessary to employ the usual method of first approximating y by an open integration formula and then iterating the solution until the desired accuracy is obtained; the solution can be obtained by mere quadratures. The method is especially successful in the important case when the differential operator has constant coefficients and is of low order. An example is given, and it is shown that for some range of parameters entering into the differential operator, the method of using the integral representation leads to numerical processes that are easier to manage than the ordinary routine for stepwise integration. (Received May 5, 1952.)

549. R. G. Selfridge: *Approximations with least maximum error.*

For certain classes of approximating functions, where each class forms an n parameter family, it has been shown that the approximation with the least maximum error to a function on a given interval is that with its error curve having absolute extrema of alternating sign, and with at least $n+1$ such absolute extrema. This paper shows that this is also a sufficient condition for least maximum error for a different class of approximating functions. An iteration process is described that has been used to find the approximations, but no proof of convergence is given. (Received April 24, 1952.)

550. W. M. Stone: *A form of Newton's method with cubic convergence.*

Using the generalized Taylor expansion of Hummel and Seebeck (Amer. Math.

Monthly vol. 56 (1949) pp. 243–247), a formula is presented for the root of an equation, $f(x)=0$, in terms of values of $f(x)$ and $f'(x)$ at points on either side of the root. Advantage lies in the property of cubic convergence without the necessity of evaluating second derivatives; if $f(x)$ is a scantily tabulated transcendental function, considerable interpolation may be avoided. (Received May 19, 1952.)

GEOMETRY

551*t*. C. M. Fulton: *The projecting tensor.*

This paper deals with a Riemannian space V_n of n dimensions imbedded in a Riemannian space V_m of m dimensions. The projecting tensor is defined as a tensor belonging to the subspace V_n. Given any tensor at a point of V_m, its projection on V_n can be found with the aid of the projecting tensor. Proper multiplication and subsequent contraction of these tensors yield the projection. The procedure also leads to a simple interpretation of the covariant derivative. (Received April 16, 1952.)

552*t*. P. C. Hammer: *Diameters of convex bodies.* Preliminary report.

A convex body C in E_n is taken to be a closed bounded convex set with interior points. A diameter of C may be considered as a chord of C which is a longest chord in its direction. The ratio in which an interior point of C divides a chord through it is taken as the ratio of the larger (or equal) part to the whole. The maximum such ratio for a point x is designated by $r(x)$. Two results are: (1) If x divides a chord ab of C in the ratio $ax/ab=r(x)$, then if y is any point on xb, $ay/ab=r(y)$. (2) Every diameter of a convex body C is intersected by c other diameters of C. Since it is known that the chord ab mentioned above is necessarily a diameter, if we designate by $F(r)$ the set of diameters of C each of which is divided by a point in its maximum ratio $r<1$ then the family of sets $F(r)$ descends as r decreases to the minimum value, r_n, of $r(x)$. (Received April 18, 1952.)

553*t*. P. C. Hammer and Andrew Sobczyk: *Planar line families.* III.

A family F of lines is said to be outwardly k-fold if it covers points exterior to some circle continuously precisely k-fold, including points at infinity. An outwardly k-fold family may be represented by the equations (1) $x \sin \alpha - y \cos \alpha - p_i(\alpha)=0$, where $i=\pm 1, \pm 2, \pm n$ if $k=2n$, or $i=0, \pm 1, \pm 2, \cdots, \pm n$ if $k=2n+1$. Necessary and sufficient conditions that equations (1) represent an outwardly k-fold line family are: (a) $p_i(\alpha)$ are continuous for $0 \leqq \alpha \leqq \pi$, (b) the upper and lower right and left derivatives of $p_i(\alpha)$ are uniformly bounded, (c) $p_i(0)=-p_{-i}(\pi)$, $i=\pm 1, \pm 2, \cdots, \pm n$, and also $i=0$ if k is odd, (d) it is possible to choose the sign of subscripts so that $p_i(\alpha)>p_j(\alpha)$ for $j<i$ and for all α in $[0, \pi]$. (Received April 18, 1952.)

TOPOLOGY

554*t*. D. O. Ellis and H. D. Sprinkle: *Topology in B-metrised spaces.* Preliminary report.

Let B be a σ-complete Boolean algebra. A set S with a distance function taking values in B and having the formal properties of a metric is called a *B-metrised space.* An example is B itself with the autometrized distance function (David Ellis, *Autom-*

etrized Boolean algebras I, Canadian Journal of Mathematics vol. 3 (1951) pp. 87–93). The distance topology induced in a B-metrized space (see David Ellis, *Geometry in abstract distance spaces*, Pub. Math. Debrecen vol. 2 (1951) pp. 1–25) by the Kantorovitch topology of B is considered. Cauchy sequences, completeness, and related questions are studied. As an interesting application the following are demonstrated: A topological space is 0-dimensional if and only if it is a space of uniform structure defined by a filter having a symmetric idempotent base. Any T_1 space, S, whose topology is compatible with such a uniform structure is metrizable over the Boolean algebra $2^{S\times S}$. One also finds that B itself under its autometrized topology is always complete (this was shown by Löwig using a different definition of Cauchy sequence than ours). (Received April 17, 1952.)

555*t*. S. T. Hu: The homotopy addition theorem.

The homotopy addition theorem is a fundamental assertion which relates the element of the homotopy group determined by a map of a cellular sphere to those determined by the partial maps on the cells of the sphere. It is used in the proof of the Hurewicz Isomorphism Theorem and is also a consequence of it. However, in the mathematical literature, one can not find an explicit proof of this elementary but important assertion concerning homotopy groups. In the present paper, an elementary proof of the homotopy addition theorem is given. It is elementary in the sense that it is based only on the definition of homotopy groups together with a few well known properties of homtopy which were elementarily proved in the mathematical literature. The proof is given in a completely detailed and formalized way. (Received May 2, 1952.)

556. Tibor Radó and P. V. Reichelderfer: *Remarks on the topology of Euclidean spaces.*

The purpose of this note is to indicate simple methods by which many of the classical theorems in the topology of Euclidean n-space may be derived from the eight basic axioms for the relative cohomology groups of compact pairs of Hausdorff spaces proposed by S. Eilenberg and N. E. Steenrod in *Foundations of algebraic topology*, Princeton University Press. First, an elementary proof is given for the theorem that the cohomology group $H^n(F)$ of a compact subset F of Euclidean n-space is trivial. Almost as a corollary it is shown that if F is a compact subset of Euclidean n-space for $n \geqq 2$ such that the complement of F is connected, then $H^{n-1}(F)$ is trivial. This result is used to derive a proof of the invariance of the interior and of the frontier in Euclidean n-space. (Received April 21, 1952.)

W. T. Puckett,
Acting Associate Secretary

BOOK REVIEWS

Arithmetical questions on algebraic varieties. By B. Segre. London, Athlone, 1951. 5+55 pp. 10s. 6d.

This book is based on three lectures given by the author in King's College, London, in March 1950. He describes it in the preface as "an account of the more important problems in which algebraic concepts and methods have proved fundamental in overcoming arithmetical difficulties, or where arithmetical notions and results have played a rôle in algebraic geometry." The exposition is throughout very condensed, as might be expected on comparing the small size of the book with the large field covered. Results are quoted largely without proof, or with only an outline of the ideas used in the proof; there is however an excellent and extensive bibliography, by means of which the reader attracted by any of the subjects touched on can have no difficulty in informing himself more fully.

The first chapter, presumably corresponding to the first lecture, deals with quadrics in an arbitrary commutative field γ. First is considered the rather exceptional case where γ has characteristic 2, and in particular where it has only two elements; here the conic is not self-dual, as all its tangents pass through a fixed point, and every line through this point is a tangent, though its point of contact may not belong to γ but to a quadratic extension of γ. Another peculiar result is that the tangent lines to a general quadric surface (in the field of order 2) form a linear congruence; the surface has a unique tangent in each of its nine points, and these are all the lines meeting a particular pair of skew lines, which are the only lines in space not meeting the quadric at all. Most of the rest of the chapter exhibits how much of the ordinary theory of quadrics in the real field remains valid for an arbitrary field, so long as the latter is not of characteristic 2; as a typical result of this kind we may quote the theorem that every quadratic form in n variables is equivalent in γ (i.e., can be transformed by a reversible linear transformation with coefficients in γ) to one of the form

$$y_1y_1' + \cdots + y_hy_h' + \phi(z_1, \cdots, z_k)$$

where the y's, y''s, and z's are some or all of the new variables (so that $2h+k\leq n$) and the equation $\phi(z_1, \cdots, z_k)=0$ has no solution in γ except $(0, \cdots, 0)$; moreover, if two expressions in this form are equivalent in γ, the numbers h, k are the same for both (this is the generalisation of Sylvester's theorem) and the forms ϕ are equivalent in γ.

The second chapter deals mainly with applications of geometry to the problem of finding rational solutions of given algebraic equations or systems of equations. As a preliminary example, a set of quadratic and cubic equations in six variables is studied, and shown to define a reducible curve in S_5, whose constituents are examined separately for rational points. Methods applicable to the general cubic in four variables are given, depending largely on the theory of the lines on the cubic surface; these exhibit ten possibilities in an arbitrary field, of which only one (the familiar figure of 27 lines) occurs in an algebraically closed field, and only four in the real field, but all ten in the rational field. Results are quoted by which the equation $\sum_{i=1}^{n} x_i^3 = 0$ can be solved completely (by polynomials with integer coefficients in $n-2$ parameters) except when $n=3, 5, 7$, or 9. A number of quartic forms in four variables, and the "biaxial" equation $\phi(x, y) = \psi(z, w)$ (where ϕ, ψ are any two forms of the same order, with nonzero discriminants) are treated in something the same manner. Finally some study is given to what the author calls Severi-Brauer varieties, namely those which are birationally equivalent to linear spaces, either in γ or in some extension of γ, the transformation being without exceptional points. The main problems here are, under what circumstances the transformation can be made in γ; if it cannot, what is the least extension of γ in which it can be made; and what is the simplest model (in some defined sense, such as a minimum order model) to which such a variety can be reduced by a birational transformation in γ, without exceptional points. In the study of these problems, the anticanonical system is the tool chiefly used.

This leads naturally up to the third chapter, which is mainly devoted to problems of rationality, especially to the type of extension of γ which may be needed for the rational parametrisation of unirational or birational varieties, defined in γ. Recent results on the general complete intersection of any number of quadrics are given; this variety is birational if the number of variables is sufficiently great compared with the number of equations. The book ends with the problem of under what circumstances an isomorphism between transcendental extensions of two fields implies the existence of, or (more strongly) is necessarily itself an extension of, an isomorphism between the ground fields; and with some applications of the results of Chapter I to the study of the birational self-transformations of a surface by means of the quadratic intersection form of its base.

PATRICK DU VAL

Finite deformation of an elastic solid. By F. D. Murnaghan. New York, Wiley, 1951. 6+140 pp. $4.00.

Apparently intended as a text, this book follows the growing custom of beginning with an introductory chapter containing pure mathematics neither necessary nor sufficient for the applications which follow. The student is deterred from accepting the results without question by frequent interjections of "(why?)," "(prove this)," and other commands, and the pedagogical usefulness of the works is attested by the numerous simple exercises. The pages, whose crowding with symbols and parenthetical expressions in the text suggest that the publisher confused the manuscript with some work on topology or algebra, are rather frightening to look at.

The author maintains that the theory of finite deformations is most easily presented and understood by the use of matrices. "Do not fall into the error of regarding them as a complicated device invented by mathematicians to make the theory of elasticity harder than it actually is," he warns. However, the book did not furnish recreational reading to the reviewer, who finds the author's former tensorial treatment in the American Journal of Mathematics, 1937, not only more complete but also by virtue of its compact explicitness easier to grasp. At one point the author reverts to tensors, although taking care to advise readers to skip this passage, and later in the solution of special problems in curvilinear coordinates he employs without derivation equations which would develop naturally if an invariant formulation had been given to start with.

About forty pages are required to derive the classical equations of finite elastic strain. Next the author develops what he calls "the integrated linear theory" of hydrostatic pressure. This theory is obtained by taking the expression for change of volume in the linearized theory, then supposing the Lamé constants are linear functions of pressure, then integrating the result. He shows that by suitable choice of the constants occurring it is possible to get very good agreement with some of Bridgman's experiments. The numerical details are presented in full.

To consider the author's theory, let us repeat part of his presentation of the classical proof of the existence of stress-strain relations (pp. 55–56). Writing T for the stress matrix, η for the strain matrix, ρ_a and ρ_x for the density before and after deformation, J for the matrix of gradients of deformed with respect to undeformed positions, ψ for the mass density of the energy of deformation, and using a star to denote transposition, he says: "Since [the principle of conservation of energy] must hold for an arbitrary volume V_* of our deformable medium, we have (why?)

$$\mathrm{Tr}\,(J^{-1}T(J^*)^{-1}\delta\eta) = \rho_*\delta\psi.$$

Since ψ is (by hypothesis) a function (written symmetrically) of η we have (by the very definition of a differential) $\delta\psi = \mathrm{Tr}\,((\partial\psi/\partial\eta)\delta\eta)$ (show this), and so

$$\mathrm{Tr}\,(J^{-1}T(J^*)^{-1}\delta\eta) = \rho_*\,\mathrm{Tr}\left(\frac{\partial\psi}{\partial\eta}\delta\eta\right).$$

Since this relation must hold for an arbitrary (symmetric) matrix $\delta\eta$, we have (why?) . . .

$$T = \rho_* J\frac{\partial\psi}{\partial\eta}J^*."$$

The reader notes that if ψ were allowed to depend on T here, the author's formula for $\delta\psi$ would become $\delta\psi = \mathrm{Tr}\,((\partial\psi/\partial\eta)\delta\eta + (\partial\psi/\partial T)\delta T)$, and the proof would fail. That is, the classical finite strain theory requires that for a given point in the medium at given temperature or entropy, ψ be determined by the strain η *alone*, independently of the existing stress. Thus the author's assumption that the Lamé constants are functions of pressure would appear inconsistent with the theory of finite strain, and some readers may prefer to regard his "integrated linear theory" as an isolated semi-empirical result. The foregoing remarks are not to be confused with the well known results, discussed by the author on p. 65, that if an isotropic elastic body is really subject to initial hydrostatic pressure p_0, but we choose to neglect that fact and treat it as if it were unstressed, we can get correct results for small strains if we simply replace λ by $\lambda + p_0$, μ by $\mu - p_0$.

The remainder of the book is devoted to the case when the strain energy is approximated as a cubic function of the strain components. From the general considerations of Reiner [Amer. J. Math. vol. 70 (1948) pp. 433–446], which are not discussed by the author, it can be shown that to this degree of approximation the classical formulae relating shear stress to angle of shear, twisting couple to angle of twist, etc., will not be altered, although the characteristic phenomena of nonlinear elasticity will appear in their simplest form as new stresses not present at all in the linear theory. The author works out the form of the cubic terms for the various types of crystals. In the following discussion of simple tension, simple shear, compression of a circular cylindrical tube and of a spherical shell, and torsion of a circular cylinder, no mention is made of the general solutions valid for arbitrary strain energy which have recently appeared in the literature [R. S. Rivlin, Philos. Trans. Roy. Soc. London. ser. A vol. 241 (1948) pp. 379–397, and other papers]. New, however, is the calculation of the second order change of dimensions in a state of simple shearing stress, as distinct from a simple shear displacement, and of the similar change of dimensions of a circular cylinder in torsion.

The only historical references in the book tell us that Jacobians are named after Jacobi, the Lamé constants after Lamé, besides giving the dates and nationalities of these two persons. Apart from a single reference to some experimental data, the only literature citations are to the author's other texts. While this practice has become the rule in volumes intended for the pedagogical and undergraduate market, its extension to serious works does not seem altogether commendable to this reviewer. The publishers present this book as an "authoritative exposition." Inclusion of the recent results in finite strain theory obtained by Signorini, Reiner, Rivlin, and Green and Shield, which seem deep and significant to the reviewer, would not have been unwelcome.

In the preface the author states: "If the mathematical treatment given here serves to stimulate the procurement of experimental knowledge of these phenomena we shall have attained our aim." Abundant and detailed experiments on the very large strain of rubber have been reported by Rivlin from 1947 onwards. In the reviewer's opinion, the results of these experiments fully confirm the predictions of the general theory of elasticity, while showing that the second order approximation employed by the author is insufficient.

C. TRUESDELL

Lectures in abstract algebra. Vol. I. *Basic concepts*. By N. Jacobson. New York, Van Nostrand, 1951. 12+217 pp. $5.00.

This is the first volume of a projected three volume work designed to give a general treatment of abstract algebra. This volume gives a comprehensive introduction to abstract algebra and its basic concepts. The next two volumes will be more specialized in nature. The second one will deal with the theory of vector spaces and the final volume with field theory and Galois theory.

The present volume is well organized and excellently written. A considerable number of exercises are given that vary greatly in difficulty.

After a short introduction on set theory and the system of rational integers the author begins with a preliminary treatment of semi-groups and groups. Non-associative binary compositions are discussed briefly. This is followed by a fairly orthodox treatment of rings, integral domains, and fields. The third chapter deals with various types of extensions of rings and fields, such as the field of fractions of an integral domain, polynomial rings, and simple field extensions.

The next chapter contains a discussion of factorization theory for commutative semi-groups and integral domains. It is shown that the unique factorization theorem holds for principal ideal domains and for Euclidean domains. Unique factorization is also discussed for polynomial rings over rings which have unique factorization.

In Chapter V the author discusses groups with operators, generalizing many of the earlier results on groups and group homomorphisms. The isomorphism theorems are proved and this leads up to a proof of the Jordan-Hölder theorem. This is followed by a discussion of the concept of direct product and a proof of the Krull-Schmidt theorem. Infinite direct products are discussed briefly.

The first part of the sixth chapter deals with the theory of modules. Ascending and descending chain conditions are discussed and a proof is given of the Hilbert basis theorem. The second half of this chapter contains a discussion of ideal theory in Noetherian rings. It is shown that every ideal can be represented as the intersection of primary ideals and two uniqueness theorems are proved about this intersection.

In the final chapter an introduction to the theory of lattices is given. Modular lattices, complemented lattices, and Boolean algebras are treated briefly. A number of results proved earlier in the book are discussed here from a lattice-theoretic point of view—the Jordan-Hölder theorem being proved for modular lattices.

With this volume the author has made an excellent beginning. The completed work should be one of the best general treatments of abstract algebra available.

W. H. MILLS

Introduction to number theory. By T. Nagell. New York, Wiley, 1951. 309 pp. $5.00.

This is essentially a revised edition of a book published in Swedish in 1950. The present edition contains more problems and an additional chapter on the prime number theorem but unfortunately replaces the useful two line biographies of some 63 mathematicians who have contributed to number theory by a corresponding "name index." Some idea of the contents can be obtained from the chapter titles: I, Divisibility; II, On the distribution of primes; III, Theory of congruences; IV, Theory of quadratic residues; V, Arithmetical properties of the roots of unity; VI, Diophantine equations of the second degree; VII, Diophantine equations of higher degree; VIII, The prime number theorem.

The first chapter is fairly standard and contains the unique factorization theorem, a treatment of the arithmetic functions $\phi(n)$, $\mu(n)$, $\tau(n)$, the linear diophantine equation and proofs of the irrationality of e and π.

Chapter II deals with the sieve of Eratosthenes, the Euler product for the zeta function, and Tchebycheff's result that the true order of $\pi(x)$ is $x/\log x$. In addition, the author gives a worthwhile general discussion of the results of modern research in the theory of primes including the Goldbach problem, the twin prime problem, and Brun's method. However, he neglects to mention A. Selberg's work on sieves, and his statements concerning $\pi(x)-\mathrm{Li}(x)$ and the gaps between consecutive primes can both be strengthened. In addition, his remark that $2^{127}-1$ is the largest known prime is no longer true as the result of recent work. (At the moment of writing, the largest prime known is $2^{1279}-1$, as communicated to the reviewer by D. H. Lehmer.)

Chapter III gives an unusually complete account of the theory of congruences including the identical congruence, divisibility of polynomials with respect to a modulus, the Chinese remainder theorem, polynomial congruences to prime power moduli, primitive roots and indices, power residues, integer representing polynomials, and a remainder theorem first appearing in a little known paper of A. Thue (Christiania Videnskabs Selskabs Forhandlinger 1902, No. 7, pp. 7–31) and subsequently rediscovered or generalized by Aubry, Scholz, Vinogradov, Brauer and Reynolds, Rédei, and others. The exposition of identical congruences might have been clarified by a suitable notational device which would distinguish between the two possible meanings of $f(x)\equiv g(x) \pmod{n}$. Also, the author's treatment of congruences to prime power moduli assumes a good knowledge of discriminants and will therefore need amplification for most elementary number theory classes in this country; the final result, due originally to the author and giving the number of solutions of a polynomial congruence to a prime power modulus, is not to be found in many other books. This proof and a number of others in this chapter are somewhat lacking in clarity and they are likely to prove difficult for beginning students; in a few places, the proofs can be simplified by appealing to previous results.

Included in the fourth chapter are the Jacobi symbol, Gauss' lemma, the quadratic reciprocity law, and the infinity of primes in various special arithmetic progressions. The chapter is marred by a number of small flaws.

Chapter V deals with the cyclotomic polynomial, its irreducibility over the rational field, and its prime divisors; the basic properties of Gaussian sums are proved with the aid of suitable trigonometric and polynomial identities.

Chapter VI contains results on the representation of integers as weighted sums of squares, the four square (Bachet-Lagrange) theorem, various results on the Pell and similar equations, and a discussion of lattice points on conics.

Chapter VII deals with some of the usual diophantine equations like $x^4+y^4=z^2$, proves the Euclidean algorithm and unique factorization in $K((-1)^{1/2})$, $K((-2)^{1/2})$, and $K((-3)^{1/2})$, proves the impossibility of the Fermat equation $x^n+y^n=z^n$ for $n=3$ and 7, and proves a result of Kummer and some properties of cubic curves. A general description of the present state of knowledge regarding lattice points on plane algebraic curves is given and the work of Thue, Siegel, Mordell, and Weil is mentioned.

The last chapter is devoted to the elementary proof of the prime number theorem recently discovered by A. Selberg and P. Erdős. This outstanding work of Selberg is well known and was mentioned on his receiving the Fields medal at the International Congress of 1950; that of Erdős received official recognition at the end of 1951 in the following citation on the occasion of the award to him of the Frank Nelson Cole Prize in the Theory of Numbers:

"To Paul Erdős, for his many papers in the Theory of Numbers, and in particular for his paper *On a new method in elementary number theory which leads to an elementary proof of the prime number theorem*, Proceedings of the National Academy, vol. 35, pp. 374–385, July 1949, in which he makes important contributions to the new elementary theory of primes inaugurated by A. Selberg."

Despite the generally recognized importance of Erdős' work, the author has so completely neglected to mention it that the only impression one can gather from the book is that Erdős is a mathematician who gave some lectures attended by van der Corput who developed the exposition in the form given by the author.

The book contains 180 exercises which are to be found at the end of five of the eight chapters. The author's statement that these are not routine is an understatement; many will prove quite difficult for students.

The reviewer feels that the appeal of this book will be to the more advanced and mature student and that it will be valued chiefly for the special topics of Chapters V–VIII now made conveniently available. Most beginning students will probably find the book too difficult and it may therefore prove unsuitable as a classroom text for a first course.

LOWELL SCHOENFELD

Ordinary non-linear differential equations in engineering and physical sciences. By N. W. McLachlan. Oxford University Press, 1950. 6+201 pp. $4.25.

In the preface to this work the author states: "owing to the absence of a concise theoretical background, and the need to limit the size of this book for economical reasons, the text is confined chiefly to the presentation of various analytical methods employed in the solution of important technical problems." It is therefore with forethought, and perhaps with malice aforethought, that the author has presented the mathematical theory in a form which is highly disorganized and which reveals ever so tellingly the inadequacy of present mathematics to explain what are now common experiences of the engineer and physicist.

The state of disorganization of the mathematics is such that one might conclude that there is no theory of differential equations and that all one can hope for in practice is that the differential equation encountered has been solved in the literature. Thus Chapter II, on *Equations readily integrable*, consists solely of the following examples: $y' = -x/y$; $y' = (x+y)/(x-y)$; $y' = 2-x/y$; Bernoulli's equation; certain Riccati equations; the simultaneous equations: $y' = z + y[(y^2+z^2)^{1/2}-2a]$, $z' = -y+z[(y^2+z^2)^{1/2}-2a]$; $y'' = y'^2(2y-1) \cdot (y^2+1)^{-1}$; $y''+ay'^2y^{-1}=0$; $ax^2y'' = (xy'-y)^2$; $ay'''+yy''+y'^2=0$; the Lane-Emden equation. Chapter III concerns *Equations integrable by elliptic functions*. Again the choice of examples is arbitrary. The other chapters are restricted to a very few special equations and add little towards any general point of view.

Granted the deficiencies of the existing mathematical theory of differential equations, the reviewer believes that the author has exaggerated the situation. Even his list of "equations readily integrable" is a very inadequate presentation of what is known. One gains the impression that solution of a differential equation is to mean only solution in terms of elementary (or elliptic!) functions, and that only a simple analytical expression for the solution can be of use. The fact that the differential equation itself defines functions is ignored, although it is implicit in the approximate methods of solution described. It is often clear that the practical application does not demand a formula for solutions but only a knowledge of certain qualitative properties: especially, existence and stability of periodic solutions. The author ignores the contributions to this aspect by Poincaré, Birkhoff, and many others in recent times.

Despite the defects mentioned, the book has a number of good points which should render it of value to engineers. In the later chapters a number of important examples are treated in great detail and, where the mathematics fails, physical experiments are called upon to solve the equations. Any method based on experiments, even with such precise instruments as the modern electronic differential analyzer, does of course beg the question, for one can never be certain as to exactly what equation has been solved. However, in wisely guided hands, the instruments can be used to obtain clues as to the form of solutions and a more precise mathematical analysis may then be able to establish the agreement of equation with experiment. This process has been carried through completely in very few cases and in the present work the mathematics is carried only to the point of a first or second approximation by a perturbation method. In the fourth chapter (*Equations having periodic solutions*) this is done for the van der Pol equation, the physical experiments concerning a thermionic valve circuit and an electric generator-motor combination. Equations of form $\ddot{y}+a\dot{y}+f(y)=A \cos bt$ are handled similarly, with a beam-spring system and a hydro-electric surge chamber as illustrations.

Chapters V and VI concern the representation of approximate solutions in the form $A(t) \sin \phi(t)$. The approximations are always obtained by some variation on the perturbation method and the only justifications for their accuracy are physical plausibility arguments or actual experiments. One is continually impressed with the agreement of this crude theory and experiment. In Chapter VII certain non-linear equations are approximated by equations of Mathieu type and known properties of the Mathieu functions are used to discuss stability; a variety of physical illustrations are described. Of special interest is the occurrence of subharmonic resonance. Chapter VIII concerns the method of isoclines and (quite briefly) numerical methods. The book concludes with a large bibliography.

Although intentionally weak on the theoretical side, this book offers much valuable information to the engineer and to the mathematician presents a definite challenge which is hard to ignore.

W. KAPLAN

NOTES

The Royal Statistical Society has announced the publication of a new journal, Applied Statistics, under the editorship of Mr. L. H. C. Tippett. Volume 1, number 1 dated March 1, 1952 was published on April 25.

The Second Graduate Fellowship Program of the National Science Foundation, providing awards for study during the 1953–54 academic year, will get under way early in October. Application forms for both predoctoral and postdoctoral graduate fellowships for the 1953–54 academic year may be obtained after October 1, 1952, from the National Science Foundation, Washington 25, D.C. Completed applications must be returned to the Fellowship Office, National Research Council, by January 5, 1953.

Twenty-five fellowships are offered by the American Association of University Women to American women for advanced study or research during the academic year 1953–54. Applications and supporting materials must reach the office in Washington by December 15, 1952. For detailed information concerning these fellowships and instructions for applying, address the Secretary, Committee on Fellowship Awards, American Association of University Women, 1634 Eye Street N.W., Washington 6, D. C.

Associate Professor V. F. Cowling of the University of Kentucky has received a Ford Foundation Faculty Fellowship and will be on leave of absence at Yale University.

Dr. Allen Devinatz of the Illinois Institute of Technology has received a National Science Foundation Postdoctoral Fellowship and will study at the Institute for Advanced Study.

The honorary degree of Doctor of Science was conferred on Dean Emeritus L. P. Eisenhart by Princeton University on June 17.

Dr. I. B. Fleischer of the University of Chicago has received a National Science Foundation Postdoctoral Fellowship and will study at the University of Nancy.

The honorary degree of Doctor of Science was conferred on Dr. Kurt Gödel of the Institute for Advanced Study by Harvard University on June 19.

Dr. D. J. Lewis of the Ohio State University has received a National Science Postdoctoral Fellowship and will study at the Institute for Advanced Study.

Dr. A. E. Livingston of the University of Oregon has received a National Science Foundation Postdoctoral Fellowship and will study at the Institute for Advanced Study.

Associate Professor V. O. McBrien of the College of the Holy Cross has received a Ford Foundation Fellowship and will be on leave of absence at Harvard University.

The honorary degree of Doctor of Science was conferred on Mr. E. C. Molina by the Newark College of Engineering on June 6.

Professor Rothwell Stephens of Knox College has received a Faculty Fellowship, Fund for the Advancement of Education.

Dr. John Wermer of Yale University has received a National Science Foundation Postdoctoral Fellowship.

Dr. Abolghassem Ghaffari of the Institute for Advanced Study has been appointed to a professorship at Teheran University.

Mr. R. H. Kyle of Princeton University has been appointed a lecturer at Trinity College, Dublin, Ireland.

Dr. Daniel Pedoe of Westfield College is on leave of absence and has been appointed to a visiting professorship at University College, Khartow, Anglo-Egyptian Sudan.

Dr. N. C. Ankeny of the Institute for Advanced Study has been appointed to an assistant professorship at Johns Hopkins University.

Professor Henry Antosiewicz of Montana State College is on leave of absence and has been appointed to a research position at American University.

Assistant Professor K. J. Arnold of the University of Wisconsin has been appointed to an associate professorship at Michigan State College.

Assistant Professor J. H. Blau of Pennsylvania State College has been appointed to an associate professorship at Antioch College.

Mr. W. W. Bledsoe has been appointed a lecturer at the University of California, Berkeley.

Mr. I. E. Block of Harvard University has accepted a position as mathematical consultant with the Philco Corporation.

Professor L. M. Blumenthal of the University of Missouri is on leave of absence and has been appointed a consultant at the Institute for Numerical Analysis, Los Angeles, California.

Mr. R. K. Brown of Rutgers University has accepted a position as a mathematician with the U. S. Signal Corps, Belmar, New Jersey.

Mr. R. M. Brown of the University of Maryland has accepted a position as a physicist with the Naval Research Laboratory.

Mr. H. D. Brunk of Sandia Corporation has been appointed to an associate professorship at the University of Missouri.

Professor N. R. Bryan of the University of Maine has retired with the title Professor Emeritus.

Assistant Professor L. J. Burton of Bryn Mawr College has been appointed to an assistant professorship at Lake Forest College.

Mr. P. L. Butzer of the University of Toronto has been appointed a lecturer at McGill University.

Professor Sarvadaman Chowla of the University of Kansas has been appointed to a professorship at the University of Colorado.

Professor J. G. van der Corput has been appointed to a visiting professorship at the University of California, Los Angeles.

Assistant Professor D. A. Darling of the University of Michigan is on leave of absence and has been appointed to a visiting assistant professorship at Columbia University.

Professor R. F. Deimel of Stevens Institute of Technology has retired with the title Professor Emeritus.

Assistant Professor Albert Edrei of the University of Colorado has been appointed to an associate professorship at Syracuse University.

Professor L. R. Ford of the Illinois Institute of Technology has retired with the title Professor Emeritus.

Professor F. L. Griffin of Reed College has retired.

Dr. P. C. Hammer of Los Alamos Scientific Laboratory has been appointed to an associate professorship at the University of Wisconsin.

Assistant Professor L. Aileen Hostinsky of Temple University has been appointed to an assistant professorship at Pennsylvania State College.

Professor Leo Katz of Michigan State College is on leave of absence and has been appointed to a visiting professorship at the University of California, Berkeley.

Mr. R. A. C. Lane of Lehigh University has accepted a position as an engineer with the Radio Corporation of America, Camden, New Jersey.

Assistant Professor Violet H. Larney of Kansas State College of Agriculture and Applied Science has been appointed to an assistant professorship at the New York State College for Teachers.

Professor Walter Leighton of Washington University is on leave of absence and has been appointed a consultant at the Office of Scientific Research.

Miss Edith Moss of Simmons College has accepted a position as Actuarial Trainee with the John Hancock Life Insurance Company, Boston, Massachusetts.

Dr. P. P. Nesbeda of the Catholic University of America has accepted a position as mathematician with the Radio Corporation of America, Camden, New Jersey.

Professor Harry Pollard of Cornell University is on leave of absence and will be at the Institute for Advanced Study.

Professor R. G. Putnam of New York University has retired.

Dr. H. E. Rauch of Rutgers University has been appointed to an assistant professorship at the University of Pennsylvania.

Dr. Daniel Resch of the University of Michigan has been appointed to an assistant professorship at Northwestern University.

Associate Professor Myron Rosskopf of Syracuse University has been appointed to an associate professorship at Columbia University.

Assistant Professor C. W. Saalfrank of Rutgers University has been appointed to a professorship at Lafayette College.

Dr. Pierre Samuel of the University of Clermont-Ferrand has been appointed to an assistant professorship at Cornell University.

Associate Professor R. R. Stoll of Lehigh University has been appointed to a professorship at Oberlin College.

Associate Professor Alexander Tartler of Lafayette College has been appointed to a professorship at Drexel Institute of Technology.

Professor M. E. Wescott of Northwestern University has been appointed to a Professorship of Applied Statistics at Rutgers University.

Professor Jacob Wolfowitz of Cornell University is on leave of absence and will be at the Institute for Numerical Analysis, Los Angeles, California, until January 31 and at the University of Illinois until June 5.

The following promotions are announced:

A. G. Anderson, Oberlin College, to an assistant professorship.

Richard Arens, University of California, Los Angeles, to an associate professorship.

B. J. Ball, University of Virginia, to acting assistant professorship.

R. W. Ball, University of Washington, to an assistant professorship.

R. H. Bardell, University of Wisconsin, to a professorship.

H. G. Bergmann, City College, New York, to an assistant professorship.

R. H. Bing, University of Wisconsin, to a professorship.

Daniel Block, Yeshiva University, to an assistant professorship.

R. H. Bruck, University of Wisconsin, to a professorship.

T. E. Caywood, Institute for Air Weapons Research, University of Chicago, to supervisor of the new program in Operations Research.

L. W. Cohen, Queens College, to an associate professorship.

Nancy Cole, Syracuse University, to an associate professorship.

R. M. Conkling, University of New Hampshire, to an assistant professorship.

J. A. Daum, Agricultural and Mechanical College of Texas, to a professorship.

Ky Fan, University of Notre Dame, to a professorship.

Samuel Goldberg, Lehigh University, to an assistant professorship.

W. O. Gordon, Pennsylvania State College, to a professorship.

Fritz Herzog, Michigan State College, to a professorship.

J. C. Kiefer, Cornell University, to an assistant professorship.

Morris Kline, New York University, to a professorship.

J. A. Marano, Manhattan College, to an assistant professorship.

W. H. Mills, Yale University, to an assistant professorship.

R. E. Ozimkoski, Fordham University, to an assistant professorship.

P. T. Rotter, Mutual Benefit Life Insurance Company, to an associate mathematician.

Albert Schild, Temple University, to an assistant professorship.

A. S. Shapiro, Cornell University, to an assistant professorship.

T. T. Tanimoto, Allegheny College, to an assistant professorship.

The following appointments to instructorships are announced: University of Arizona: Dr. G. M. Petersen; University of California, Berkeley: Dr. R. G. Stoneham; Cornell University: Mr. N. T. Hamilton, Mr. Hyman Kamel, Dr. Ilse L. Novak; Harvard University: Mr. Robert Osserman; Lehigh University: Dr. J. A. Schatz; Princeton University: Dr. M. T. Wechsler; Rutgers University: Dr. K. G. Wolfson; Swarthmore College: Dr. David Rosen; Washington State College: Dr. D. W. Bushaw; Yale University: Dr. Lawrence Markus.

Associate Professor Emeritus Joseph Barnett of the Oklahoma Agricultural and Mechanical College died on April 12, 1952 at the age of sixty-nine years. He had been a member of the Society for fourteen years.

Associate Professor Emeritus E. F. A. Carey of Montana State University died on May 19, 1952. He had been a member of the Society for forty-four years.

Professor Marie Litzinger of Mt. Holyoke College died on April 7, 1952 at the age of fifty-two years. She had been a member of the Society for twenty-seven years.

Dean E. D. Meacham of the University of Oklahoma died on June 28, 1952 at the age of sixty-four years. He had been a member of the Society for thirty-seven years.

NEW PUBLICATIONS

Boltyanskiĭ, V. G. See Yaglom, I. M.

Carathéodory, C. *Conformal representation.* (Cambridge Tracts in Mathematics and Mathematical Physics, no. 28.) 2d ed. Cambridge University Press, 1952. 10+115 pp. $2.50.

von Cues, N. *Schriften des Nikolaus von Cues. Die mathematischen Schriften.* Trans. by J. Hofmann with an introduction and remarks by J. E. Hofmann. Hamburg, Meiner, 1952. 52+268 pp.

Foà, E. *Fondamenti di termodinamica.* Ed. by A. Giulianini. Bologna, Zanichelli, 1951. 13+255 pp. 3000 lire.

Giulianini, A. See Foà, E.

Gloden, A. *Table de factorisation des nombres* N^4+1 *dans l'intervalle* $3000<N\leq 6000$. Luxembourg, Gloden, 1952. 1+51 pp. 80 Belgian fr.

Hald, A. *Statistical tables and formulas.* New York, Wiley, 1952. 97 pp. $2.50.

Hald, A. *Statistical theory with engineering applications.* New York, Wiley, 1952. 12+783 pp. $9.00.

Hasse, H. *Über die Klassenzahl abelscher Zahlkörper.* Berlin, Akademie, 1952. 12+190 pp.

Hofmann, J. See von Cues, N.

Hofmann, J. E. See von Cues, N.

Lévy, J. See Poincaré, H.

Lorch, E. R. *Theory of functions.* New York, Columbia University, 1951. 3+139 pp. $4.45.

Maxwell, J. C. *The scientific papers of James Clerk Maxwell.* Ed. by W. D. Niven. New York, Dover, 1952. 32+607+8+806 pp. $10.00.

Meister, Fr. *Magische Quadrate.* Zürich, Wurzel, 1952. 71 pp.

Myškis, A. D. *Lineĭnye differencial'nye uravneniya s zapazdyvayuščim argumentom.* Moscow, Gosudarstvennoe Izdatel'stvo Tehniko-Teoretičeskoi Literatury, 1951. 255 pp. 8.80 rubles.

Niven, W. D. See Maxwell, J. C.

Pamyati S. V. Kovalevskoĭ. Sbornik stateĭ. Moscow, Izdatel'stvo Akademiĭ Nauk SSSR., 1951. 155 pp., 1 plate. 8.6 rubles.

Parodi, M. *Sur quelques propriétés des valeurs caractéristiques des matrices carrées.* (Memorial des Scientifiques Mathématiques, no. 118.) Paris, Gauthier-Villars, 1952. 64 pp. 800 fr.

Pogorelov, A. V. *Izgibanie vypuklyh poverhnosteĭ.* Moscow, Gosudarstvennoe Izdatel'stvo Tehniko-Teoretičeskoi Literatury, 1951. 184 pp. 6.57 rubles.

Poincaré, H. *Oeuvres de Henri Poincaré.* Vol. 7. Published under the auspices of the Académie des Sciences by the Section de Géométrie. With the collaboration of J. Lévy. Paris, Gauthier-Villars, 1952. 8+635 pp.

Riesz, F., and Sz.-Nagy, B. *Leçons d'analyse fonctionnelle.* Budapest, Académie des Sciences de Hongrie, 1952. 8+449 pp. $7.70.

Sz.-Nagy, B. See Riesz, F.

Tables of Coulomb wave functions. Vol. 1. (National Bureau of Standards Applied Mathematics series, no. 17.) Washington, U. S. Government Printing Office, 1952. 27+141 pp. $2.00.

Targ, S. M. *Osnovnye zadači teorii laminarnyh tečeniĭ.* Moscow, Gosudarstvennoe Izdatel'stvo Tehniko-Teoretičeskoi Literatury, 1951. 420 pp. 15.85 rubles.

Tricomi, F. *Funzioni analitiche.* Reprint of 2d ed. Bologna, Zanichelli, 1952. 7+134 pp. 1500 lire.

Tricomi, F. *Funzioni ellittiche.* 2d ed. Bologna, Zanichelli, 1951. 9+343 pp. 4500 lire.

Weyl, H. *Symmetry.* Princeton University Press, 1952. 8+168 pp. $3.75.

Yaglom, I. M., and Boltyanskiĭ, V G. *Vypuklye figury.* Moscow, Gosudarstvennoe Izdatel'stvo Tehniko-Teoretičeskoi Literatury, 1951. 343 pp. 6.60 rubles.

Journals Published by the

American Mathematical Society

Bulletin of the American Mathematical Society

This journal is the official organ of the Society. It reports official acts of the Society and the details of its meetings, with abstracts of all research papers presented. It contains some of the officially invited addresses presented before the Society. It contains also announcements and notes on matters of current interest, and reviews of advanced mathematical books.

The subscription price is $7.00 per annual volume of six numbers.

Invited addresses offered for publication should be sent to G. B. Price, 215 Strong Hall, University of Kansas, Lawrence, Kan.; other articles to W. T. Martin, Room 2-255, Massachusetts Institute of Technology, Cambridge 39, Mass.; book reviews to R. P. Boas, Jr., Lunt Hall, Northwestern University, Evanston, Ill.; and items for Notes to the Bulletin of the American Mathematical Society, 80 Waterman St., Providence 6, R.I. All other communications to the editors should be addressed to the Managing Editor, G. B. Price, 215 Strong Hall, University of Kansas, Lawrence, Kan.

Proceedings of the American Mathematical Society

This journal is devoted entirely to research in pure and applied mathematics, and is devoted principally to the publication of original papers of moderate length.

The subscription price is $11.00 per annual volume of six numbers.

Papers in algebra and theory of numbers should be sent to Nathan Jacobson Leet Oliver Memorial Hall, Yale University, New Haven 11, Conn.; in analysis to A. C. Schaeffer, University of Wisconsin, Madison 6, Wis.; in other fields to G. A. Hedlund, Leet Oliver Memorial Hall, Yale University, New Haven 11, Conn. All other communications to the editors should be addressed to the Managing Editor, G. A. Hedlund, Leet Oliver Memorial Hall, Yale University, New Haven 11, Conn.

Transactions of the American Mathematical Society

This journal is devoted entirely to research in pure and applied mathematics, and includes in general longer papers than the Proceedings.

Normally two volumes of three numbers each are published annually. The subscription price is $6.00 per (semi-annual) volume of three numbers.

Papers in algebra, number theory, logic and foundations should be sent to Saunders MacLane, University of Chicago, Chicago 37, Ill.; in probability, theory of measure, real variables to J. L. Doob, Department of Mathematics, University of Illinois, Urbana, Ill.; other papers in analysis and applied mathematics to L. V. Ahlfors, Department of Mathematics, Harvard University, Cambridge 38, Mass.; in geometry and topology to G. T. Whyburn, School of Mathematics, University of Virginia, Charlottesville, Va. All other communications to the editors should be addressed to the Managing Editor, G. T. Whyburn, School of Mathematics, University of Virginia, Charlottesville, Va.

Memoirs of the American Mathematical Society

This series is devoted to the publication of research monographs in pure and applied mathematics. The memoirs are issued serially, but not periodically. Each issue consists of either a single monograph or a group of cognate papers and is priced separately according to size and cost of publication. The editorial standards for acceptance are the same as for the Transactions and manuscripts intended for publication should be submitted to the appropriate editor of the Transactions.

Mathematical Reviews

This journal contains abstracts and reviews of the current mathematical literature of the world. It is sponsored by thirteen mathematical organizations, located both in the United States and abroad.

Mathematical Reviews is published monthly, excepting July. The subscription price is $20.00 per annual volume of eleven numbers.

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SOME NOTIONS AND PROBLEMS OF GAME THEORY

J. C. C. McKINSEY

The theory of games is a relatively new branch of mathematics, which treats of situations involving conflict among rational agents. In a typical problem to which the theory is applicable, each of several people can to some extent, but only partially, influence the outcome of a certain event; no one of them by himself can determine the outcome completely; and they differ in their preferences among the various possible outcomes: the theory of games is then concerned with the problem of what each person should do in order to maximize his expectation of good.

The theory was originally developed—first by von Neumann,[1] and later by Morgenstern and von Neumann[2]—to provide a mathematical basis for economics. More recently it has been successfully applied to problems of military tactics.[3] The mathematical statisticians,[4] finally, have found that some of the fundamental notions of this theory are extremely useful for their discipline: procedures analogous to those used in the theory of games, if they do not yet tell the statistician exactly what he ought to do, have at least taught him that certain things are better left undone.

In this lecture I shall summarize the mathematical aspects of this theory. The literature of the subject is already too extensive for me to be able to cover it completely in the time at my disposal, but I hope to explain some of the important notions involved, and to indicate some of the outstanding problems.

Before going to a general description of games it is well to explain some technical terminology. In the theory of games one means by a *game*, roughly speaking, a body of rules which specify unambiguously what, under various conditions, each of certain persons, called the *players* of the game, is permitted to do, what chance devices are

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An address delivered before the Pasadena meeting of the Society, December 1, 1951, by invitation of the Committee to Select Hour Speakers for Far Western Sectional Meetings; received by the editors November 29, 1951.

[1] See von Neumann [19]. (The numbers in brackets refer to items in the bibliography.)

[2] See von Neumann and Morgenstern [20].

[3] For a nontechnical account of the relation of the theory of games to military matters, see McDonald [15] and [16]. For a detailed analysis of an interesting problem in this domain (the problem of how an airplane should cruise in searching for a submarine) see Morse [17].

[4] For the application of the theory of games to statistical problems, see Arrow, Blackwell, and Girshick [1], and Wald [25] and [26].

to be used, and the like. A *play* of a game, on the other hand, is a particular way of playing the game from beginning to end. Thus a chess tournament consists of a number of *plays* of the *game* of chess.

What are called *finite* games can conveniently be represented with the help of topological trees.[5] I shall give an example of such a representation, for the case of a certain three-person game. Suppose that the first move of the game is made by player P_1, who has to choose from among four alternatives. We represent this situation by the bottom part of Figure I, where the "1" at the lowest point of the figure indicates that the first move is made by P_1, and the four rising lines indicate that P_1 has four alternatives at this move.

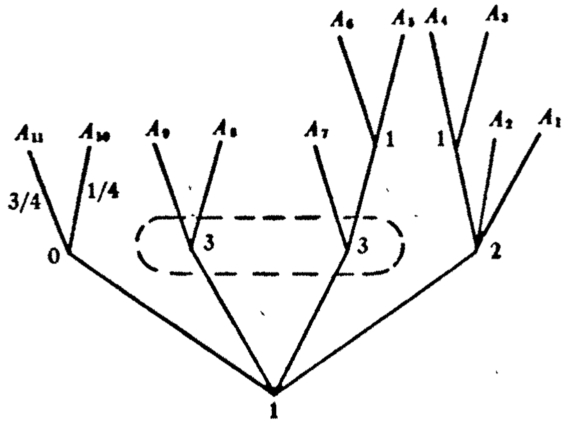

FIG. I

We number the alternatives at this first move (and, indeed, at each of the moves) in a counterclockwise sense with roman numerals. Thus alternative I at the first move is the one corresponding to the segment leading to the vertex marked "2" in the figure.

Now suppose that, if P_1 chooses I on this first move, then the next move is to be made by player P_2, who in turn has a choice from among three alternatives; and suppose that in addition it is given that if P_1 chooses II or III on the first move, then the next move is made by player P_3, who in either case is to choose from two alternatives. This is indicated in Figure I by putting a "2" at the end of the segment corresponding to the first choice by P_1, drawing three lines upward from this point, by putting a "3" on the vertices corresponding to moves II and III of P_1, and drawing two lines upward from each of these points.

---

[5] An exact definition of a game was first given in von Neumann and Morgenstern [20, Chapter II]; but Kuhn, in [12], gave a much simpler definition. My discussion is based on that of Kuhn.

Suppose, moreover, that, if P_1 chooses IV on the first move, then the next move is made by chance—an "umpire," let us say, chooses from two alternatives by means of some chance device—and suppose that the chance device is such that it assigns probability 1/4 to the first chance alternative, and 3/4 to the second. We represent this in Figure I by putting a "0" at the vertex corresponding to the move made by chance; and putting "1/4" and "3/4" on the appropriate lines rising from this vertex.

Finally, let us suppose that if P_1 chooses I and P_2 chooses III, or if P_1 chooses II and P_3 chooses I, then the next move is to be made by P_1 again; that he then has a choice of two alternatives; and that these are the only possible moves in the game. Then the tree in Figure I represents the complete structure of the moves of the game.

When games are represented in this way, we see that any unicursal path from the bottom to the top of the tree represents a play of the game; clearly there are just as many plays as there are top points of the tree. Thus there are precisely eleven possible ways of playing the game represented in Figure I.

It is apparent that, in order to describe the game, it is also necessary to state what will be the payments to or from the various players in case the game terminates at the various points $A_1, \cdots, A_{11}$. This can be done by giving a function H, which is defined over $A_1, \cdots, A_{11}$, and assumes vectors $\langle x, y, z\rangle$ as values; where, for instance, if

$$H(A_6) = \langle 5, 3, -8\rangle,$$

this means that, in case the play terminates at A_6, then P_1 and P_2 will be paid 5 and 3 respectively, and 8 will be taken away from P_3.

In order to complete the description of the game, however, it is also necessary to specify how much the players know about the previous choices at the time they make their moves. (This specification is clearly very important, since the absence of information is, generally speaking, a handicap in playing a game.) Thus suppose, for instance, that when P_3 makes his choice he does not know whether P_1 has chosen II or III on his first move; we can indicate this, as in Figure I, by enclosing the two corresponding points in a region bounded by a dotted line.

It can even happen that P_1, on his second move, does not "remember" what he did on his first move. Such a situation can be realized in practice by making P_1 consist of a "team" of two men, who are kept isolated from each other, so that the second man, when it is his time to move, does not know what the first man has done. Such a situation would be represented graphically by enclosing within a dotted line the two uppermost vertices of Figure I.

We call such a set of vertices among which a player cannot distinguish when he makes his move an *information set*. The totality of information sets constitutes a partition of the moves of the game (i.e., of the vertices of the tree); if a player at a certain point is completely informed about the past course of the play, the corresponding information set contains only the single point—in which case we usually omit the dotted line.

If each information set contains just one vertex, we say that the game is one with *perfect information*. Thus a game with perfect information is one in which each player, at the time when he makes each of his moves, is completely informed about the past course of the play.

In general, an n-person finite game is one that can be represented as above by a finite tree. The partition of the vertices into information sets is subject to the obvious restrictions that all the vertices in a given information set must correspond to the same player, and must present the same number of alternatives. In addition it is convenient to impose the following condition:

CONDITION A. No information set intersects any play in more than one point.

If Condition A is not satisfied, certain difficulties arise, which will be discussed later.

An enormous simplification of the theory of finite games was effected by von Neumann's introduction of the notion of a strategy. By a *strategy* for a player is meant a function which is defined over the class of his information sets, and for each information set picks out one of the available alternatives. Thus, for the game represented in Figure I, let α be the set consisting of the bottom vertex, and let β be the set consisting of the two top vertices; then a strategy for player P_1 is a function f such that

$$f(\alpha) \in \{\mathrm{I}, \mathrm{II}, \mathrm{III}, \mathrm{IV}\}$$

and

$$f(\beta) \in \{\mathrm{I}, \mathrm{II}\}.$$

Thus P_1 has eight strategies for this game. The number of strategies for a finite game is of course always finite.

Now imagine that, before starting a play of a game, each player picks a strategy and plays according to it. Then, if there are no chance moves involved in the game, the outcome, including what payments will be made, is already determined; thus the players might as well pick their strategies, and then leave to a computing machine the task of calculating the outcome. On the other hand, if the game involves also chance moves, then a choice of strategies by the players does not completely determine the outcome, but it at least determines the expectation of each player; here again, then, after picking their strategies the players can leave the actual play to a machine—though now we shall have to suppose that the machine is provided with suitable chance mechanisms (or with tables of random numbers) which it will employ at appropriate points in the play.

In terms of this notion of strategies, we can now describe, corresponding to a given finite game, an equivalent game of much simpler character. For each player P_i ($i=1, \cdots, n$), let the set of strategies available to P_i be α_i. Let $M_1, \cdots, M_n$ be functions such that, when player P_1 chooses strategy x_1, player P_2 chooses strategy x_2, and so on, then the amount

$$M_i(x_1, \cdots, x_n)$$

is the expectation of the ith player. Then the given game is equivalent to the following n-move game: on the first move P_1 chooses an element x_1 from α_1; on the second move P_2, without being informed what choice P_1 has made, chooses an element x_2 from α_2; $\cdots$; and on the last move, P_n, without being informed what choices have been made by the other $n-1$ players, chooses an element x_n from α_n. After all the choices have been made, player P_i (for $i=1, \cdots, n$) is paid the amount

$$M_i(x_1, \cdots, x_n).$$

This new form of the game is called the *normalized form*; the form considered earlier (where there can be many moves, chance moves, etc.) is called the *extensive form*. The functions $M_1, \cdots, M_n$ are called the *payoff functions* of the game in normalized form.

A game in normalized form, with payoff functions $M_1, \cdots, M_n$, is called *zero-sum* if for every choice $x_1, \cdots, x_n$ of strategies for the n players we have

$$\sum_{i=1}^{n} M_i(x_1, \cdots, x_n) = 0.$$

Thus a zero-sum game is one where the sum of the payments made, at least on the average, is zero.

It is clear that the normalized form of a game is zero-sum if the sum of the payments to the various players is zero for each possible outcome. This condition is not necessary, however, if the game in extensive form involves chance moves, since the payoff functions are defined as expectations.

All the familiar parlor games are zero-sum, since wealth is neither created nor destroyed in the course of playing them—but is merely transferred from one player to another. But this is not to say that only zero-sum games are important. For it appears that economic processes, when considered as games, are ordinarily not zero-sum, since wealth is sometimes created during these processes; and in military processes wealth is often destroyed.[6]

Although the theory of non-zero-sum games is extremely important in practical applications, it appears very difficult to formulate useful concepts for this theory; and the situation is similar with respect to games with more than two players. The conceptual part of the theory of zero-sum, two-person games, on the other hand, is already in a very satisfactory state; and this theory presents a great variety of exactly formulated mathematical problems, some of which turn out to be quite difficult. During the last few years the attention of mathematicians interested in game theory has accordingly been largely devoted to problems about zero-sum two-person games; I want to review very briefly some of the known results in this domain.

Let Γ be a (finite) zero-sum two-person game in normalized form. We can suppose that the strategies of P_1 are represented by the numbers

$$1, \cdots, m,$$

and those of P_2 by

$$1, \cdots, n.$$

Moreover, there are two functions M and M' such that, if P_1 chooses strategy i and P_2 chooses strategy j, then P_1 receives

$$M(i, j)$$

and P_2 receives

$$M'(i, j).$$

---

[6] It should be remarked that in military situations it often becomes exceedingly difficult to state what sort of assumptions should be made about the utility of the objects created or destroyed—and one is puzzled how to compare the utility scales of the contestants. Thus in such situations it is not always easy to say whether we are dealing with a zero-sum game or not.

Since Γ is zero-sum we have, for all $i \leq m$ and $j \leq n$,

$$M(i, j) + M'(i, j) = 0,$$

and hence

$$M'(i, j) = - M(i, j).$$

Thus we can say simply that P_1 receives $M(i, j)$ and P_2 receives $-M(i, j)$. Hence the game is determined by the "payoff matrix"

$$\begin{bmatrix} M(1, 1) & \cdots & M(1, n) \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ M(m, 1) & \cdots & M(m, n) \end{bmatrix};$$

if P_1 chooses the ith row and P_2 chooses the jth column, then P_2 is to pay P_1 the amount $M(i, j)$.

Now for any element i which P_1 may choose, he can be sure of getting at least

$$\min_{j \leq n} M(i, j).$$

Since P_1 is at liberty to choose i, therefore, he can make his choice in such a way as to insure that he gets at least

$$\max_{i \leq m} \min_{j \leq n} M(i, j).$$

Similarly, P_2 can make his choice in such a way that P_2 is sure to get at least

$$\max_{j \leq n} \min_{i \leq m} -M(i, j),$$

and hence so that P_1 will get at most

$$- \max_{j \leq n} \min_{i \leq m} -M(i, j),$$

which equals

$$\min_{j \leq n} \max_{i \leq m} M(i, j).$$

If it happens that

$$\min_{j \leq n} \max_{i \leq m} M(i, j) = \max_{i \leq m} \min_{j \leq n} M(i, j), \tag{1}$$

then P_1 can himself play so as to get this common value (which we denote by "v"), and P_2 can keep P_1 from getting more than v. In this case, there are *optimum strategies* i_0 and j_0 for the two players such that, for all i and j,

$$M(i, j_0) \leqq M(i_0, j_0) \leqq M(i_0, j) \tag{2}$$

and

$$M(i_0, j_0) = v.$$

Thus P_1, unless he has good reason to believe that P_2 will behave in some particular foolish way, cannot do better than to choose i_0; and similarly, P_2 cannot do better than to choose j_0.

It is easily shown that conditions (1) and (2) are equivalent: thus condition (1) is satisfied if and only if the matrix of the game has a *saddle-point*—i.e., an element which is at the same time the minimum of its row and the maximum of its column. The matrix

$$\begin{pmatrix} 2 & 3 \\ 1 & -5 \end{pmatrix},$$

for example, has a saddle-point in its upper left-hand corner; hence P_1 can expect to get at least 2 in this game (by choosing the first row), and P_2 can keep P_1 from exceeding 2 (by choosing the first column).

It is interesting to note in this connection that Morgenstern and von Neumann have shown (in [20, pp. 112–124]) that the matrix of the normalized form of every two-person, zero-sum game with complete information has a saddle-point. Thus there exist optimum strategies for playing chess, for example, or backgammon. The theorem does not apply to games like bridge or poker, however, where one does not know what cards have been dealt one's opponents.

On the other hand, not every matrix has a saddle-point; a simple example is the following:

$$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

The question then arises whether there are nevertheless optimum ways of playing such games. The answer is affirmative, provided we introduce a new notion: namely, that of using a chance mechanism for the choice of strategies.

Suppose that, in the game whose matrix is

$$\begin{bmatrix} M(1, 1) & \cdots & M(1, n) \\ \vdots & & \vdots \\ M(m, 1) & \cdots & M(m, n) \end{bmatrix},$$

the first player, instead of choosing the m strategies available to him in some predetermined manner, selects them at random with probabilities $x_1, \cdots, x_m$, and that the second player selects his strategies with probabilities $y_1, \cdots, y_n$. This means that P_1 chooses a vector $X = \langle x_1, \cdots, x_m\rangle$ from the set $\mathfrak{S}_m$ of all vectors $\langle x_1, \cdots, x_m\rangle$ whose components satisfy

$$x_i \geqq 0 \text{ for } i = 1, \cdots, m, \quad \sum_{i=1}^{m} x_i = 1;$$

and similarly that P_2 chooses a vector $Y = \langle y_1, \cdots, y_n\rangle$ from the analogously defined set $\mathfrak{S}_n$. (Elements of $\mathfrak{S}_m$ and $\mathfrak{S}_n$ are called *mixed strategies*; we sometimes call the original strategies $1, \cdots, m$ and $1, \cdots, n$ *pure strategies*.) Then the expectation $E(X, Y)$ of the first player is given by the equation

$$E(X, Y) = \sum_{j=1}^{n} \sum_{i=1}^{m} M(i, j) x_i y_j.$$

Reasoning as we did in the case of pure strategies, we see that, by playing properly, P_1 can insure that his expectation will be at least

$$\max_{X \in \mathfrak{S}_m} \min_{Y \in \mathfrak{S}_n} E(X, Y);$$

and the second player can insure that the expectation of the first will be at most

$$\min_{Y \in \mathfrak{S}_n} \max_{X \in \mathfrak{S}_m} E(X, Y).$$

At this point, however, the situation diverges radically from the case of pure strategies, for now it can be shown that, for every payoff matrix M, we have

$$\max_{X \in \mathfrak{S}_m} \min_{Y \in \mathfrak{S}_n} E(X, Y) = \min_{Y \in \mathfrak{S}_n} \max_{X \in \mathfrak{S}_m} E(X, Y).$$

Thus if we permit the players to use mixed strategies, then every finite zero-sum, two-person game has optimum strategies for the two players. This theorem, which was first proved by von Neumann,[7] is sometimes called the "minimax theorem"; it is the fundamental theorem of finite games.

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[7] The earliest proof of this theorem is to be found in von Neumann [19]; this proof makes use of the Brouwer fixed-point theorem. An elementary algebraic proof was given later in Ville [24]. In von Neumann and Morgenstern [20, pp. 153–158], will be found a simple proof depending on the theory of convex sets. Other proofs are to be found in Loomis [14] and Brown and von Neumann [6].

Thus if we let $\mathcal{D}_1$ and $\mathcal{D}_2$ be the sets of optimum mixed strategies for P_1 and P_2 respectively, we see that $\mathcal{D}_1$ and $\mathcal{D}_2$ are not empty. Since $\mathcal{D}_1 \subseteq \mathcal{S}_m$ and $\mathcal{D}_2 \subseteq \mathcal{S}_n$, it is clear that $\mathcal{D}_1$ and $\mathcal{D}_2$ are respectively subsets of $(m-1)$-dimensional and $(n-1)$-dimensional Euclidean space. It is easily verified that $\mathcal{D}_1$ and $\mathcal{D}_2$ are closed and convex; and a proof that $\mathcal{D}_1$ and $\mathcal{D}_2$ are polyhedral has been given by Shapley and Snow (in [22]), who have also established a constructive method for finding the vertices of the polyhedra; from the convexity this of course enables one to find all the members of $\mathcal{D}_1$ and $\mathcal{D}_2$. It should be remarked, however, that the method of Shapley and Snow becomes laborious in the case of games with matrices of large order, since the method involves inverting a substantial proportion of the submatrices of the matrix of the game.

A more rapid (iterative) procedure for computing a pair of optimum strategies for a given game has been developed by Brown (see [5]), and shown to be convergent by Julia Robinson.[8]

Mathematicians have also interested themselves in the problem of what sets $\mathcal{D}_1$ and $\mathcal{D}_2$ can be regarded as the sets of optimum strategies for the two players of some zero-sum, two-person game. This problem has an elegant solution, which has been found by Bohnenblust, Karlin, and Shapley (in [3]), and independently by Gale and Sherman (in [8]).

Although the theory of finite zero-sum two-person games might thus seem to be already in a satisfactory form, it should be noticed that neither of the above methods of finding optimum mixed strategies is practicable when the number of pure strategies is more than, say, 100. On the other hand, when we normalize almost any game given in extensive form, the number of pure strategies is found to be quite large. Thus, even for the very simple game of ticktacktoe, there are more than $10^{10,000}$ pure strategies for each player; and clearly the number of pure strategies increases at a tremendous rate with the number of moves in the game. Two partial solutions have been given for the difficulties that arise from this circumstance.

The first partial solution rests on the fact that certain information in a game may be in a sense superfluous. Thus suppose, for example, that the first two moves of a certain game are both made by P_1, and that P_1, when he makes his second move, is informed what he did on his first move; then we see that this information is actually of no help to P_1, since his knowing what strategy he is playing insures that he knows what he did on his first move; thus we can obtain an equivalent game by decreasing the information given P_1 in his second move, which will decrease the number of information sets for P_1, and hence the number of strategies available to him. A simple method of deciding what information sets can safely be coalesced has been given by Krentel, McKinsey, and Quine (in [11]); this method, however, applies only to a certain subclass of the totality of finite games. An interesting condition applying to all finite games has recently been found by Norman Dalkey;[9] in this general case, however, there may be more than one game which is equivalent to a given game, and which is "completely deflated" (i.e., in which no more information sets can be coalesced without losing equivalence); and the problem remains to pick out, from among the completely deflated games equivalent to a given game, one which will make the number of pure strategies as small as possible.

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[8] See Robinson [21].

The second partial solution, which goes in a different direction altogether, was first made use of by Morgenstern and von Neumann (see [20, pp. 186–218]) in a discussion of a certain form of poker, and was later treated in a general way by Kuhn in [12]. In order to explain this method, it is necessary to introduce two new notions about games. By a game *with perfect recall* we mean a game in which each player, at each of his moves, remembers everything he did and knew at his previous moves; thus every two-person game which can be played by just two people (rather than by teams) is a game with perfect recall; rummy, for instance, is a game with perfect recall, but bridge is not—since in bridge each player is a pair of people, neither of whom is informed what cards the other holds. (I have given here only an intuitive explanation of this notion; it can easily be made mathematically precise, however, by expressing a condition on the information partition for the given game.) By a *behavior strategy* for a player of a game is meant a set of probability distributions, one for each of the player's information sets; and such that, if α is an information set which presents r alternatives, then the probability distribution corresponding to α is a member of $\mathcal{S}_r$. It is clear that, instead of using mixed strategies in playing a given game, one can also use behavior strategies; it is intuitively evident, moreover, that one can always do at least as well with a mixed strategy as with a behavior strategy. Now Kuhn[10] has shown, conversely, that, in the case of a game with perfect recall, one can always do as well with a behavior strategy as with a mixed strategy. This fact greatly decreases the difficulty of calculating optimum ways of playing games with perfect recall, for in general the number of parameters to be determined in calculating optimum behavior strategies is much smaller than the number of parameters to be determined in calculating optimum mixed strategies.

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[9] This result is to appear in the forthcoming second volume of *Contributions to the theory of games* (Annals of Mathematics Studies, No. 28), which will be published by the Princeton University Press.

[10] See Kuhn [12]. Actually Kuhn established a considerably stronger result than the one stated above.

It should be remarked that the above two methods cannot very well be used in combination. For the first method, which coalesces some of the information sets of a player, leads from games with perfect recall to games without perfect recall. Thus the question arises concerning when it is better to use one method, and when the other. In general, it appears that some more attention could profitably be devoted to the problem of how to solve games given in extensive form.

I turn now to games which are not finite: for example, to games in which choices are to be made from an infinite, instead of from a finite, number of alternatives. A great part of the research in this direction has been devoted to games which are already in normalized form. Thus suppose that P_1 chooses a real number x from the closed unit interval $(0 \leqq x \leqq 1)$, that P_2, without being informed what choice P_1 has made, then chooses a number y from the closed unit interval, and that P_2 then pays P_1 an amount $M(x, y)$, where M is a function of two real variables defined over the closed unit square. Such games are called *continuous games*. By a *mixed strategy* for such a game is meant a cumulative distribution function over the unit interval. If P_1 chooses x by means of a cumulative distribution function f, and P_2 chooses y by means of a cumulative distribution function g, then the expectation $E(f, g)$ of P_1 is given by the formula

$$E(f, g) = \int_0^1 \int_0^1 M(x, y)df(x)dg(y).$$

It has been shown by Ville (in [24]) that if M is continuous, then

$$\max_f \min_g E(f, g) = \min_g \max_f E(f, g),$$

so that a continuous game with a continuous payoff function has optimum mixed strategies for the two players. This result has been strengthened and generalized by Wald (in [25]), and by Karlin (in [10]).

As might be expected, mathematicians have devoted considerable attention to the problem of finding optimum mixed strategies for particular continuous games. Interesting results of this sort have been obtained in two rather wide special cases: (1) the case where $M(x, y)$ is convex in y for each x (and the analogous case where $M(x, y)$ is concave in x for each y); and (2) the case where $M(x, y)$ is a polynomial. In the first case, it has been shown by Bohnenblust, Karlin, and Shapley (in [4]) that if $M(x, y)$ is strictly convex in y for each x, and is differentiable over the closed unit square, then there is a unique optimum strategy for P_2, which assigns probability 1 to one point, and hence probability 0 to all other points (thus what is called a *step-function with one step*) and there is a step-function with two steps which is an optimum strategy for P_1. In the second case also, which has been treated by Dresher, Karlin, and Shapley (in [7]), it has been shown that each player always has an optimum strategy which is a step-function (the number of steps depending on the degree of M); and constructive methods have been developed for finding the appropriate step-functions. The methods of Dresher, Karlin, and Shapley are also applicable to many games with payoff functions of the type called *polynomial-like*: i.e., which can be represented in the form

$$M(x, y) = \sum_{j=1}^{n} \sum_{i=1}^{m} a_{ij} r_i(x) s_j(y)$$

where $r_1, \cdots, r_m$ and $s_1, \cdots, s_n$ are continuous.

There are also many interesting rather special results known in this connection. Gross[11] has given an example of a continuous game with a rational payoff function which has no step-function as an optimum strategy. Blackwell and Girshick have given an example[12] of a continuous game where every pure strategy is employed in the unique optimum strategy. Bellman and Blackwell (in [2]) have given an interesting analysis of the role of bluffing in two-person poker.

It goes without saying that there remain a great variety of unsolved problems in this domain. Formulations of some of these will be found in Kuhn and Tucker's introduction to a recent collection of papers on games (see [13]).

A problem which appears to be particularly important, but unfortunately very difficult, is that of dealing in some way with games which have more than one move, and where the choices are made from infinite sets. Thus suppose, for instance, that a game has four moves: in the first move P_1 chooses a number x_1 from the unit interval; in the second move, P_2, knowing x_1, chooses a number y_1 from the unit interval; in the third move, P_1, knowing y_1, but having forgotten x_1, chooses a number x_2 from the unit interval; and in the fourth move P_2, knowing y_1 and x_2, but not x_1, chooses a number y_2 from the unit interval. (The payoff is then some function of the four variables x_1, x_2, y_1, y_2.) A pure strategy for P_1 is now an ordered couple $\langle a, f\rangle$, where a is a real number and f is a function of one variable (it depends on y_1); and a pure strategy for P_2 is an ordered couple $\langle g, h\rangle$, where g is a function of one variable (it depends on x_1) and h is a function of two variables (it depends on y_1 and x_2). Thus in this game a mixed strategy is a distribution function over function space. It is not clear even over which subsets of this function space the probability distributions are to be regarded as being defined; and certainly we do not know conditions under which optimum strategies for such a game exist, or how to find them when they do.

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[11] See Gross [9]. This proof makes use of the fact, established in Tarski [23], that only algebraic numbers are definable in elementary algebra.

[12] In a private communication.

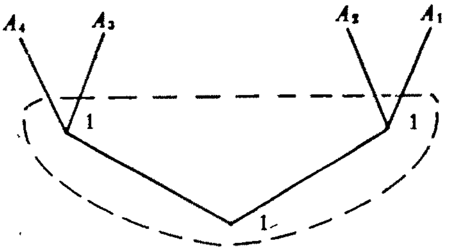

FIG II

Another important problem, to which attention has been called by Helmer,[13] is the problem of finding some rational way of dealing with conflict situations which are not technically two-person zero-sum games, though they closely resemble such games. I shall begin with a detailed examination of a situation which violates Condition A above. Just in order to have a term, let us agree to mean by a "pseudo-game" something that is like a game, except that Condition A is not satisfied. A very simple pseudo-game is one whose graphical representation is given in Figure II. Here we can suppose, if we please, that there are two players, P_1 and P_2; but all the moves are made by P_1. On the first move, P_1 is to choose from two alternatives, and on the second move he is again to choose from two alternatives. The dotted line enclosing the three vertices indicates the sole information set of the game; thus P_1, when he moves, does not know whether he is moving for the first or the second time. Since plays intersect the information set in two points, Condition A is not satisfied, and hence this is only a pseudo-game.

It is apparent that, in order to realize this situation in practice, it would be necessary to make P_1 a team consisting of two human beings. For if P_1 consisted of a single human being, then, on the second move, P_1 would remember that he had already moved, so that his information would be better than that indicated by the diagram.

Hence, suppose that there are two men, $P_1^{(1)}$ and $P_1^{(2)}$, who are together to constitute P_1, and suppose they are put into separate rooms and not allowed to communicate with each other during the play. After the beginning of the play, the umpire is to go into one room, and, ask the man there to choose one of the alternatives I and II, and then into the other room, and ask the other man to choose one of the alternatives I and II. P_2 will then pay P_1 an amount $M(x, y)$, where x is the alternative chosen by the man in the first room, and y is the alternative chosen by the man in the second room.

If we were to define a strategy for a pseudo-game in the same way we defined it for a game, then there would be only two strategies for P_1 in this game: one strategy would make him always pick I, and the other would make him always pick II. The first strategy would make him always end up at A_1, and the second would make him always end up at A_4. Any mixed strategy, consequently, would make him end up sometimes at A_1 and sometimes at A_4—but never at A_2 or A_3. Since it can happen, however, that the payoffs to P_1 at A_2 and A_3 are both higher than at A_1 or A_4, it is seen that in some cases P_1 can do better than by playing any mixed strategy: for example $P_1^{(1)}$ and $P_1^{(2)}$ could agree that $P_1^{(1)}$ is to choose I, and $P_1^{(2)}$ is to choose II—in which case the play would terminate either at A_2 or A_3.

Thus it might be thought that for this game we should distinguish four strategies:

1. $P_1^{(1)}$ and $P_1^{(2)}$ both pick I;
2. $P_1^{(1)}$ and $P_1^{(2)}$ both pick II;
3. $P_1^{(1)}$ picks I, and $P_1^{(2)}$ picks II;
4. $P_1^{(1)}$ picks II, and $P_1^{(2)}$ picks I.

But now the difficulty arises that, if P_1 plays strategy 3 or 4, then it is impossible to tell whether the play will terminate at A_2 or at A_3; for this will depend on whether the umpire goes first into the room occupied by $P_1^{(1)}$ or first into the room occupied by $P_1^{(2)}$. Since the payoffs at A_2 and A_3 are not necessarily equal, however, and since we do not know even the probability that the umpire will go first into the room occupied by $P_1^{(1)}$, we cannot in general calculate the expectation of P_1 if strategy 3, or strategy 4, is used. Thus our theory of games cannot be carried over in any obvious way to pseudo-games.

It is in order to avoid difficulties of this sort that we require that games satisfy Condition A. By imposing this condition we reduce the theory of games to something manageable; but the difficulties have not been legislated out of existence: they continue to exist, even if no longer countenanced as part of the theory of games. For conflict situations describable only as pseudo-games do arise in practice, and we sometimes have to cope with them—regardless of what we choose to call them. Suppose, for example, that two policemen, A and B, are trying to shoot an escaped insane criminal, C. Here it is natural to regard A and B as one "player," and C as the other "player." A and B may find it advantageous to separate; and after they have done so, A, when he sees C and shoots at him, may very well not know whether B has previously shot. It does not seem unreasonable to inquire what sort of actions A and B should take, even though this is not technically a game. This class of problems seems to be equivalent to the problem of defining optimum strategies for games in normalized form, where the values of the elements of the payoff matrix are not known exactly, but are merely required to satisfy certain inequalities.

It can also happen that, due for instance to difficulties of communication, it is not possible for the various members of a team to agree upon a strategy before the beginning of a play—in which case again, the theory of games in normalized form is no longer applicable. Situations of this sort are especially apt to occur in connection with military affairs.

In concluding this survey, I turn to a consideration of the problems which arise in connection with what are called "general" games: i.e., games which are not necessarily zero-sum, and which may have more than two players. As has been mentioned earlier, such games are particularly important so far as regards practical applications, but they present great conceptual difficulties; in particular, game theorists have not been able even to come to any general agreement as to what should be meant by a solution of such a game.

I shall discuss briefly the two most ambitious attempts[13] that have been made to deal with this subject: that of von Neumann, and that of Nash. I shall not try to give anything like a complete and adequate account of the theories in question, however, but shall merely say enough to give an intuitive idea of them.

von Neumann's treatment[14] of general games is based on the notion of a characteristic function for such a game, which is introduced as follows. If N is the set of all players of a game, and if S is any subset of N, then it can happen that the players form themselves into two "coalitions," S and $N-S$, which cooperate among themselves and oppose each other. If this happens, then the game becomes essentially a two-person game, and we can define a function v by agreeing that if S is any subset of N, then $v(S)$ is the total amount the members of S could expect to obtain, if they all cooperated together, and all the other players formed themselves into an opposing coalition, and tried to hurt the members of S as much as possible.

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[13] More recently a very interesting partial theory of zero-sum n-person games has been developed by L. S. Shapley. This theory was outlined by Shapley at the September 4, 1951 meeting of the Econometric Society at Minneapolis. A brief account will be found in Econometrica vol. 20 (1952) p. 91.

[14] Roughly the last two-thirds of von Neumann and Morgenstern [20] is devoted to an exposition of this theory of general games.

By a *solution* of a general n-person game, von Neumann means a certain set of ordered n-tuples of real numbers. I am not going to repeat here the exact conditions which must be satisfied by such a class of n-tuples, except to remark that the conditions are expressed solely in terms of the characteristic function; so that two games whose characteristic functions coincide will have the same solutions. The intuitive meaning of a solution is roughly the following: it represents the set of all possible ways, for a given organization of society, in which money can be distributed among the players at the end of a play.

Now consider a two-person non-zero-sum game in normalized form, where P_1 has just one strategy (so the outcome of a play cannot be affected by P_1), and P_2 has two strategies; and suppose that, in case P_2 uses his first strategy, then the payoff to P_1 is 10, and to P_2 is 0; and in case P_2 uses his second strategy, the payoff to P_1 is 0 and to P_2 is $-1{,}000$. It is convenient to represent such games by a pair of payoff matrices:

$$\begin{array}{cc} \textit{Payoff to } P_1 & \textit{Payoff to } P_2 \\ (10\quad 0) & (0\quad -1{,}000). \end{array}$$

Thus we again think of P_1 as being able to choose a row (but there in only one row in this case, so he has no real choice), and of P_2 as being able to choose a column.

The characteristic function of this game is a function defined over the four subsets of the set of two players. Since neither P_1 nor P_2 can, through his own endeavors, make sure of getting more than 0, we see that

$$v(\{P_1\}) = v(\{P_2\}) = 0. \tag{3}$$

Moreover, P_1 and P_2 can together behave in such a way that the sum of the payments to them will be 10 (in order to achieve this end, it is only necessary that P_2 choose the first column); hence

$$(4) \qquad v(\{P_1, P_2\}) = 10.$$

Finally, as in any game, if Λ is the empty set, we have

$$(5) \qquad v(\Lambda) = 0.$$

Equations (3), (4), and (5) completely specify the characteristic function.

For this game it turns out that there is just one solution in the sense of von Neumann, and this solution is the set of all couples $\langle x_1, x_2\rangle$ of non-negative real numbers such that $x_1+x_2=10$. This may be interpreted in the following way: before playing this game the two players will (or perhaps merely "should") get together and agree that P_2 will choose the first column, so that P_1 will be paid 10 and P_2 will be paid 0; they will decide, moreover, how this 10 is to be divided between them into x_1 and x_2; this division can be arbitrary, subject only to the restriction that x_1 and x_2 are both to be non-negative and are to have a sum of 10.

Two objections have been urged against von Neumann's notion of a solution. In the first place, it is felt by some people that knowing the solution in this sense of the word would not be much help in playing the game. The solution leaves a wide range of indeterminancy, and tells us nothing about the way in which the agreement between P_1 and P_2 is finally to be reached.

Secondly, it seems doubtful that the characteristic function adequately represents all the complications of the game. Thus, for example, one has the intuitive feeling that, in the game described above, P_1 is in a better bargaining position than is P_2, and most people feel they would rather play the part of P_1 than that of P_2: for P_1 will automatically be paid 10, unless P_2 is willing to injure himself very greatly by choosing the second column, and hence himself losing 1,000; therefore it hardly seems likely that P_2 will be able to persuade P_1 to pay him part of the 10, in order to insure that P_2 will play the first column. On the other hand, the characteristic function of this game, from (3), is symmetric in P_1 and P_2; so that no asymmetry can appear in any notion of a solution, so long as the solution is defined solely in terms of the characteristic function.

An attempt to avoid these difficulties, and to devise an altogether new way of dealing with general games, has been made by Nash (in [18]). Nash distinguishes first between what he calls "noncooperative" and "cooperative" games. A *noncooperative* game is one in which no communication is allowed between the players, and in which, in particular, they are not allowed to make agreements about side-payments; a *cooperative* game is one in which communication is allowed.

For the noncooperative game, Nash introduces the notion of an equilibrium point, which is a generalization of the notion of a saddle-point for a two-person zero-sum game. An *equilibrium point* for an n-person game is an ordered n-tuple $\langle x_1, \cdots, x_n\rangle$ of strategies such that, for $i=1, \cdots, n$, if every player P_j, for $j\neq i$, chooses x_j, then P_i cannot do better than to choose x_i. Nash shows that every finite game has an equilibrium point (in mixed strategies). He also points out that social groups who repeatedly play a certain game often fall into the habit of playing an equilibrium point; in such cases, of course, a newcomer to the group (unless he is able to persuade others to deviate) cannot do better than to stay with the equilibrium point also, and in general he will lose if he deviates from it.

Nash calls an n-person game *solvable* in case the set, S, of all its equilibrium points satisfies the following condition: if $1\leqq i\leqq n$, and if $\langle x_1, \cdots, x_n\rangle\in S$ and $\langle y_1, \cdots, y_n\rangle\in S$, then $\langle x_1, \cdots, x_{i-1}, y_i, x_{i+1}, \cdots, x_n\rangle\in S$. If a game is solvable in this sense, the set of its equilibrium points is called its *solution*. He deals with the cooperative games, in turn, by reducing them to noncooperative games in the following way: the negotiations of the cooperative game are included as formal moves in a noncooperative game (these moves consist of such procedures as, for example, one player's making an offer of a side-payment to another).

It must be remarked that Nash's theory—though it represents a considerable advance—has some grave deficiencies and certainly cannot be regarded as a definitive solution of the conceptual problems of this domain. In the first place, so far as regards the noncooperative games, it is unfortunately not the case that every game is solvable in the sense of Nash. Thus consider the two-person game whose matrices are as follows:

$$\begin{array}{cc} \textit{Payoff to } P_1 & \textit{Payoff to } P_2 \\ \begin{pmatrix} 10 & -30 \\ 8 & -18 \end{pmatrix} & \begin{pmatrix} -20 & -30 \\ 6 & 12 \end{pmatrix}. \end{array}$$

Here there is an equilibrium point in the upper left-hand corner, and one in the lower right-hand corner; but the other two points in the matrices are not equilibrium points, and hence the game is not solvable. P_1 would of course prefer the equilibrium point in the upper left-hand corner, and P_2 would prefer the equilibrium point in the lower right-hand corner. The theory of Nash seems to throw little light on the question how to play a game having such a pair of payoff matrices.

In the second place, even if the theory of noncooperative games were in a completely satisfactory state, there appear to be difficulties in connection with the reduction of cooperative games to noncooperative games. It is extremely difficult in practice to introduce into the noncooperative games the moves corresponding to negotiations, in a way which will reflect all the infinite variety permissible in the cooperative game, and this without giving one player an artificial advantage (due to his having the first chance of making an offer, let us say).

Thus it seems that, despite the great ingenuity that has been shown in the various attacks on the problem of general games, we have not yet arrived at a satisfactory definition of a solution of such games. It is rather likely, as has been suggested by Bellman,[11] that it will be found necessary to distinguish many types of games, and define "solution" differently for different types; the theory of Nash of course represents a step in this direction. This whole aspect of the theory of games presents a challenging problem to the mathematician, and in my opinion an extremely important one—since the application of game theory to a very wide class of practical situations must wait for such a definition.

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STANFORD UNIVERSITY

THE SUMMER MEETING IN EAST LANSING

The fifty-seventh Summer Meeting and thirty-third Colloquium of the American Mathematical Society were held at Michigan State College, East Lansing, Michigan, Tuesday to Friday, September 2–5, in conjunction with meetings of the Mathematical Association of America, The Institute of Mathematical Statistics, and the Econometric Society.

Over 600 people registered for the meeting, including the following 409 members of the Society:

C. R. Adams, R. B. Adams, G. E. Albert, M. W. Al-Dhahir, B. E. Allen, E. B. Allen, W. R. Allen, C. B. Allendoerfer, A. G. Anderson, R. D. Anderson, T. W. Anderson, R. V. Andree, J. J. Andrews, H. A. Antosiewicz, B. M. Armstrong, K. J. Arnold, Nachmann Aronszajn, M. C. Ayer, W. L. Ayres, H. M. Bacon, I. A. Barnett, D. Y. Barrer, R. C. Bartels, P. T. Bateman, W. F. Bauer, H. W. Becker, E. M. Beesley, E. G. Begle, J. H. Bell, Arthur Bernhart, Benjamin Bernholtz, D. L. Bernstein, W. A. Beyer, R. H. Bing, D. W. Blackett, David Blackwell, H. D. Block, I. M. Blyth, R. P. Boas, H. F. Bohnenblust, R. C. Bose, J. W. Bower, Barron Brainerd, A. T. Brauer, J. L. Brenner, Leonard Bristow, J. C. Brixey, F. E. Browder, M. C. Brown, R. H. Bruck, C. C. Buck, R. C. Buck, C. E. Buell, C. E. Burgess, J. H. Bushey, H. E. Campbell, K. H. Carlson, J. W. Carr, R. E. Carr, E. D. Cashwell, Lamberto Cesari, Abraham Charnes, Y. W. Chen, S. S. Chern, Joshua Chover, A. B. Clarke, F. M. Clarke, H. E. Clarkson, Nathaniel Coburn, D. E. Coffey, Harvey Cohn, R. H. Cole, E. G. H. Comfort, N. B. Conkwright, Geraldine Coon, T. F. Cope, A. H. Copeland, Max Coral, R. R. Coveyou, V. F. Cowling, H. S. M. Coxeter, J. W. Coy, C. C. Craig, M. V. Cross, A. B. Cunningham, D. A. Darling, Robert Davies, W. M. Davis, G. P. Dinneen, W. J. Dixon, J. M. Dobbie, C. L. Dolph, W. F. Donoghue, J. L. Doob, A. C. Downing, D. W. Dubois, R. J. Duffin, W. L. Duren, Ben Dushnik, P. S. Dwyer, J. M. Earl, W. F. Eberlein, J. G. Elliott, D. O. Ellis, Benjamin Epstein, D. H. Erkiletian, M. E. Estill, H. P. Evans, W. H. Fagerstrom, D. T. Finkbeiner, C. D. Firestone, W. H. Fleming, M. M. Flood, K. W. Folley, M. K. Fort, J. S. Frame, Evelyn Frank, Bernard Friedman, W. B. Fulks, R. E. Fullerton, H. K. Fulmer, I. S. Gál, H. L. Garabedian, H. M. Gehman, J. J. Gergen, Murray Gerstenhaber, Leonard Gillman, M. A. Girschick, Wallace Givens, Casper Goffman, S. I. Goldberg, R. A. Good, A. W. Goodman, Lillian Gough, S. H. Gould, Cornelius Gouwens, A. A. Grau, L. M. Graves, R. E. Graves, R. L. Graves, F. L. Griffin, H. C. Griffith, R. K. Haddad, Franklin Haimo, P. R. Halmos, P. C. Hammer, H. A. Hanson, Frank Harary, T. E. Harris, O. G. Harrold, G. E. Hay, R. M. Hayes, L. J. Heider, R. G. Helsel, M. S. Hendrickson, A. S. Henriques, I. N. Herstein, Fritz Herzog, G. W. Hess, D. G. Higman, T. H. Hildebrandt, J. G. Hocking, J. L. Hodges, J. E. Hoffman, R. V. Hogg, F. E. Hohn, D. L. Holl, T. C. Holyoke, Harold Hotelling, Ralph Hull, C. C. Hurd, J. W. Hurst, W. R. Hutcherson, C. G. Jaeger, R. D. James, T. A. Jeeves, W. E. Jenner, L. W. Johnson, R. E. Johnson, B. W. Jones, P. S. Jones, Joachim Kaiser, Shizuo Kakutani, Jan Kalicki, Wilfred Kaplan, Leo Katz, D. E. Kearney, W. H. Keen, M. W. Keller, L. M. Kelly, S. H. Khamis, D. E. Kibbey, W. M. Kincaid, J. M. Kingston, Evelyn Kinney, S. C. Kleene, Erwin Kleinfeld, L. A. Knowler, F. W. Kokomoor, T. C. Koopmans, H. W. Kuhn, E. G. Kundert,

R. M. Lakness, H. G. Landau, R. E. Langer, Leo Lapidus, E. H. Larguier, H. D. Larsen, C. G. Latimer, L. M. LeCam, J. R. Lee, G. F. Leger, E. L. Lehmann, Walter Leighton, Frank Levin, D. J. Lewis, F. A. Lewis, H. M. Lieberstein, A. J. Lohwater, C. I. Lubin, J. M. McArtney, Dorothy McCoy, N. H. McCoy, R. W. MacDowell, C. C. MacDuffee, G. W. Mackey, Saunders MacLane, H. M. MacNeille, H. B. Mann, E. C. Marth, W. T. Martin, Imanuel Marx, F. J. Massey, K. O. May, J. P. Mayberry, L. E. Mehlenbacher, C. W. Mendel, B. E. Meserve, D. M. Mesner, E. J. Mickle, W. E. Milne, Hazleton Mirkil, Benjamin Ernest Mitchell, Josephine Mitchell, Don Mittleman, E. E. Moise, H. F. Montague, M. D. Montgomery, J. C. Moore, W. K. Moore, D. C. Morrow, Frederick Mosteller, T. S. Motzkin, E. J. Moulton, S. T. C. Moy, H. T. Muhly, M. E. Munroe, C. W. Munshower, W. L. Murdock, G. G. Murray, S. B. Myers, Zeev Nehari, A. L. Nelson, W. J. Nemerever, C. J. Nesbitt, Jerzy Neyman, Ivan Niven, E. A. Nordhaus, E. S. Northam, Ilse Novak, R. E. O'Donnell, E. G. Olds, Ingram Olkin, Richard Otter, J. C. Oxtoby, N. H. Painter, O. O. Pardee, Emanuel Parzen, M. H. Payne, L. L. Pennisi, F. W. Perkins, H. P. Pettit, George Piranian, Everett Pitcher, J. C. Polley, Pasquale Porcelli, J. E. Powell, G. B. Price, G. C. E. Prins, L. E. Pursell, Gustave Rabson, C. B. Rader, Tibor Radó, J. F. Randolph, G. E. Raynor, M. O. Reade, Mina Rees, Paul Reichelderfer, W. T. Reid, H. J. Reiter, R. R. Reynolds, H. B. Ribeiro, H. G. Rice, C. E. Rickart, F. A. Rickey, P. R. Rider, F. D. Rigby, J. H. Roberts, R. A. Rosenbaum, Murray Rosenblatt, P. C. Rosenbloom, Arthur Rosenthal, M. F. Rosskopf, E. H. Rothe, Herman Rubin, Herbert Ruderfer, Walter Rudin, Charles Saltzer, Hans Samelson, James Sanders, I. R. Savage, A. C. Schaeffer, R. D. Schafer, A. T. Schafer, J. A. Schatz, Henry Scheffé, E. V. Schenkman, E. R. Schneckenburger, Lowell Schoenfeld, Augusta Schurrer, C. H. W. Sedgewick, I. E. Segal, Esther Seiden, D. B. Shaffer, D. H. Shaffer, M. E. Shanks, L. S. Shapley, Seymour Sherman, R. L. Shively, Edward Silverman, M. L. Slater, M. F. Smiley, G. W. Smith, S. S. Smith, V. E. Smith, Ernst Snapper, H. H. Snyder, W. S. Snyder, E. S. Sokolnikoff, D. E. Spencer, V. E. Spencer, C. E. Springer, E. P. Starke, F. H. Steen, Arthur Stein, H. B. Stelson, C. F. Stephens, B. M. Stewart, R. W. Stokes, R. R. Stoll, M. H. Stone, D. D. Strebe, P. C. Sweetland, Gabor Szegö, Alfred Tarski, M. E. Taylor, W. C. Taylor, H. P. Thielman, W. J. Thron, F. H. Tingey, H. E. Tinnappel, E. W. Titt, C. J. Titus, John Todd, Leonard Tornheim, E. A. Trabant, C. K. Tsao, A. W. Tucker, Bryant Tuckerman, J. W. Tukey, W. T. Tutte, Eugene Usdin, E. P. Vance, N. H. Vaughan, D. F. Votaw, L. I. Wade, G. W. Walker, R. M. Walter, S. S. Walters, D. L. Webb, G. C. Webber, C. P. Wells, I. L. Wente, John Wermer, G. W. Whitehead, W. F. Whitmore, W. M. Whyburn, L. R. Wilcox, J. D. Wilkes, J. E. Wilkins, G. M. Wing, Y. K. Wong, M. A. Woodbury, Oswald Wyler, L. C. Young, Daniel Zelinsky, J. L. Zemmer, J. A. Zilber, J. W. Zimmer.

The Colloquium Lectures, on ***Arithmetical classes and types of algebraic systems***, were presented by Professor Alfred Tarski of the University of California on Tuesday afternoon and Wednesday, Thursday, and Friday mornings. Presiding officers were, in order, Professors S. C. Kleene, C. C. MacDuffee, C. B. Allendoerfer, and R. C. Buck.

The Committee to Select Hour Speakers for Annual and Summer Meetings invited one speaker. On Thursday, September 4, Professor E. E. Moise of the University of Michigan gave an address entitled

On the topology of 3-manifolds. Professor R. H. Bing presided.

The organized social events provided for the Society consisted of: an informal coffee hour on Monday evening in the Conference Room of the Physics-Mathematics Building, an informal tea on Tuesday afternoon in Phillips Lower Lounge, trips to the Oldsmobile plant in Lansing on Wednesday and Thursday, and a trip to Henry Ford Museum and Greenfield Village on Thursday. A banquet was held on Wednesday evening in the Union Building. Professor C. B. Allendoerfer, of the University of Washington, was toastmaster and the main speaker of the evening was President John A. Hannah of Michigan State College. A concert by members of the Department of Music of Michigan State College was given on Thursday evening in the Music Building, followed by a party sponsored by the Institute of Mathematical Statistics.

Activities for women included campus tours on Tuesday and Wednesday.

The Council met on Tuesday evening, September 2.

The Secretary announced the election of the following fifty-three persons to ordinary membership in the Society:

Mr. Brian Abrahamson, Graduate Student, University of Chicago;
Mr. Maurice J. Aegerter, Student, University of Wisconsin;
Mr. George Bachman, New York University;
Mr. William Aaron Beyer, Graduate Assistant, Pennsylvania State College;
Dr. Victor Wayne Bolie, Mathematician, Collins Radio Company, Cedar Rapids, Iowa;
Mr. Carmen Nicholas Campopiano, Teaching Assistant, Rutgers University;
Mr. Stanley Howard Cohn, Instructor, Fournier Institute of Technology, Lemont, Illinois;
Mr. Pierre Euclide Conner, Jr., Tulane University;
Mr. Walter Cornetz, Communications Engineer, Grumman Aircraft Engineering Corporation, Bethpage, Long Island, New York;
Dr. Charles Raymond DeCarlo, Applied Science Representative, International Business Machines Corporation, Pittsburgh, Pennsylvania;
Sister Mary Perpetual Help Dowling, Incarnate Word College, San Antonio, Texas;
Associate Professor Marcellus M. Dreiling, St. Joseph's College, Collegeville, Indiana;
Mr. Frederick Robert Eisler, Electronics Engineer, Western Electric Company, Jersey City, New Jersey;
Mr. David Israel Epstein, New York University;
Mr. Sidney Marshall Foulks, Research Physicist, Magnolia Petroleum Company, Dallas, Texas;
Mr. William Franklin Furr, Instructor, Southern University, Baton Rouge, Louisiana;
Dr. R. Wayne Gardner, Olivet Nazarene College, Kankakee, Illinois;
Mr. Herbert B. Goertzel, Mathematics Assistant, Oak Ridge National Laboratory, Oak Ridge, Tennessee;
Dr. Richard Pettengill Gosselin, Mathematician, Institute for Air Weapons Research, University of Chicago;
Mr. Edward Everett Grace, Teaching Fellow, University of North Carolina;

Assistant Professor Alfred Thomas Hind, Jr., Clemson College;

Mr. John Edward Hoffman, Graduate Assistant, University of Oklahoma;

Mr. Mark Holzman, Research Engineer, Western Geophysical Company, Los Angeles, California;

Mr. William James Hood, Graduate Student, Stanford University;

Mr. Jiro Ishihara, Digital Computer Laboratory, Massachusetts Institute of Technology;

Mr. Ted R. Jenkins, Teaching Fellow, University of Washington;

Mr. Raghunath Prasad Kanoria, Research Student, Calcutta University, Calcutta, India;

Mr. Frank Kozin, Illinois Institute of Technology;

Professor Arwid Ottomar Count Kumin-Kumins, Annhurst College, South Woodstock, Connecticut;

Mr. R. Sherman Lehman, Research Assistant, Stanford University;

Mr. Hiramie Theodore McAdams, Research Chemist, Aluminum Research Laboratories, East St. Louis, Illinois;

Mr. Paul Brainerd McKowen, Aerodynamicist, Bell Aircraft Corporation, Buffalo 5, New York;

Mr. William Harold Mays, Instructor, Pikeville College, Pikeville, Kentucky;

Mr. Norman Wilfred Moir, Jr., Assistant Instructor, Temple University;

Mr. James C. Morelock, Graduate Fellow, University of Florida;

Assistant Professor Vernon Howard Morrill, Hardin-Simmons University, Abilene, Texas;

Dr. William Nachbar, Applied Mathematician, Boeing Airplane Company, Seattle, Washington;

Mr. Louis Legendre Pennisi, Instructor, University of Illinois;

Mr. William James Perry, Instructor, Pennsylvania State College;

Mr. Philip C. Rapp, Supervisor, Analysis Section, Bell Aircraft Corporation, Buffalo 5, New York;

Assistant Professor Dolores Agnes Rickens, College of St. Elizabeth, Convent Station, New Jersey;

Mr. John Theodore Robacker, Mathematician, Northrop Aircraft Company, Los Angeles, California;

Mr. Albert S. Rosenthal, Mathematician, Naval Air Development Center, Johnsville, Pennsylvania;

Dr. William Keyser Saunders, Mathematician, National Bureau of Standards, Washington, D. C.;

Mr. Henry Sharp, Duke University;

Mr. Allen Lowell Shields, Fellow, Massachusetts Institute of Technology;

Mr. Robert Simon, Senior Project Analytical Engineer—Mathematical Consultant, Chance Vought Aircraft, Dallas, Texas;

Mr. David Slepian, Research Mathematician, Bell Telephone Laboratories, Murray Hill, New Jersey;

Mr. Bernard Sohmer, New York University;

Dr. David D. Strebe, Instructor, University of Buffalo;

Mr. Alfred L. Toth, Instructor, United States Military Academy;

Miss Ruth Ann Wagner, Mathematician, The Rand Corporation, Santa Monica, California;

Mr. Michael Yanowitch, Mathematician, Reeves Instrument Corporation, New York, New York.

It was reported that the following person had been elected as a nominee of the institution as indicated:

Kenyon College: Mr. Roger Harold Geeslin.

The Secretary announced that the following person had been admitted to the Society in accordance with a reciprocity agreement with the Société Mathématique de France: Professor Robert Faure, University of Hanoi, Tonkin, French Indo China. Palais Bella Bista, Square D'Oftende, Nice, France.

The University of California, Davis, California, DePaul University, Chicago, Illinois, and the University of New Hampshire, Durham, New Hampshire, were elected to institutional membership.

The following appointments of representatives of the Society were reported: Professor J. R. Kline at the inauguration of Lewis Webster Jones as President of Rutgers University on May 8, 1952; Professor E. G. Begle at the convocation ceremonies at the United States Military Academy celebrating the Sesquicentennial year of the Academy on May 20, 1952; Professor M. H. Martin as the Society's representative at the National Congress of Applied Mechanics in Istanbul on August 20–28, 1952.

The following additional appointments by the President were reported: as a Committee on a Summer Mathematical Institute: A. A. Albert (Chairman), Samuel Eilenberg, Saunders MacLane, D. C. Spencer, and Oscar Zariski; as a committee to study problems of nominations by petition: William Feller (Chairman), Warren Ambrose, J. W. Green, G. A. Hedlund, M. R. Hestenes, and S. B. Myers; as a committee on arrangements for the Annual Meeting to be held at Washington University on December 27–29, 1952: R. R. Middlemiss (Chairman), T. L. Downs, H. Margaret Elliott, H. M. Gehman, Francis Regan, Marlow Sholander, and J. W. T. Youngs; Professor M. H. Martin appointed a representative to the U. S. National Committee on Theoretical and Applied Mechanics for a four-year term beginning January 1, 1953; Professor R. V. Churchill re-appointed a member of the Editorial Committee on Applied Mathematics Symposium Proceedings for a three-year term beginning July 1, 1952 (Committee now consists of: Eric Reissner (Chairman), A. H. Taub, and R. V. Churchill); Professor Eric Reissner re-appointed as Chairman of the Editorial Committee on Applied Mathematics Symposium Proceedings for the period July 1, 1952 to June 30, 1953; Professors B. P. Gill and C. H. W. Sedgewick as auditors for 1952.

The following dates of meetings have been approved by the Council: April 24–25, 1953, at the University of Chicago and May 2, 1953, at Stanford University. The Council voted to accept an invitation to hold the Sixth Applied Mathematics Symposium at the Corona Laboratories of the National Bureau of Standards on August 26–28, 1953. The topic of this Symposium will be *Numerical Analysis.*

An invitation to hold the Summer Meeting for 1954 at the University of Wyoming, Laramie, Wyoming, was accepted.

Invitations to give addresses in 1952 were announced: N. Aronszajn, November meeting at Lafayette; H. H. Goldstine, November meeting at Raleigh; and A. M. Gleason, Annual Meeting at St. Louis.

The Secretary reported that the Visiting Lecturer of the Society, Professor Marshall H. Stone of the University of Chicago, had completed his program of lectures. During the academic year 1951–52, Professor Stone visited almost all parts of the country, traveling over 30,000 miles, and delivered a total of 81 lectures.

The Executive Director reported that a combined directory of the membership of the Society and of the Mathematical Association of America is under preparation. This directory will be sent to members of the Society one year and the Association the next year. The directory will be a separate publication and will not be bound with the Bulletin.

The Council voted to have Society membership cards made up. These will be issued to members of the Society on request. Requests for these cards should be sent to the Headquarters of the Society.

The Council voted to recommend to the Trustees that the Society keep available, by reprinting if necessary, volumes of the Transactions for the previous ten years, that no reprinting of back numbers of the Bulletin be done, but that an attempt be made to keep back volumes of the Bulletin for the previous ten years available by ordering sufficient copies from the printer at the time of publication. Until this time, the Society has kept available all back numbers of both the Transactions and Bulletin. This has necessitated reprinting numerous older numbers and has tied up a large part of the Society's funds.

The Council voted to approve the following resolution, similar to one approved by the Mathematical Association of America.

The Council of the American Mathematical Society has voted to instruct the Secretary to send copies of the following resolution to the Governor of the State of Oklahoma, to the presiding officers of the two houses of the Oklahoma Legislature, to the Attorney General of Oklahoma, and to the Presidents of Oklahoma A. & M. College and the University of Oklahoma.

"The Council of the American Mathematical Society has considered the effect upon mathematics and mathematicians of the Oklahoma Loyalty Oath Law of April 9, 1951 (Enrolled House Bill No. 8, 23 Legislature, Laws 1951, 510.S.1941 (Supp.) SS 37.1–37.8).

"Without presuming to censure the State of Oklahoma for this action, the Council wishes to convey to the heads of government in Oklahoma its view that many mathematicians regard this type of legislation as dangerous to individual liberty and to the principles which protect freedom of thought in a university from political infringement, and as ineffective for the purpose of eliminating disloyal influences from the universities. Even those who sign such oaths often do so more in a spirit of tolerance for the patriotic motives which conceived the law than in respect for the law itself, or more in fear than in concurrence.

"Whenever such laws are enacted, it is to be expected that at least a few of the American idealists, the sincerely religious people, and the courageous lovers of liberty will rebel, with tragic effects upon themselves, their families, and upon the universities where they work. It is not to be expected that such legislation will be effective in eliminating from the faculties men who are dangerous to the national welfare, so that the injury caused is a useless waste, as at Oklahoma A. & M. College, where a department of mathematics which had achieved much recognition for its mathematical work was seriously damaged."

A general business meeting of the Society was held on Wednesday, September 3. Professor T. H. Hildebrandt presided. The members present voted to amend the by-laws as follows:

(a) to delete the post of Librarian as specified in Section 1 of Article I, Section 1 of Article VII, and Section 6 of Article VII, and to delete Section 3 of Article I concerning the Society's exchanges.

(b) to amend Article III to create an elected Committee on Printing and Publishing, consisting of three members, and to amend Article XI by adding a Section 3 specifying the duties of this Committee.

(c) to amend Article IX of the by-laws by revising Sections 1, 2, 7, 9, and 11. These revisions eliminate fractional years of membership and also eliminate reduced dues for the early years of membership. Dues bills will now be distributed before the end of the calendar year. The Bulletin and Proceedings will be sent, as privileges of membership, only to those members who have discharged their dues for the year in question.

(d) to eliminate from the by-laws, as of December 31, 1952, certain provisions which will have become obsolete as of that time. These are the last sentence of Section 1 of Article IV, and the last three paragraphs of Section 1 of Article VII.

The by-laws of the Society, as revised, appear at the end of this number of the Bulletin.

The Secretary reported that the Council, with the approval of the Board of Trustees, has voted that any member on the rolls of the Society in 1952 and entitled to reduced dues under the dues schedule in effect at the beginning of 1952 shall continue to have his dues fixed according to the old schedule rather than the new schedule specified in Section 2 of Article IX.

Abstracts of the papers read follow. Presiding officers at the sessions for contributed papers were Professors W. T. Reid, H. T. Muhly, R. H. Bing, P. R. Halmos, L. R. Wilcox, Dr. Seymour Sherman, Professors W. M. Whyburn, Saunders MacLane, O. G. Harrold, C. E. Rickart, A. H. Copeland, G. W. Whitehead, G. S. Young, W. F. Eberlein, P. C. Rosenbloom, Fritz Herzog, B. M. Stewart, and L. M. Kelly.

Papers whose abstract numbers are followed by the letter "*t*" were read by title. Paper number 575 was presented by Professor Good, 582 by Professor Kalicki, 584 by Professor Olkin, 598 by Professor Aronszajn, 610 by Professor Dolph, 615 by Dr. Fleming, 626 by Professor Halmos, 628 by Professor Herzog, 629 by Professor Piranian, 663 by Professor Titt, 664 by Dr. Titus, 674 by Mr. Shaffer, 679 by Professor Hohn, 691 by Professor Hutcherson, 694 by Mr. Kelly, 695 by Professor Radó, 698 by Professor Motzkin, 701 by Mr. Suppes, 703 by Professor Rosenblatt, and 726 by Professor Shanks.

Dr. Blair was introduced by Professor J. W. T. Youngs, Mr. Kozin, by Professor Gordon Pall, Mr. Burrill by Professor M. F. Smiley, Mr. Hammersley by Mr. John Todd, Professor Kimball by Professor D. L. Holl, Dr. Pennisi by Professor L. M. Graves, Professor Samuelson by Dr. G. E. Forsythe, Mr. Davis by Professor R. M. Thrall, and Mr. Sharp by Professor J. H. Roberts.

ALGEBRA AND THEORY OF NUMBERS

557*t*. Frederick Bagemihl and Paul Erdős: *Intersections of prescribed cardinality*. Preliminary report.

The authors prove the following theorem, which generalizes a recently published result [F. Bagemihl, Ann. of Math. vol. 55 (1952) pp. 34–37]: Let $\mathfrak{A}$ and $\mathfrak{B}$ be sets, and to every $a \in \mathfrak{A}$ let there correspond a subset, $\Gamma(a)$, of $\mathfrak{B}$. Suppose that the following conditions are satisfied: (I) $|\mathfrak{A}| \leqq \aleph_\alpha$; (II) $|\Gamma(a)| \geqq \aleph_\alpha$ for every $a \in \mathfrak{A}$; (III) if $a \in \mathfrak{A}$, there is a cardinal number $\mathfrak{n}_a \geqq 1$ such that, if $a' \in \mathfrak{A}$ and $a' \neq a$, then $|\Gamma(a) \cap \Gamma(a')| \leqq \mathfrak{n}_a$; and the complex of cardinal numbers $\mathfrak{n}_a$ $(a \in \mathfrak{A})$ is strongly less than $\aleph_\alpha$ (i.e., every sum of fewer than $\aleph_\alpha$ terms belonging to the complex is less than $\aleph_\alpha$); (IV) there is a cardinal number $\mathfrak{m} \geqq 1$ with the following properties: (a) $\mathfrak{b}^{\mathfrak{m}} < \aleph_\alpha$ for every $\mathfrak{b} < \aleph_\alpha$; (b) if $\mathfrak{B}'$ is an arbitrary subset of $\mathfrak{B}$, with $|\mathfrak{B}'| = \mathfrak{m}$, and $\mathfrak{m}(\mathfrak{B}')$ is the number of elements a of $\mathfrak{A}$ for which $\mathfrak{B}'$ is a subset of $\Gamma(a)$, then the complex of cardinal numbers $\mathfrak{m}(\mathfrak{B}')$, obtained by letting $\mathfrak{B}'$ run through all the subsets of $\mathfrak{B}$ having $\mathfrak{m}$ elements, is strongly less than $\aleph_\alpha$. With every $a \in \mathfrak{A}$, associate a cardinal number $\mathfrak{a}_a$ such that (V) $\mathfrak{m} + \mathfrak{n}_a - 1 \leqq \mathfrak{a}_a \leqq \aleph_\alpha$. Then there exists a subset $\mathfrak{B}^*$ of $\mathfrak{B}$, with $|\mathfrak{B}^*|\leqq\aleph_0$ such that $|\mathfrak{B}^*\cap\Gamma(a)|=\mathfrak{a}_a$ for every $a\in\mathfrak{A}$. An illustration of this theorem is the following: With every straight line l in the plane, associate a cardinal number $\mathfrak{a}_l$ $(2\leqq\mathfrak{a}_l\leqq\aleph)$. Then there exists a point set which is intersected by every line l in precisely $\mathfrak{a}_l$ points. (Received May 28, 1952.)

558*t*. H. W. Becker: *Dual Williams vectors (Pythagorean tetrahedrons).*

In the notation of previous abstracts (Bull. Amer. Math. Soc. (1952)) if u, v, x, y, s, t are each of the form $\sum_s q_m r^{n-2m} s^{2m+p}$, $p=0$ or 1, then their dual forms $\sum_s (-)^m q_m r^{n-2m} s^{2m+p}$ also define W.V. under some permutation of the six components, such as $(tux)(vsy)$, or $(ty)(uv)(xs)$. When not self-dual, these W.V. have properties surprisingly different from their duals. A family of W.V. solutions consists in all those derivable from a given parametric solution under $r, s\rightarrow f_r(r, s), f_s(r, s)$, and their transforms $u, v, x, y, s, t\rightarrow ty, ux, vy, xy, xs, uv$, etc. There are six families whose simplest examples are among the 17 sub-miniature W.V. now known, with $t<10^4$. Four families are Carmichael vectors: the Fermat I and II, Dickson's *History* II, p. 175 and repeatedly thereafter; the Hillyer triad, p. 174 (under $r, s\rightarrow r\pm s$ the Euler triad, p. 474); and the Rolle, p. 447 (using first powers instead of squares of the two parameters) or equivalently, Fermat, p. 172. The other two are the Euler, p. 668 (ii); and Lenhart, pp. 506–507. The Hillyer family is self-dual, in that its members are either self-dual or duals of each other. The Rolle and Euler families are duals, when not self-dual. Dualities exist between members of any pair of the Fermat I and II and Lenhart families. So, new discoveries may be expected as twins. (Received July 21, 1952).

559*t*. H. W. Becker: *The tetradization transform on Carmichael vectors.*

A C.V. depends on solution of $y=4(abcd(a^2-b^2)(c^2-d^2))^{1/2}$, $t=(a^2+b^2)(c^2+d^2)$ in integers. A W.V. depends on integer $s=2(i\phi k\lambda)^{1/2}$, $x=((i^2-\phi^2)(k^2-\lambda^2))^{1/2}$. The Petrus transform of a W.V. is a C.V., with T, $U=t^2\pm y^2$, $X=v^2-u^2$, $V=2uv$, $Y=2xs$, $Z=2ty$, $T=(A^2+B^2)(C^2+D^2)$, $A=i$, $B=\phi$, $C=k$, $D=\lambda$. It follows that if (a, b, c, d) is a C.V., then intermediate between it and its Petrus transform (A, B, C, D) there is a tetrad of C.V.: (a, b, A, B), (a, b, C, D), (c, d, A, B), and (c, d, C, D). Numbering the respective t's and y's accordingly, then $tT=t_1t_4=t_2t_3$, $yY=y_1y_4=y_2y_3$. In the Rolle C.V., $a=A=r$, $b=B=s$, $c=(r^2+s^2)^2$, $d=4rs(r^2-s^2)$, $C=(r^4+6r^2s^2-s^4)^2$, $D=s(3r^4-6r^2s^2-s^4)^2$, so actually only (c, d, C, D) is interposed. With the Hillyer C.V., only two *new* C.V. are introduced, the other two being members of the triad. But with Fermat I and II C.V., the tetrad is entirely new and distinct, rounding up into formulation many specimens hitherto mavericks. Miniature C.V. are those with $t<10^8$; 43 examples are tabulated. A complete census depends on, and contributes to, the classification of Pythagorean triangles according to area/□. Apparently the most populous class is that of 1254, including the Pythagorean couples 11, 8; 19, 8; 22, 3; 38, 11; 147, 128; 486, 475; 507, 32; 1350, 19; 1862, 1859; · · · besides 185^2, $1254\cdot 4^2$ etc. on substituting these couples into the Rolle c, d and C, D above. (Received July 21, 1952.)

560. D. W. Blackett: *Simple and semisimple near-rings.*

A "near-ring" is an algebraic system which satisfies all the ring axioms except possibly one of the distributive laws (say the right one) and the commutative law of addition. The right ideals of a ring are both the right modules of the ring and the kernels of right homomorphisms of the ring. For a near-ring the right modules are not identical with the kernels of right homomorphisms. These kernels will be called the right ideals of the near-ring. For those near-rings in which the additive identity annihilates the near-ring from the left, every right ideal is also a right module. A near-ring is called semisimple if the additive identity annihilates the near-ring from the left, the near-ring contains no nonzero nilpotent right modules, and the right modules satisfy the descending chain condition. Every semisimple near-ring has only a finite number of nonisomorphic types of irreducible right representation spaces. If there is only one such type, the semisimple near-ring is called simple. Every semisimple near-ring may be represented uniquely as a direct sum of simple sub-near-rings. Each of these sub-near-rings is the sum of the right modules which are irreducible spaces of a particular type. (Received June 25, 1952.)

561. R. L. Blair: *Stone's topology for a binary relation.*

Let $\mathfrak{R}^*$ be the transitive relation generated by a binary relation $\mathfrak{R}$ on a set L. If $T\subseteq L$, let $T^- = [x\in L; x\mathfrak{R}^*T]$ and $T^+ = [x\in L; T\mathfrak{R}^*x]$. An element $x\in L$ is *s-irreducible* (relative to $\mathfrak{R}$) in case $L\mathfrak{R}^*x$ does not hold, and, for every two-element subset $[a, b]$ of L, $[a, b]^-\subseteq[x]^-$ implies $a\mathfrak{R}^*x$ or $b\mathfrak{R}^*x$. A Stone topology $\gamma(\mathfrak{R})$ is introduced in the set S of all s-irreducible elements of L by defining the closure of $S_1\subseteq S$ to be $S\cap S_1^{-+}$ (*cf.* M. H. Stone, Trans. Amer. Math. Soc. vol. 41 (1937) pp. 375–481). When L is the set of ideals of a ring and $\mathfrak{R}$ denotes set inclusion, $\gamma(\mathfrak{R})$ contains the spaces of prime (McCoy, Amer. J. Math. vol. 71 (1949) pp. 823–833) and primitive (Jacobson, Proc. Nat. Acad. Sci. U.S.A. vol. 31 (1945) pp. 333–338) ideals as, in general, proper subspaces. Additional applications are made to lattices, including lattices of continuous functions. The preceding observations motivate a study of lattices each of whose elements is a meet of meet-irreducible elements. In this connection the following result is proved: If every maximal dual ideal of a complete complemented modular lattice L is completely prime, then L is atomic. This, incidentally, yields an answer to a question raised by Birkhoff and Frink (Trans. Amer. Math. Soc. vol. 64 (1948) pp. 299–316) concerning a type of representation of a lattice by sets. (Received July 21, 1952.)

562*t*. Volodymyr Bohun-Chudyniv: *On orthogonal and non-orthogonal closed systems of K-nions and their applications.*

The aims of this paper are: (1) To define the notion of a closed system of K-nions of 2^{k+2} rank and Kth degree composed of Hamilton's systems of quaternions $[1, i_\alpha, i_\beta, i_\gamma]$ and demonstrate a method for determining such systems on the basis of $n=2^{k+2}-1$ imaginary units. (2) To define the notion of the volume of a closed system and to demonstrate a method for determining its dimension. (3) To define the notion of a closed manifold of systems of 2^{k+2} rank and Kth degree; a method is given for determining its value. (4) To indicate the criteria of the classification of such systems. (5) To give a generalized expression of Euler's formula for square polynomials for an arbitrary $n>4$ (Euler has given a formula for $n=4$). (6) To give schemes and the method for determining, with the help of these systems, square matrices composed of integers (2^{2k+4}) and $(2^{k+2}-1)^2$. These matrices satisfy all conditions of the Euler problem. L. Euler (Novi Comm. Acad. Petrop. vol. 15 (1770) p. 75; Comm. Arith. vol. I, pp. 427–443) gave only the schemes for $k=0$. Bibliography: (I) Hamilton, *Lectures on Quaternions*, Dublin, 1853; (II) H. Grassmann, *Die lineale Ausdehnungslehre*, Leipzig, 1844; (III) Frobenius, Grelli Journ. vol. 84 (1878) p. 59; (IV) Cayley,

J. Reine Angew. Math. vol. 50, pp. 192, 202; (V) D. A. Grave, Akad. Nauk S.S.S.R. (1938), Trak po Alg. Anal., I, str. 48–57; (VI) V. Bohun-Chudyniv, *On a method and general scheme for the solution of the Euler problem* (Bull. Amer. Math. Soc. Abstract 58-3-269). (Received July 17, 1952.)

563. A. T. Brauer: *On the least primitive root* (mod p).

J. Vinogradoff [Journ. Soc. Phys. Math. de Perm. vol. 1 (1918) pp. 94–98] proved the following theorem by analytic methods. Let p be a prime, g the smallest positive primitive root mod p, and k the number of different prime divisors of $p-1$, then $g<\{2^k(p-1)p^{1/2}\log p\}/\phi(p-1)$. In this paper the following much larger bound is obtained, but by purely elementary methods using a theorem of A. Brauer and T. L. Reynolds [Canadian Journal of Mathematics vol. 3 (1951) pp. 367–374]. If p is a prime of form $4n+1$ and $r=2^k$, then $g<p^{(r-1)/r}$. The same bound holds for the absolute value of the absolute smallest primitive root of primes of form $4n+3$. (Received September 2, 1952.)

564. J. L. Brenner: *The matrix equation* $AX=B$.

Necessary and sufficient conditions are given for the solvability of the matrix equation $AX=B$. Here A, B are given finite matrices; the elements of A, B, X come from a principal ideal ring R, or a ring S of slightly more general nature. The method is to find a complete set of invariants of $\{AX\}$ when X runs through the set of all matrices which have two-sided inverses. This amounts to finding a canonical form for A. If the canonical form for B can be obtained from the canonical form for A by replacing some columns of the latter by zero columns, then $AX=B$ is solvable. (Received July 17, 1952.)

565. H. E. Campbell: *A converse of Cartan's criterion for solvability.*

Let R_x be the transformation $a\rightarrow ax$ in a Lie algebra $\mathfrak{L}$ over a field of characteristic zero. Then Cartan's criterion for solvability is that if the trace $t(R_x^2)=0$ for all x of $\mathfrak{L}$, then $\mathfrak{L}$ is solvable. A construction is given to prove that if $\mathfrak{N}$ is solvable over a field F of characteristic zero, then $\mathfrak{N}$ can be imbedded in a Lie algebra $\overline{\mathfrak{N}}$ over a field $\overline{F}\supseteq F$ such that $t(R_xR_y)=0$ for all x, y of $\overline{\mathfrak{N}}$. More generally if $\mathfrak{L}=\mathfrak{S}+\mathfrak{N}$ where $\mathfrak{S}$ is semi-simple and $\mathfrak{N}$ is solvable, then $\mathfrak{L}$ can be imbedded in $\overline{\mathfrak{L}}=\mathfrak{S}+\overline{\mathfrak{N}}$ over a field $\overline{F}\supseteq F$ such that $t(R_xR_y)=0$ for all x of $\overline{\mathfrak{N}}$ and all y of $\overline{\mathfrak{L}}$. Also any faithful representation $x\rightarrow S_x$ of $\mathfrak{L}=\mathfrak{S}+\mathfrak{N}$ by matrices can be extended to a faithful representation $x\rightarrow\overline{S}_x$ over a field $\overline{F}\supseteq F$ such that $t(\overline{S}_x\overline{S}_y)=0$ for all x in $\mathfrak{N}$ and all y in $\mathfrak{L}$. (Received July 17, 1952.)

566t. Leonard Carlitz: *A note on the multiplication formulas for the Bernoulli and Euler polynomials.*

As Nielsen has pointed out, if a normalized polynomial of degree m satisfies the multiplication formula for the Bernoulli polynomial for a single value >1 of the multiplier, then it is identical with $B_m(x)$; a similar result holds for $E_m(x)$. In the case of the mixed equation the situation is somewhat different. We show that if normalized polynomials satisfy $g_{m-1}(kx)=-(2k^{m-1}/m)\sum_{s=0}^{k-1}(-1)^s f_m(x+s/k)$ for two distinct even values of k, then $f_m(x)=B_m(x)+a_m$, $g_{m-1}(x)=E_{m-1}(x)$, where a_m is an arbitrary constant. A like result holds in the case of the Bernoulli and Euler polynomials of higher order. More generally a result of this sort holds for the Eulerian polynomials

$\phi_m(x, \alpha)$ defined by $((1-\alpha)/(1-\alpha e^t))e^{xt} = \sum_{m=0}^{\infty} (\phi_m(x, \alpha)/m!)t^m$, where $\alpha \neq 1$ is a root of unity. (Received July 21, 1952.)

567*t*. Leonard Carlitz: *Congruences connected with the power series expansions of the Jacobi elliptic functions.*

Extending the results of a previous paper (Duke Math. J. vol. 16 (1949) pp. 297–302), it is shown first that $A^m(u)(A^{p-1}(u)-1)^r \equiv 0 \pmod{(p^m, p^r)}$, where u is an indeterminate, p is a prime >2, and sn x = sn $(x, u) = \sum_{m=1}^{\infty} A_m(u)x^m/m!$. Similar results are also obtained for powers of sn x. Next if $A_m(u) = \sum a_{mi}u^i$, then it is proved that $a_{m+t,i} \equiv (-1)^{t/2}a_{mi} \pmod{p^m, p^{r-j}}$, where $p^{r-1}(p-1) \mid t$, $p^{j-1} \leq i < p^j$. If β_{mi} denotes the general coefficient in the expansion of $x/\text{sn}\, x$, then $\beta_{m+t,i}/(m+t) \equiv (-1)^{t/2}(\beta_{mi}/m) \pmod{(p^{m-1}, p^{r-j})}$, provided $p-1 \nmid m$; a similar result is also obtained when $p-1 \mid m$. (Received June 16, 1952.)

568*t*. Leonard Carlitz: *Generalised Dedekind sums.*

Put $c_r(h, k) = \sum_{\mu=1}^{k} \overline{B}_{p+1-r}(\mu/k)\overline{B}_r(h\mu/k)$, where $\overline{B}_r(x)$ denotes the Bernoulli function, p is a fixed odd integer >1, and $(h, k) = 1$. T. M. Apostol has proved a transformation formula for the function $G_p(x) = \sum_1^{\infty} n^{-p}x^n/(1-x^n)$ (Duke Math. J. vol. 17 (1950) pp. 147–157). Using this result we prove a reciprocity formula involving $c_r(h, k)$ which includes Apostol's reciprocity theorem for $c_p(h, k)$. We also generalize some theorems on Dedekind sums proved by Rademacher and Whiteman (Amer. J. Math. vol. 63 (1941) pp. 377–407). (Received July 21, 1952.)

569*t*. Leonard Carlitz: *Invariantive theory of equations in a finite field.*

Consider the reversible transformation T: $\eta_i = f_i(\zeta_1, \cdots, \zeta_r)$ $(i = 1, \cdots, r)$, where $f_i(x_1, \cdots, x_r) \in GF[q, x_1, \cdots, x_r]$, and ζ_i, $\eta_i \in GF(q)$. Two polynomials g, $h \in GF[q, x_1, \cdots, x_r]$ are *equivalent* provided $Tg = h$ for some T; thus the totality of polynomials is separated into a number of *classes*. The determination of this number is a simple combinatorial problem. Applications are made to the equation $f(\zeta_1, \cdots, \zeta_r) = \alpha$. (Received July 21, 1952.)

570*t*. Leonard Carlitz: *Some sums connected with quadratic residues.*

The well known result $\sum_1^{(p-1)/2} (r/p) > 0$ is generalized in the form $(-1)^k \sum_{r=1}^{(p-1)/2} (r/p)E_{2k}(2r/p) > 0$ $(p \equiv 3 \pmod 4)$, $(-1)^k \sum_{r=1}^{(p-1)/2} (r/p)E_{2k}(2r/p) > 0$ $(p \equiv 1 \pmod 4)$, where $E_m(x)$ denotes the Euler polynomial. Similar results involving the Bernoulli polynomials also hold. (Received June 16, 1952.)

571. Harvey Cohn: *Stable lattices*. II.

Continuing in the terminology of the previous abstract (Bull. Amer. Math. Soc. Abstract 57-3-203), the author considers lattices representing all integers in a totally real field K, for which the norm function, $\phi = |x_1x_2 \cdots x_n|$, has a *stable* minimum (under infinitesimal variations of the coefficients a_{ij}). Here $x_i = \sum a_{ij}m_j$ and a_{ij} is the i-conjugate of the j-basis element. The condition for stability is that for any two conjugates, i and I, a unit u exist for which u_i/u_I is negative. When K is quadratic this just means there exists a unit of negative norm. When $K = R(\cos 2\pi/N)$, the condition is satisfied for an extensive class of N including (at least) prime powers. (Research sponsored by Army Ardnance.) (Received July 17, 1952.)

572. H. S. M. Coxeter: *Hyperbolic honeycombs and indefinite quadratic forms.* Preliminary report.

Many indefinite forms $f(x, y, z, \cdots)$ are found to yield symmetrical patterns when the variables are interpreted as homogeneous coordinates in a hyperbolic space whose absolute quadric has the equation $f=0$. The symmetry operations of the pattern represent automorphs of the form. For instance, when $f=x^2+y^2-z^2-xz-yz$ [Markoff, Math. Ann. vol. 56 (1903) pp. 233–251], the solutions of the Diophantine equation $f=-1$ are the vertices of a tessellation of squares and hexagons filling the hyperbolic plane. Again, when $f=x^2+y^2+z^2-t^2-xt-yt-zt$ [Dickson, *Studies in the theory of numbers*, p. 143], the integral solutions of $f=-1$ are the vertices of a honeycomb of cubes and truncated octahedra filling hyperbolic 3-space. (Received June 19, 1952.)

573. D. W. Dubois: *Structure of Archimedean partly ordered fields.*

A field F (commutative) is called a *partly ordered field* if F has a nonempty subset P of positive elements which is closed under addition, multiplication, and division. "$x>y$" and "$y<x$" are abbreviations for "$x-y\in P$." The partial order is *Archimedean* if for every $y\in F$, and $x\in P$, there exists an integer n with $nx>y$. The following structure theorem is proved: *If F is Archimedian partly ordered, then there is a family $\mathfrak{S}$ of simple Archimedean orderings of F such that the positive elements of F are those which are positive according to every ordering belonging to $\mathfrak{S}$.* This structure is revealed by showing that a certain Banach algebra A is a subdirect sum of fields isomorphic with the field R of all real numbers, where A is the completion of F relative to the norm $p(x)=\inf\{\gamma>0; -\gamma<x<\gamma\}$, γ being a rational (every partly ordered field contains the ordered field of rationals). (Received July 11, 1952.)

574t. Oscar Goldman: *Ordered abelian groups.* Preliminary report.

If G is a totally ordered abelian group, denote by G_+ its set of non-negative elements. Let L be the set of all complex-valued functions f defined on G_+ for which $\|f\|=\sum_{t\in G_+}|f(t)|<\infty$. Then L forms a commutative Banach algebra under $\|f\|$ as norm, and convolution as multiplication; we call L the group algebra of the *ordered* group G. The algebraic structure of L reflects some of the order properties of G, as for example: L has no zero divisors if, and only if, the ordering of G is Archimedean. The space of maximal ideals M of L, with the usual topology, has some interest. M is compact, and, if one assumes the ascending chain condition on isolated subgroups of G, M is also connected. (The converse of the latter statement has not yet been settled.) The elements of M are in one-to-one correspondence with the *order-characters* of G, i.e., the complex-valued multiplicative functions on G_+ of modulus not exceeding one, and assuming the value 1 at the zero element of G. Because of this, M has a sort of analytic structure, and L is an algebra of analytic functions. (Received July 21, 1952.)

575. R. A. Good and D. R. Hughes: *Associated groups for a semigroup.*

Let S be a semigroup, B a subsemigroup of S satisfying $BSB\subseteq B$; then B is called a bi-ideal of S. An ordered sequence (finite or denumerably infinite) of subsemigroups of S, each of which is a bi-ideal of the preceding semigroup, is called a bi-ideal chain of S. If and only if S possesses a Kerngruppe in the sense of Suschkewitsch it possesses a finite bi-ideal chain with a member B_n such that B_n possesses no proper bi-ideals and is thus a group. B_n is a bi-ideal of S and is called an associated group for S. The index of an associated group is defined; all associated groups for S have the same index and are isomorphic to one another. Every group homomorph of S is a homomorph of the (abstract) associated group for S; under certain conditions the associated group for S is a homomorph of S, and is thus a maximal group homomorph of S. (Received July 21, 1952.)

576. Franklin Haimo: *Some fixpoint properties of normal automorphisms*. Preliminary report.

Let G be a group with center Z. Let $B_1(G)$ be the set of all elements of G which are the common fixpoints for all the central automorphisms of G. If $B_k(G)$ is defined, construct $B_{k+1}(G)$ by $B_{k+1}(G)/B_k(G)=B_1(G/B_k(G))$. A typical result is the following: For a central automorphism α of G, let T_α be the set of all $x^{-1}\alpha(x)$, where x runs through G. Then $T_\alpha \subset Z$. Suppose that α restricted to T_α is the automorphism $x \rightarrow x^{-1}$, and suppose that there exists an integer $n \geqq 2$ such that $G=B_n(G)$. Then α^2 is the identity automorphism on G. (Received July 21, 1952.)

577t. Melvin Henriksen and J. R. Isbell: *On the continuity of the real roots of an algebraic equation.*

While the root of an algebraic equation is a continuous multiple-valued function of its coefficients, there is no general theory of decomposition into single-valued functions. This paper shows that when a polynomial $P(x, w)=w^n+\sum_{k=0}^{n-1} a_k(x)w^k$ has coefficients which are continuous functions on a Hausdorff space X, and for each $x \in X$ there is at least one real root of $P(x, w)=0$, there is an open set U on which a real root can be given by a continuous function. In case the number of real roots is constant over X, one of them can be given by a function continuous on X. With these results one can prove for normal spaces the following theorem, stated by Hewitt [*Rings of real-valued continuous functions*. I, Trans. Amer. Math. Soc. vol. 64 (1948) p. 76]: Let $C(X, R)$ be the ring of all continuous real-valued functions on a completely regular space X; let M be a maximal ideal in $C(X, R)$. Then $C(X, R)/M$ is a real closed field. Hewitt's proof relies on the erroneous assumption that the least real root of a real polynomial is a continuous function of its coefficients [see Mathematical Reviews vol. 10 (1949) p. 126]. Since the present proof relies on the Tietze extension theorem, the question for completely regular spaces is not settled. (Received June 30, 1952.)

578. I. N. Herstein: *A generalization of a theorem of Jacobson*. III.

In the paper *A generalization of a theorem of Jacobson* [Amer. J. Math. (1951) pp. 756–763] it was proved that if R is a ring with center Z such that $x^n-x \in Z$ for all $x \in R$ where n is a fixed integer larger than 1, then R is commutative. In the second paper in this series this result was extended to the case where $n=n(x)$ might vary but where $n(x) \leqq M$ for all $x \in R$. In this paper the following unrestricted theorem is proved: Let R be such that $x^{n(x)}-x \in Z$ for all $x \in R$, $n(x)$ depending on x. Then R is commutative. The result is first established for the division ring case using a recent field theoretic theorem due to Marc Krasner (forthcoming). By the Jacobson structure theorems the semi-simple case is disposed of. The situation then hinges on the subdirectly irreducible case. The key result here is that all commutators are in the center. From there, by a series of theorems, the problem is reduced to the situation where $n(x)$ is constant. The theorem is known to be true here, settling the general situation completely. (Received July 17, 1952.)

579. D. G. Higman: *Focal series in finite groups.*

For S a subgroup of the [finite] group G, we define a descending chain of normal subgroups ${}_iS$ of S, which we term the *focal series* of S in G, by the recursive formulas: ${}_0S=S$, ${}_{i+1}S=$ the subgroup generated by all commutators $c=[s, g]$ with c and s in S, g in G. Let $\mathfrak{p}$ be a set of primes, and denote by $P(\mathfrak{p})$ the subgroup generated by all those elements in G whose orders have no prime divisors in $\mathfrak{p}$. The central result of our investigation is: *if for some i, $\mathfrak{p}$ contains all the prime divisors of the index $[S: {}_iS]$, then every prime which divides $[P(\mathfrak{p})\wedge S:P(\mathfrak{p})\wedge {}_iS]$ also divides $[P(\mathfrak{p}):P(\mathfrak{p})\wedge S]$.* Hence, for instance, if $[G:S]$ and $[S:{}_iS]$ are relatively prime, there exists a normal subgroup N of G such that $G=NS$ and ${}_iS=N\wedge S$. (Received July 16, 1952.)

580*t*. J. R. Jackson: *A partial ordering defined by certain matrices.*

Let (p_{mn}) be an $M\times N$ matrix with real entries. Write $m\prec^* m'$ if, for some n $p_{mn}<0<p_{m'n}$; and let "$\prec$" be the extension of "$\prec^*$" to be transitive. It is shown that the relation "$\prec$" is a partial ordering of the integers $1, 2, \cdots, M$, if and only if the following condition is satisfied: (p_{mn}) can be replaced, by means of interchanges of rows and interchanges of columns, by a matrix (q_{mn}) with the property that if $q_{\mu\nu}>0$, then whenever $m\leqq\mu$ and $n\leqq\nu$, we have $q_{mn}\geqq 0$. The result is of interest from the point of view of a mathematical description of industrial production scheduling. (Received July 17, 1952.)

581*t*. R. E. Johnson: *Representations of prime rings.*

This paper continues the study of prime rings started in an earlier work (*Prime rings*, Duke Math. J. vol. 18 (1951) pp. 799–809). It is assumed that the prime ring R has a set $\mathfrak{R}$ of r-ideals and a set $\mathfrak{L}$ of l-ideals satisfying the seven closure assumptions of the earlier paper. If we select atoms M from $\mathfrak{L}$ and N from $\mathfrak{R}$ such that $M\cdot N\neq 0$, then the ring $K=M\cap N$ is an integral domain, and M is an (R, K)-module while N is a (K, R)-module. It is shown that the lattice $\mathfrak{R}$ is isomorphic to a lattice of K-submodules of M, and dually for $\mathfrak{L}$ and N. Finally, it is shown that for any set $\{x_1, \cdots, x_n\}$ of K-linearly independent elements of N, and any other set $\{y_1, \cdots, y_n\}$ of elements of N, there exist an a in R and a nonzero k in K such that $x_i a=ky_i$, $i=1, \cdots, n$. (Received July 22, 1952.)

582. Jan Kalicki and D. S. Scott: *Equational completeness of abstract algebras.* Preliminary report.

Terminology will be that of Jónsson and Tarski, *Boolean algebras with operators*, Part I (Amer. J. Math. vol. 73 (1951) in particular pp. 894–897). Let Eq (K) be the set of all equations $f=g$ between algebraic functions f and g over a similarity class K. Let Id $(\mathfrak{A}_K)$ be the set of equations $f=g$ in Eq (K) such that $f\mathfrak{A}_K=g\mathfrak{A}_K$ holds in the algebra $\mathfrak{A}_K$ of the similarity class K. $\mathfrak{A}_K$ is *equationally complete* $(\mathfrak{A}_K\in EC)$ if (1) Id $(\mathfrak{A}_K)\neq$ Eq (K) and (2) for any algebra $\mathfrak{B}_K$ if Id$(\mathfrak{A}_K)\subset$Id $(\mathfrak{B}_K)$, then Id $(\mathfrak{B}_K)$ $=$Eq (K). It has been shown that: (a) If Id $(\mathfrak{A}_K)\neq$Eq (K), then there is an algebra $\mathfrak{B}_K\in EC$ such that Id $(\mathfrak{A}_K)\subseteq$Id $(\mathfrak{B}_K)$. (b) An algebra $\mathfrak{A}=\langle A, f\rangle$ with one associative binary operation f is equationally complete if and only if one of the following conditions holds for any $w, x, y, z\in A$: (1) $f(x, y)=x$; (2) $f(x, y)=y$; (3) $f(x, y)=f(w, z)$; (4) $f(x, f(y, z))=f(f(x, y), z)$, $f(x, y)=f(y, x)$, and $f(x, x)=x$; (5) $f(x, f(y, z))=f(f(x, y), z)$, $f(x, y)=f(y, x)$, $f(x^p, y)=y$ where $x^1=x$, $x^{i+1}=f(x^i, x)$, and p is a prime. (c) As a simple corollary to (b) it is seen that every equationally complete algebra with one associative binary operation has a finite equational basis. (Received July 21, 1952.)

583*t*. Irving Kaplansky: *Dual modules over a valuation ring.*

Several authors (Prüfer, Krull, Vilenkin, and others) have studied linearly compact modules over the p-adic integers, that is, topological modules with the finite intersection property for cosets of closed submodules. The author shows that these are dual to discrete modules; the duality must be taken, not relative to the reals mod one, but to its primary subgroup (Prüfer's group of type p^∞). Thus the theory of linearly compact modules coincides with that of ordinary modules. This is the starting point for a more general study of such duality. (Received June 12, 1952.)

584. Leo Katz and Ingram Olkin: *Properties and factorizations of matrices defined by the operation of pseudo-transposition.*

A matrix C, $n\times n$, is (p, q) pseudo-orthogonal if it satisfies the relation $CJC'=J$, where $J=I_p+(-I_q)$, $+$ is the direct sum, and $p+q=n$. Under a pseudo-orthogonal transformation, the quadratic form $\sum_1^p x_i^2-\sum_1^q y_i^2$ is invariant. Lee (Quarterly Journal of Mathematics, 1944) and Hsu (Quarterly Journal of Mathematics, 1946) have obtained factorizations of such matrices. By defining the operation of pseudo-transposition ($C^*=JC'J$ is the pseudo-transpose of C), unified definitions, analogous to the definitions using ordinary transposition, are obtained for pseudo-symmetric, pseudo-skew, and pseudo-orthogonal matrices. Analogues of certain theorems involving transposition are proved. These include a new factorization of a pseudo-orthogonal matrix in terms of a pseudo-skew matrix and the analogue of the Toeplitz factorization. The matrices considered are real. (Received July 21, 1952.)

585*t*. Frank Kozin: *On functions of three vectors.*

Let $f(x, y, z)$ be a real function of three vectors belonging to vector spaces of dimensions >0. If f is subadditive in x and y, and superadditive in x and z, then f is additive in x and y as well as in x and z. Let $\|a_{ik}\|$ be a square matrix of order n. If $x=\{a_{11}\}$, $y=\{a_{12}, \cdots, a_{1n}\}$, $z=\{a_{21}, \cdots, a_{n1}\}$, and $f(x, y, z)$ denotes the determinant $|a_{ik}|$ where a_{ik} $(i, k=2, \cdots, n)$ are kept constant, then the preceding theorem yields the sufficiency of Menger's postulates for determinants (C.R. Acad Sci. Paris vol. 234 (1952) pp. 1941–1943). (Received July 18, 1952.)

586*t*. W. J. LeVeque: *Note on S-numbers.*

The transcendental number ξ is said to be an S-number if there are constants Γ and γ such that the inequality $|x_0+x_1\xi+\cdots+x_n\xi^n|>\Gamma(\max(|x_0|, \cdots, |x_n|))^{-\gamma n}$ holds for all n and all sets of rational integers $(x_0, \cdots, x_n)\neq(0, \cdots, 0)$. Mahler showed that almost all real numbers (complex numbers) are S-numbers, and conjectured that this is even true with $\gamma=1$ $(\gamma=1/2)$. It is shown here that almost all real numbers (complex numbers) are S-numbers for which it is possible to take $\gamma=2$ $(\gamma=3/2)$; this improves a result due to Koksma, that it is possible to take $(\gamma=3)$ $(\gamma=5/2)$. (Received June 20, 1952.)

587*t*. W. J. LeVeque: *Continued fractions and approximations in $k(i)$.*

The results of the above paper are applied to two problems. The trick is to find an algorithm for producing the best Gaussian rational approximations to an arbitrary complex number ξ; a regular continued fraction is constructed, all of whose convergents are best approximations to ξ, and among whose convergents are to be found all solutions of the inequality $|\beta(\beta\xi-\alpha)|<0.48\cdots$ in Gaussian integers α and β.

The second question, how well most complex numbers can be approximated by Gaussian rationals, is answered by the following theorem: the inequality $|\beta(\beta\xi-\alpha)| < f(|\beta|)$ has only finitely many solutions α, β for almost all ξ if the integral $\int^{\infty} tf^2(t)dt$ converges, and has infinitely many solutions for almost all ξ if this integral diverges. Here f is a positive function satisfying suitable monotonicity conditions. Metric results are also obtained concerning the order of magnitude of the norms of the denominators of the continued fraction mentioned above. (Received June 20, 1952.)

588t. W. J. LeVeque: *Geometric properties of Farey sections in $k(i)$.*

The work of Cassels, Ledermann, and Mahler on complex Farey sections is extended in several directions, the principal new tool being a simple geometric interpretation of the boundaries of the regions associated with Farey elements, in terms of spheres tangent to the complex plane at the Farey points. In particular, the theorem proved by Cassels et al., that these regions are star domains about the Farey points, is here proved much more simply. (Received June 20, 1952.)

589t. W. J. LeVeque: *On n-dimensional uniform distribution modulo* 1.

Koksma has considered the distribution modulo 1 of sequences $\{f(n, \theta)\}$ depending on a real parameter θ, and has shown that under suitable restrictions such a sequence is uniformly distributed modulo 1 for almost all θ in an appropriate interval. The present author gave a different proof of certain of Koksma's results, basing the proof on properties of quasi-orthogonal functions pointed out by Kac, Salem, and Zygmund. This method is here applied to m-fold sequences $\{f_1(n_1, \cdots, n_r, \theta_1, \cdots, \theta_s), \cdots, f_m(n_1, \cdots, n_r, \theta_1, \cdots, \theta_s)\}$ depending on r integral variables and s continuous variables. Analogous but more complicated theorems are proved. (Received June 20, 1952.)

590t. Jakob Levitzki: *On the structure of P-soluble rings.*

A ring is called a P-ring if it has no nilpotent ($\neq 0$) elements modulo its maximal nil ideal. The sum of all right P-ideals of a ring S is a two-sided ideal called the P-socle of S. Iterating transfinitely the process of forming the P-socle one obtains an ascending chain of ideals $\{A_\sigma\}$ culminating, say, at A_τ. If $S = A_\tau$, then S is called P-soluble. In the present paper P-soluble I-rings (i.e. rings with nonzero idempotents in every non-nil right ideal) are studied, and e.g., a necessary and sufficient condition for the P-solubility of an FI-ring (i.e., a ring whose homomorphic images are I-rings) is found. This condition can be formulated as a weak minimum condition imposed on a certain class of principal right ideals. It turns out that an FI-ring is P-soluble if and only if every primitive image is P-soluble. Special cases are FI-rings with bounded index and more generally such with a bounded index modulo every primitive ideal, which were studied in the author's paper *On the structure of algebraic algebras and related rings* (to appear in Trans. Amer. Math. Soc.). These results are applied to the problem of the local finiteness of algebraic algebras and yield a generalization of Kaplansky's results on the problem of Kurosch. (Received August 12, 1952.)

591t. J. E. McLaughlin: *The normal completion of complemented modular lattices.* Preliminary report.

Let L be a complemented modular point lattice, L_ι its ideal completion, L_ν its normal completion. Then it is shown that L_ν is modular if and only if L_ν is isomorphic to a sublattice of L_ι. It is also shown that if L is orthocomplemented, then

L_v is orthocomplemented. This gives as a corollary the following partial extension of the Stone-Glivenko Theorem for Boolean algebras: the normal completion of an orthocomplemented modular point lattice with the finite dependence property on points is an orthocomplemented modular point lattice. An example is given of an orthocomplemented modular point lattice whose normal completion is not modular. (Received July 21, 1952.)

592. E. V. Schenkman: *A generalisation of the central elements of a group.*

For elements a and g in a group G, set $g^{-1}ag = a(g)$ or $a^{(1)}(g)$ and set $a^{(n)}(g) = a^{(n-1)}(g)$. If, for some n and every $g \in G$, $a^{(n)}(g) = a$, a is said to be weakly self-conjugate. The following results then hold: (1) if in a locally finite group G every element of prime power order p is weakly self-conjugate, then the set of such elements comprise a normal subgroup of G. (2) An element is in the nil radical of a locally finite solvable group if and only if it is weakly self-conjugate. (Received July 21, 1952.)

593. H. E. Stelson: *Finding the root of an equation by iteration.*

Assume that a real root of $x = f(x)$ has been isolated and that the sequence $x_1, x_2, x_3, \cdots$ has been computed from an initial value, x_0, near the root. For $f(x)$ a decreasing function (an increasing function can be transformed into a decreasing function), we have a root between x_n and x_{n+1}. An upper and lower limit for the root may be obtained by use of the formula $x = x_i - (\Delta x_i)^2/\Delta^2 x$ for the points $(x_0, f(x_0))$, $(x_1, f(x_1))$ and for the points $(x_0, f(x_0))$, $(x_2, f(x_2))$. These limits may be improved by use of the formula $x = x_i - (\Delta x_i + \Delta x_{i+1})/(\Delta x_{i+1}/\Delta x_i - (\Delta x_i/\Delta x_{i+1})^2 \Delta^2 x_{i+1}/\Delta^2 x_i)$ for the points $(x_0, f(x_0))$, $(x_1, f(x_1))$, $(x_2, f(x_2))$ and $(x_1, f(x_1))$, $(x_2, f(x_2))$, $(x_3, f(x_3))$ which is derived by successive approximations. $f''(x)$ must be of constant sign throughout the interval considered. Accuracy can be estimated by means of R_{n+1} for Newton's ordinary interpolation formula for divided differences. (Received March 20, 1952.)

594*t*. G. L. Walker: *Fermat's theorem for algebras.*

If A is an algebra over F and x is an indeterminate, let $I(A)$ be the set of all $f(x)$ in $F[x]$ such that $f(a) = 0$ for all a in A. $I(A)$ is then the principal ideal in $F[x]$ consisting of zero and all polynomial identities for A in a single indeterminate. We prove: if A is a total matrix algebra of order m^2 over $GF(p^n)$, then $I(A)$ is the principal ideal generated by $f = f_n f_{2n} \cdots f_{mn}$ where $f_i = x^{p^i} - x$. When $m = 1$, this is the Fermat theorem. This result is extended to semi-simple algebras of characteristic p and finally to the theorem: if A is an algebra over F with radical N, if $I(A - N) = (f)$, f in $F[x]$, and $I(N) = (x^r)$, then $(g_1) \leq I(A) \leq (g_2)$ where $g_1 = [f(x)]^r$ and $g_2 = \text{lcm } x^r, f(x)$. An example shows that these bounds on $I(A)$ cannot be improved without additional hypothesis. (Received July 18, 1952.)

595*t*. A. D. Wallace: *Topological semigroups* (*Local stability*. I).

A TSG is a map ($=$continuous function) $\mu: S \times S \to S$ such that S is Hausdorff and μ is associative. As usual $xy = \mu(x, y)$. If $a \in S$ and $T \subset S$, let T^a be the intersection of all sets $U \cdot T$, U an open set including a. If S is locally compact and T is compact then $T^a = a \cdot T$. (If S is a TG and e is the unit, then $T^e = \bar{T}$). Also $\square^a = \square$, $T_1^a \subset T_2^a$ if $T_1 \subset T_2$. $(T^b)^a \subset T^{ab}$, $(T_1 \cup T_2)^a = T_1^a \cup T_2^a$. (If e is a left unit for S then $T \subset T^e$ so e is a closure operator for S.) If S is a regular space, then $(T^a)^- \subset (T^-)^a$ so T^a is closed if T

is closed. Note that $aT\subset T^{*}$. We say that T is locally left stable at a if $T=T^{*}$. If S is compact and is locally left and right stable at a set dense in S, then S is a topological group. We note that these concepts and results are partially applicable to topological dynamics, i.e., when μ: $S\times X\to X$ is such a map that $\mu(s_1, \mu(s_2, x))=\mu(s_1s_2, x)$. (Received July 7, 1952.)

596*t*. L. R. Wilcox: *Addition and harmonic addition in a field*. Preliminary report.

A natural beginning of the study of the algebra of electric networks under the binary operations of series and parallel connections is the determination of interrelations between the operations of addition and ***harmonic addition*** $a * b=ab(a+b)^{-1}$ in a commutative field F'. First, an ideal element ∞ is appended to F' satisfying various conditions such as $a\cdot\infty=\infty\,(a\neq 0)$, $a+\infty=\infty$, etc. If $a, b\in F=F'+\{\infty\}$, then $a+b$, ab, $a * b$, $-a$ are defined so that (a) $F-\{0\}$ is a group under $*$ with ∞ as identity and $-a$ serving as inverse of a, and (b) duality between $+$, $*$ is complete. Central in the theory is a relation $\sim$ between pairs of elements of F, where $(a, b)\sim(c, d)$ means either $(a+d) * (c+b)=a * c+d * b$ or $b=a=-c=-d$. Many properties of $\sim$ are found, including those expressing reflexivity, symmetries, transitivity, and self-duality. Conversely, the properties found characterize fields in the following sense. Given a system $(F, +, *)$ having these properties, define multiplications $\times_k$ by $a\times_k b=2^{-1}\{c * (k-a)+b * (k-b)-(a+b) * (k-(a+b))\}$, where $2^{-1}x$ means $x * x$. (Here the system must not have characteristic 2 in an obvious sense.) For fixed $k\neq 0$, ∞, $(F-\{\infty\}, +, \times_k)$ is a field in which k is the multiplicative unit and in which $k\times_a k$ serves as reciprocal of $a\neq 0$. In each such field, harmonic addition coincides with $*$. (Received July 21, 1952.)

ANALYSIS

597. H. A. Antosiewicz: *Forced periodic solutions of systems of differential equations.*

The vector differential equation $dx/dt=A(t)x+p(x, t)+q(x, t, k)$ is considered under the following assumptions: (i) x is a vector in n-dimensional space E_n; (ii) $A(t)$ is a continuous and periodic matrix of period 1; (iii) $p(x, t)$ is continuous in (x, t) and periodic in t of period 1 for x in some fixed closed sphere $\overline{S}(0, R)$ and every t, satisfies a Lipschitz condition in $\overline{S}$, and $p(0, t)=0$; (iv) $q(x, t, k)$ is continuous in (x, t, k) and periodic in t of period 1 for x in $\overline{S}$ and every t and every real k, satisfies a Lipschitz condition in $\overline{S}$, and $q(x, t, 0)=0$. It is shown that if the characteristic exponents of $dx/dt=A(t)x$ are all less than unity, then there exists a positive definite quadratic form $V(x)$ such that for sufficiently small k a region V of E_n bounded by $V(x)=C$ is mapped into itself by $T{:}x(1)\to x(0)$. It follows from Brouwer's fixed point theorem that there exists in V at least one periodic solution of period 1. This periodic solution is shown to be asymptotically stable in Liapounoff's sense. (Received July 21, 1952.)

598. Nachman Aronszajn and A. N. Milgram: *Differential operators on Riemannian manifolds.*

The consideration of differential problems with general boundary conditions in a Euclidean domain requires the study of intrinsic differential operators on the boundary. The authors attack this question by considering differential operators on a general Riemannian manifold $\mathfrak{m}_n$. The notions of differential operators and o-forms (i.e. exterior differential forms with bilinear differential operators for coefficients) and their restriction to submanifolds are defined. The following results, among others, are obtained: (1) a closed o-form is an exact differential of an o-form; (2) the representation of an operator and its adjoint in terms of covariant derivatives; (3) analysis of boundary operators on an $(n-1)$-dimensional submanifold $\mathfrak{m}_{n-1}$ in connection with a differential operator defined in a neighborhood of $\mathfrak{m}_{n-1}$, in particular a complete description of all self-adjoint or positive systems among all normal systems of boundary operators. (Received July 14, 1952.)

599*t*. I. E. Block: *On Hilbert transforms.*

Let $\Lambda^*(-\infty, +\infty)$ be the class of continuous functions $f(x)$ such that $|f(x+h)+f(x-h)-2f(x)| \leq Ah$ uniformly in x as $h \to 0$. Such a class was defined by Zygmund (Duke Math. J. vol. 12 (1945) pp. 47–76) to replace the class Lip 1 of functions. The following theorem is proved: Let $f(x)$ belong to $L^p(-\infty, +\infty)$, $p>1$, and $\Lambda^*(-\infty, +\infty)$. Then the Hilbert transform of $f(x)$ belongs to $L^p(-\infty, +\infty)$ and $\Lambda^*(-\infty, +\infty)$. The method of proof follows the proof of Zygmund for a similar theorem for functions in Λ^* on the unit circle. (Received June 9, 1952.)

600. R. P. Boas: *Functions of bounded characteristic and functions of exponential type.*

The author coordinates a number of recent results concerning functions of bounded characteristic and functions of exponential type. Considerable simplification is achieved in the discussion of functions of exponential type with zeros z_n such that $\sum |I(1/z_n)|$ converges. Systematic use is made of a theorem of Ahlfors and Heins [Ann. of Math. (2) vol. 50 (1949) pp. 341–346] on the existence of $\lim r^{-1} \log |f(re^{i\theta})|$ for a function of exponential type which is bounded on the real axis. This theorem is extended to functions satisfying less restrictive conditions on the real axis; thus results of B. Levin [Doklady Akad. Nauk SSSR. N.S. vol. 71 (1950) pp. 601–604] are made sharper. The author does not consider Levin's proofs completely convincing. (Received July 16, 1952.)

601*t*. Joseph Bram: *Normality of rings of operators of type* I.

A ring A of operators on a Hilbert space H is called normal if for every subring B containing the center Z of A, we have $(B' \cap A)' \cap A = B$, where E' denotes the set of all operators on H which commute with every operator in E. von Neumann showed that every factor of type I is normal. To show that every ring of type I is also normal, it suffices to prove the result for the case in which A is, for some cardinal n, a "full matrix ring of order n" over a commutative ring, since every ring of type I is a direct sum of such rings. The space H is replaced by a smaller space H_1 on which A is represented faithfully and on which A' is exactly the center of A. It is known that every subring of A is weakly closed even when acting on H_1. The desired result is seen to be true on H_1, hence also on H. (Received June 1, 1952.)

602. George Brauer: *Sets of convergence of ordinary Dirichlet series.* Preliminary report.

It is shown that for any real number α, each set of type F_σ on the line $\sigma = \alpha$ is the set of convergence of an ordinary Dirichlet series of the form $\sum_{n=1}^{\infty} a_n n^{-s}$, $s = \sigma + i\tau$. Each denumerable set on the line $\sigma = \alpha$ is the set of divergence of an ordinary Dirichlet series. (Received July 15, 1952.)

603. F. E. Browder: *Assumption of boundary values in the Dirichlet problem for the general linear elliptic differential equation.*

Let K be a suitably differentiable linear elliptic differential operator of order $2m$ on the bounded domain D of E^n. Suppose that u is a solution of the Dirichlet problem for the equation $Ku=h$ with boundary value function g, P a point of the boundary of D. It is shown that if g and h are sufficiently differentiable in the neighborhood of P and if the boundary of D is sufficiently smooth at P, then u assumes its full set of boundary data at P. For $n=2$, a sufficient smoothness condition is that the boundary of D should be locally connected at P while P is not an isolated point of the boundary. (Received July 21, 1952.)

604t. F. E. Browder: *Higher order boundary value problems for linear elliptic differential equations.*

Let K be a suitably differentiable, strongly elliptic, linear differential operator of order $2m$ on the bounded domain D of E^n, $K=L+A$ where L is self-adjoint and A of less order than $2m$. Let γ be a set of boundary conditions and Γ the subset of $C^{2m}(\overline{D})$ satisfying the boundary conditions γ. Suppose that on Γ L is self-adjoint with respect to the inner product of $L^2(D)$ and $(u, v\in\Gamma)$, further that $(-1)^m\int_D u\cdot\overline{K}v\,dx=$ the sum of an m-inner product on D, a positive boundary integral, and a boundary integral of lower order. The semi-boundedness for K on Γ is established together with existence and differentiability theorems, for solutions of $Ku=h$ with boundary conditions γ, which are analogous to the author's previous results on the Dirichlet problem. (Received July 21, 1952.)

605t. F. E. Browder: *Linear parabolic differential equations of arbitrary order.*

If K is a suitably differentiable linear elliptic differential operator of order $2m$ on the bounded domain D of E^n, the solution $u(x, t)\,[xD,\ 0<t<\infty]$ of the equation $(-1)^{m+1}Ku=\partial u/\partial t$ is said to be a solution of the initial value problem with data g and f if for each t, u satisfies the Dirichlet boundary data of g and as t goes to zero $u(x, t)$ approaches $f(x)$ in $L^2(D)$. The existence of a solution to the initial value problem is established if K is self-adjoint and it is shown that the solution is $2m$ times differentiable in x and infinitely differentiable in t. The Green's function is constructed and applied to other boundary value problems. The methods used are based upon the author's results on linear elliptic equations. (Received July 21, 1952.)

606. R. O. Buck: *Essentially admissible sequences.*

Let S be the space of complex sequences $\alpha=\{a_n\}$ for which $\|\alpha\|=\sup|a_n|^{1/(n+1)}<\infty$. Let A be the set of those α which are essentially admissible in the sense that there exists an entire function $f(z)$ of exponential type obeying the growth restriction $f(iy)=O(\exp c|y|)$ for some $c<\pi$, and such that $f(n)=a_n$ for all large n. Necessary and sufficient conditions are obtained that a sequence α belong to the closure of A. It is also shown that A has no interior points, and that there are arbitrarily large spheres in S, disjoint from A. (Received July 21, 1952.)

607. C. W. Burrill: *Characterising systems of continuous functions.* Preliminary report.

Several writers have studied the problem of characterizing a topological space by means of the continuous functions on the space. The principal contributions of this paper are the following. (1) A completely regular (Hausdorff) space X is characterized by the lattice of continuous real-valued functions together with the sublattice of bounded continuous real-valued functions in case the space satisfies the following condition: for every $b \in \beta(X) - X$ there exists a continuous real-valued function g such that for every real number α, $b \notin \epsilon[x \in X; g(x) \leq \alpha]$. Here $\beta(X)$ is the Čech β-compactification of X and ϵ is the closure operation in $\beta(X)$. The proof closely follows that of I. Kaplansky in Bull. Amer. Math. Soc. vol. 53 (1947) pp. 617–623. (2) If Z is a completely regular (Hausdorff) space satisfying condition C, then certain ordered sets of equivalence classes of continuous real-valued functions will characterize Z. Here C is a condition of E. W. Chittenden: *if $a \in Z$ then there exists a continuous real-valued function f such that $f(x) = 0$ if and only if $x = a$.* (Trans. Amer. Math. Soc. vol. 31 (1929) p. 317.) Property C holds if Z satisfies the first axiom of countability. (Received July 21, 1952.)

608*t*. P. L. Butzer: *Linear combinations of Bernstein polynomials.*

Denote $\mathfrak{B}_n^{[0]} = [\mathfrak{B}_n^f(x)]^{[0]} = B_n^f(x)$ and $(2^k - 1)\mathfrak{B}_n^{[2k]} = 2^k \mathfrak{B}_{2n}^{[2k-2]} - \mathfrak{B}_n^{[2k-2]}$, $k = 1, 2, 3, \cdots$, where $B_n(x) = B_n^f(x) = \sum_{\nu=0}^{n} f(\nu/n) C_{n,\nu} x^\nu (1-x)^{n-\nu}$ is the Bernstein polynomial corresponding to the function $f(x)$ defined on $[0, 1]$. The polynomial $\mathfrak{B}_n^{[2k]}(x)$ can be rewritten in the form $\alpha_k B_{2^k n}(x) + \alpha_{k-1} B_{2^{k-1} n}(x) + \cdots + \alpha_0 B_n(x)$ with constants $\alpha_i = \alpha_i(k)$ satisfying $\sum_{i=0}^{k} \alpha_i = 1$. The linear combination $\mathfrak{B}_n^{[2k]}(x)$ is shown under certain conditions to approach $f(x)$ more closely than the Bernstein polynomials. For example, if $f(x)$ is defined on $[0, 1]$ with $|f(x)| \leq M$, then at points where the $2k$th derivative $f^{(2k)}(x)$ exists, it is shown that $|\mathfrak{B}_n^{[2k-2]}(x) - f(x)| = O(n^{-k})$ and $|\mathfrak{B}_n^{[2k]}(x) - f(x)| = o(n^{-k})$, $n \to \infty$, $k = 1, 2, 3, \cdots$. Correspondingly, for the Bernstein polynomials, at points where $f''(x)$ exists, the order of approximation to $f(x)$ is only $O(n^{-1})$, and the existence of derivatives of higher order of $f(x)$ cannot improve this order (Voronowskaja, S. Bernstein, C. R. (Doklady) Acad. Sci. URSS. (1932) pp. 79–92). Now if $f^{(2k)}(x)$ is continuous with modulus of continuity $\omega_{2k}(\delta)$ on $[0, 1]$, it is shown that $|\mathfrak{B}_n^{[2k]}(x) - f(x)| \leq \max \cdot \{C n^{-k} \omega_{2k}(n^{-1/2}), C' n^{-k-1}\}$ where $C = C(k)$, $C' = C'(k; f)$. If $f(x)$ satisfies only a Lipschitz condition of order α, $0 < \alpha \leq 1$, then, in general, $\mathfrak{B}_n^{[2k]}$ does not approximate $f(x)$ more closely than $B_{2^k n}(x)$ for which $|f(x) - B_{2^k n}(x)| = O(n^{-\alpha/2})$, (Popoviciu, Mathematica Cluj vol. 10 (1935) pp. 49–54). (Received June 25, 1952.)

609. Joshua Chover: *On locally-homogeneous measures and Hilbert space decompositions.* Preliminary report.

A condition of "local-homogeneity," which is a weakening of the condition of translation invariance, and an algebraic concept of support are introduced for measures on a group, and examples are discussed. Secondly, an abstract equivalence relation, which is coarser than the usual one, is defined for nonzero measures, and in particular for those with supports. The local-homogeneity condition serves to define a special instance of this relation which is useful in application to operator theory. By means of this relation, and using the analogy between ideals of measures and vector subspaces, it is shown that any self-adjoint operator on Hilbert space whose weighted spectrum is "absolutely continuous" with respect to the domain of the relation induces a unique decomposition of the space into invariant subspaces which have certain measure theoretical properties. This decomposition permits the rigorous definition of a concept of generalized eigenvalue expansion such as is required for various operators in the Dirac model for quantum mechanics. (Received July 7, 1952.)

610. R. V. Churchill and C. L. Dolph: *Inverse transforms of products of Legendre integral transforms.*

Let $f(n)$ $(n=0, 1, 2, \cdots)$ denote the Legendre integral transform of a function $F(x)$ on the interval $(-1, 1)$, $f(n)=\int_{-1}^{1}F(x)P_n(x)dx$. If $g(n)$ is the transform of a function $G(x)$, then the product $f(n)g(n)$ is the transform of the function $H(x)$ where $H(\cos\theta)$ is represented by the integral $\pi^{-1}\int_0^{\pi/2}\sin\alpha d\alpha\int_0^{2\pi}F[\sin\alpha\cos(\beta+\theta/2)]\cdot G[\sin\alpha\cos(\beta-\theta/2)]d\beta$. The integral H corresponds to the convolution integrals for Fourier and Laplace transforms. Sufficient conditions for the above inverse transform of $f(n)g(n)$ are that $F(x)$ and $G(x)$ be continuous $(|x|\leqq 1)$. The methods and results of J. P. Vinti [Proceedings of the American Mathematical Society vol. 2 (1951) pp. 19–23] played an essential role in deriving the above formula. This convolution property not only makes it possible to extend tables of Legendre transforms but it also leads to closed forms of solutions of boundary value problems. (Received July 21, 1952.)

611. E. G. H. Comfort: *A new inequality.*

The inequality $(x^r+y^r)(x+y)^r-(x+y)^{2r}-(2^{r+1}-2^{2r})x^ry^r\geqq 0$ is proved for $x\geqq 0$, $y\geqq 0$, $0\leqq r\leqq 1$. If $xy>0$, a rearrangement of the inequality shows that the difference $(x^r+y^r)-(x+y)^r$ is not less than a positive constant, which is independent of x and y, multiplied by the rth power of the harmonic mean of x and y. (Received July 14, 1952.)

612*t*. John DeCicco: *The theory of residues of polygenic functions.*

For a polygenic function $F(z)$ of class $C^{(m)}$ in the region $0<|z-z_0|<\rho$, the residue function is $R(z_0, r)=(1/2\pi i)\int_C f(z)dz$, where C is a positively sensed circle with center at z_0 and positive radius $r<\rho$. If $f(z)$ is q harmonic so that $\nabla^{2q}f(z)=0$ in the region, then $R(z_0, r)$ is an even polynomial in r of degree $2q-2$ at most. The residue of order q of a polygenic function $f(z)$ at a point z_0 is the limit (if it exists) of the derivative of order q of $R(z_0, r)$ with respect to r^2 as r approaches zero. If $f(z)$ is polygenic of class $C^{(m)}$ at a point z_0 and if $1<n+2\leqq m$, then $f(z)/(z-z_0)^{n+1}$ possesses residues at z_0 of orders 0 to $[(m-n)/2]$ inclusive. If $0\leqq q\leqq[(m-n)/2]$, the residue of order q is $(1/(n+q)!)(\nabla^{2q}/2^{2q})(\partial^n f/\partial z_0^n)$. The concepts can be extended to the residues of a polygenic function for a finite set of distinct points. If $F(z)=f(z)/(z-z_1)\cdots(z-z_n)$, where $f(z)$ is polygenic of class $C^{(2)}$ for the set S of distinct points $z_1, z_2, \cdots, z_n$, the residue of order zero of $F(z)$ is the divided difference $[z_1z_2\cdots z_n]$ of $f(z)$, and the residue of order one of $F(z)$ is the divided difference $[z_1z_2\cdots z_n]$ of $\partial^2 f/\partial z\partial\bar{z}$ minus the sum of the divided differences $[z_1\cdots z_{k-1}z_{k+1}\cdots z_n]$ of the function $(\partial f/\partial\bar{z})/(z-z_k)^2$ for $k=1, \cdots, n$. These results are extended to confluent divided differences. (Received June 18, 1952.)

613. R. J. Duffin: *Discrete potential theory.*

A study is made of analogy and counter-analogy between the Laplacian operator and the corresponding difference operator relative to the regular rectangular lattice in space. Courant has shown that the analogue of the potential of a point charge may be represented as a triple Fourier transform. This function is investigated here in detail and its complete asymptotic expansion at infinity is determined. To do this it is first of all necessary to develop a theory of the asymptotic expansions of multiple Fourier transforms. In the two-dimensional case, it is found possible to evaluate explicitly the potential of a point charge. As an analogue of Gauss's mean value theorem, it is shown that the value of a discrete harmonic function is to a certain approximation the mean of its values at the lattice points in a concentric sphere. An analogue of Harnack's inequality for positive harmonic functions is found. The following corollary corresponds to Liouville's theorem: A positive function which satisfies the Laplace difference equation at all lattice points is a constant. (Received July 18, 1952.)

614t. R. S. Finn: *On the solutions of elliptic systems of first order equations.*

Solutions of elliptic systems of the form (1) $\alpha\phi_x+\beta\phi_y=\psi_y$, $\gamma\phi_x+\delta\phi_y=-\psi_x$ are considered. At each point $(x+iy)$ an upper bound for the eccentricity ϵ of the mapping $(x+iy)\rightarrow(\phi+i\psi)$ is determined by the magnitudes of the coefficients α, β, γ, δ. It is shown that if $\epsilon<K\log(1+x^2+y^2)$, then a nonconstant entire solution omits at most one finite value. A similar theorem has been proved by H. Grötzsch (Leipziger Berichte vol. 80 (1928)) using other methods, under the assumption of uniformly bounded eccentricity. More generally, the domain of values achieved by any solution in a neighborhood of an isolated point is exactly that of an analytic function, provided that $\epsilon<K\log(1/x^2+y^2)$ near the point. Theorems of similar type hold for the complex gradient $\phi_x-i\phi_y$ of a solution of an elliptic equation (2) $A\phi_{xx}+2B\phi_{xy}+C\phi_{yy}=0$. If (2) is nonlinear, and if $\epsilon<K(1+\phi_x^2+\phi_y^2)^n\log(1+x^2+y^2)$ for some integer n, then an entire solution for which $\phi_x-i\phi_y$ omits a nondegenerate continuum is a linear function $\phi(x, y)=ax+by+c$. Applications are made to solutions of elliptic variational problems $\delta\iint F(p, q)dxdy=0$. It is shown that if $F_{pp}+F_{qq}<K$, the condition $|\phi|<K(\log 1/x^2+y^2)^{1/2}$ is sufficient for removability of an isolated singularity. Estimates are obtained for the rate of growth of entire solutions and various generalizations of Liouville's theorem for harmonic functions are given. The proofs depend on a conformally invariant formulation of the equations and on a technique due to M. Shiffman for estimating growth of double-integral quadratic functionals. (Received July 18, 1952.)

615. W. H. Fleming and L. C. Young: *A notion of boundary for generalised parametric surfaces.*

A linear functional L on the space of continuous parametric integrands $f(x, J)$ defined by $L(f)=\iint f[x(u, v), J(u, v)]du dv$, where $x(u, v)=[x^1(u, v), \cdots, x^m(u, v)]$ is an "elementary" (e.g., Lipschitzian) mapping of $[0\leqq u\leqq 1; 0\leqq v\leqq 1]$ into m-space and $J(u, v)=[\partial(x^r, x^s)/\partial(u, v)]$, is called an *elementary (parametric) surface*. Weak limits of sequences of elementary surfaces are called *generalised (parametric) surfaces* [L. C. Young, Bull. Soc. Math. France vol. 79 (1951) pp. 59–85]. The *g-boundary* λ_L of a generalized surface L is obtained by restricting the domain of L to the space of exact integrands. If $\lambda_L=0$, L is termed *closed*; L is *basic closed* if $L=L_1+L_2$ with L_1, L_2 closed implies $L_1=kL$. Every generalized surface whose g-boundary λ bounds some elementary surface is weak limit of positive linear combinations of elementary surfaces bounded by integral multiples of λ. Given $b>0$, a sufficient condition that an exact $\phi(x, J)$ exists such that $f_0(x, J)+\phi(x, J)$ is positive definite for $|x|\leqq b$ is that $L(f_0)>0$ for all basic closed L. For $m=3$, the latter inequality holds if f_0 is semidefinite, semiregular, and satisfies a certain condition involving its zeros. (Received July 14, 1952.)

616t. K. O. Friedrichs: *On the differentiability of solutions of elliptic differential equations.*

Let the vector-valued function $u(x)$, defined in an open set $\mathfrak{D}$ of m-dimensional space $x=\{x_1, \cdots, x_m\}$, be a solution of the differential equation $Lu(x) = \sum_{\rho,\sigma=0}^{r} \bar{D}^\rho a_{\rho\sigma}(x) D^\sigma u(x) = f(x)$ of order $2r$, in which D is the gradient and $\bar{D}$ the negative divergence, so that $D^\sigma u$ stands for all m^σ derivatives of u of order σ while $\bar{D}^\rho$ transforms systems of m^ρ functions into one function. Furthermore let $a^{\rho\sigma}(x)$ be an m^ρ by m^σ matrix, and $f(x)$ a given function. Assume that $\int u^r(x) a^{rr}(x) u^r(x) dx \geqq a \int u^r u^r dx$ for every system $u^r(x)$ of m^r functions which vanish outside a compact subset of $\mathfrak{D}$, so that L is elliptic in the restricted sense. With reference to a Hilbert norm, we assume that u admits the operator D in the "strong sense" r times and that $a^{\rho\sigma}(x) D^\sigma u(x)$ admits the operator $\bar{D}$ in the "weak sense" ρ times. *Theorem* 1. If $f(x)$ admits D in the strong sense s times, the solution $u(x)$ admits D in the strong sense $r+s$ times provided $a^{\rho\sigma}$ is in $\mathfrak{C}_s$. *Corollary*. $u(x)$ is in $\mathfrak{C}_k$ if $s+r>k+m/2$. The main tools are integral inequalities and approximation with the aid of mollifiers. (Received June 16, 1952.)

617. W. B. Fulks: *The Neumann problem for the heat equation.*

The author considers the solutions of the following boundary value problem. Let $U(P) = U(x, t)$ be a solution of (1) $U_{xx} = U_t$ in R: $\{0<x<1; 0<t<T\leqq\infty\}$ which takes on prescribed initial values $U(x, 0)$ and whose normal derivative takes on prescribed boundary values $U_x(0, t)$ and $U_x(1, t)$. He is particularly concerned with generalizations of this problem. Let $Q=(y, s)$ be a point on the boundary B of R, and let $F(P, Q)$ be the (explicitly known) Green's function associated with the problem. He then obtains necessary and sufficient conditions on $U(P)$ in order that a solution of (1) can be represented in the form (2) $\int_B F(P, Q) d\mu(Q)$. He further shows that $U(x, 0+) = \mu'(x)$ and $U_x(0+, t) = \mu'(t-0)$ wherever these derivatives exist. He then gives necessary and sufficient conditions on $U(P)$ for (3) $U(P) = \int_B F(P, Q) f(Q) dQ$ and from (2) and (3) he extracts conditions which uniquely determine the solution of this boundary value problem. (Received July 10, 1952.)

618. I. S. Gál: *On the principle of condensation of singularities.*

In a recent paper [Ann. Institut Fourier, vol. 3, p. 24] the author introduced the concept of "*asymptotically subadditive*" sequences of homogeneous and bounded operations mapping a normed vector space into another one. As was shown there the principle of uniform boundedness remains valid for these operations. It is now proved that the principle of condensation of singularities can also be extended in the same generality: Let $\{u_{mn}(x)\}$, $m, n=1, 2, \cdots$, be a double sequence of homogeneous and bounded operations mapping a complete vector space E into a normed vector space E'. Suppose that every single sequence $\{u_{mn}(x)\}$, $n=1, 2, \cdots$ (m fixed), is asymptotically subadditive, that is to say $\|u_{mn}(x+y)\| \leqq \|u_{mn}(x)\| + o(|u_{mn}| \cdot \|y\|)$ uniformly in $x, y \in E$ and $\inf_y [\|u_{mn}(x+y)\| + \|u_{mn}(x)\| - \|u_{mn}(y)\|] \geqq o(|u_{mn}|)$ as $n \to \infty$ where the infimum is taken for every $y \in E$ with $\|y\| \leqq 1$. The *generalised principle of condensation* is as follows: If, for every $m=1, 2, \cdots$, there is an $x_m \in E$ such that $\limsup \|u_{mn}(x_m)\| = +\infty$ as $n \to \infty$, then there is a common element $x \in E$ having the property that $\limsup \|u_{mn}(x)\| = +\infty$ as $n \to \infty$ for every $m=1, 2, \cdots$. (Received July 23, 1952.)

619t. Seymour Ginsburg: *A cardinal number associated with a family of sets.*

For each ordinal number $a<\alpha$, let U_a be a family of subsets of a set R_a, partially ordered by set inclusion. Let $md(U_a)$ be the smallest cardinal number which is the power of the union of a maximal family of incomparable elements of U_a. Let $cmd(U_a)$ be the smallest cardinal number which is the $md(V)$ of some coinitial subfamily V of U_a. The following two results are shown: (1) $\prod_{a<\alpha} cmd(U_a) = cmd(\prod_{a<\alpha} U_a)$, where by $\prod_{a<\alpha} U_a$ is meant the cartesian product. (2) If U_a is a ramified family of sets, then $md(U_a) = cmd(U_a)$. (Received July 15, 1952.)

620t. Seymour Ginsburg: *A class of everywhere branching sets.*

Let P be an everywhere branching partially ordered set (see Day, *Oriented systems*, Duke Math. J. vol. 11 (1944) p. 201 ff.). Let $A(x) = \{y \mid y \leq x, y \in P\}$. P is said to have sufficiently many non-coinitial subsets (smns) if, for each pair of elements p and q, in P at least one of the following two conditions is false: (1) $A(p)$ is coinitial in $A(q)$, and (2) $A(q)$ is coinitial in $A(p)$. If P is the dual of a ramified system, then P is shown to contain a coinitial subset S which has smns. A residual subset Q of P is called maximal residual if Q is no proper coinitial subset of any residual subset of P. Let $F(P)$ be the family of maximal residual subsets of P, partially ordered by set inclusion. $F(P)$ has smns. If P has smns, then P is coinitially similar to $F(P)$. If two everywhere branching partially ordered sets, P_1 and P_2, are coinitially similar, then $F(P_1)$ is isomorphic to $F(P_2)$. (Received July 15, 1952.)

621t. Seymour Ginsburg: *A theorem on similarity transformations.*

Each set considered is simply ordered, and each function a similarity transformation. The product of two sets is to be ordered by first differences. Theorem: Let g map $C \times D$ into $E \times F$ and have the property that for each element y in C, there are at least two elements in E, $u(y)$ and $v(y)$, where $u(y) < v(y)$, such that the set intersection of $g(y \times D)$ and $(u(y) \times F)$ is nonempty, and the set intersection of $g(y \times D)$ and $(v(y) \times F)$ is nonempty. Then (a) for a given element p in C and q in E, where $u(p) \leq q \leq v(p)$, a function h, of C into E, can be found so that $h(p) \neq q$, and for each element y in C, either $h(y) = u(y)$ or $h(y) = v(y)$; and (b) if $C = E$, then a function h of E into E can be found so that for each element y in E, $h(y) \neq y$ and $u(y) \leq h(y) \leq v(y)$. Applications are the following: (1) Let E have the property that for each function f of E into E, a point p can be found so that $f(p) = p$. Then (E, E) is an A-pair (see Arens: *Ordered sequence spaces*; Portugalia Math. vol. 10 (1951) pp. 25–28). (2) Let p be a fixed point of E, and F a set in which each point is fixed. Then for each element y in F, (p, y) is fixed in $E \times F$. (Received July 15, 1952.)

622t. Seymour Ginsburg: *Sets of incomparable elements in coinitial subsets of a partially ordered set.*

For each ordinal number $a < \alpha$, let P^a be a partially ordered set with no minimal element. By $cs(P^a)$ is meant the smallest cardinal number m which has the following property: For some coinitial subset Q^a of P^a, m is the smallest cardinal number which is greater than the power of each family of incomparable elements in Q^a. If $cs(P^0) = n+1$, n being finite, then it is shown that P^0 contains a coinitial subset Q^0 which is a degenerate ramified system with at most n incomparable elements. Using combinatorial methods, a study is made of conditions under which $cs(\prod_{a<\alpha} P^a) \leq \aleph_0$, where $\prod P^a$ is the cardinal product of the P^a, and $cs(P^a) \leq \aleph_0$ for each a. For example, if $\alpha = 2$, then $cs(\prod_{a<\alpha} P^a) \leq \aleph_0$. Particular attention is focused on the case where each set P^a is simply ordered. (Received July 15, 1952.)

623t. Casper Goffman: *Lower semicontinuity of area functionals.* I.

Fréchet showed a number of years ago that a lower semicontinuous functional

$u(x)$ on a metric space S, satisfying a certain condition C, has a unique extension to a lower semicontinuous $U(x)$ on the completion $\bar{S}$ of S, also satisfying C. If S is the space of polyhedral mappings, properly metrized, and $u(x)$ is elementary area, then $\bar{S}$ consists of all continuous mappings and $U(x)$ is Lebesgue area. Now, elementary area is lower semicontinuous (with condition C) relative to other metrizations of the polyhedral mappings, and the Fréchet process may be applied to each of them. The long range objective is to make a thorough investigation of the possibilities. This first paper considers the nonparametric case. The elementary area $u(p)$ of a polyhedral function $p(x, y)$ is lower semicontinuous (with condition C) if the distance $d(p, q)$ is taken as $\iint |p(x, y)-q(x, y)|\,dxdy$. The functional $U(f)$, assured by Fréchet's Theorem, now has all summable functions as domain. It agrees with Lebesgue area for continuous functions. Much of the Radó-Tonelli theory for continuous functions is extended to this case, the salient ideas such as expressions of Goëcze, bounded variation, and absolute continuity in sense of Tonelli being appropriately modified. (Received July 11, 1952.)

624*t*. Casper Goffman: *One-one measurable transformations.*

This paper is a continuation of one given at the International Congress of Mathematicians (Cambridge, 1950) entitled *Lusin's theorem for one-one measurable transformations.* It is shown, for example, that if I_n, I_m are n- and m-dimensional unit cubes, $n<m$, and if $(f(x), f^{-1}(y))$ is a one-one measurable transformation between them, then, for every $\epsilon>0$, there is a homeomorphism $(g(x), g^{-1}(y))$ between I_n and a subset of I_m such that $f(x)=g(x)$ and $f^{-1}(y)=g^{-1}(y)$ on sets whose n-dimensional and m-dimensional measures, respectively, exceed $1-\epsilon$. The method of proof combines constructions similar to those used by Nöbeling and Besicovitch in surface area problems and theorems on the extension of homeomorphisms from given homeomorphisms on special zero-dimensional sets. The results of both papers will appear in Acta Mathematica. (Received July 11, 1952.)

625*t*. A. W. Goodman: *The rotation theorem for starlike univalent functions.*

Stroganoff [Trudy Mat. Inst. Steklova vol. 5 (1934) pp. 247–258] proved that among the functions $f(z)$, $f(0)=0$, $f'(0)=1$, regular, univalent and starlike in $|z|<1$, the extreme value of $\arg f'(re^{i\theta})$ for fixed r is assumed by $z/(1-z)^2$ at an appropriate point z. In this paper a short and elementary proof of the same result is obtained. The analogous problem for functions starlike in $|z|>1$ is discussed. (Received July 16, 1952.)

626. P. R. Halmos, Günter Lumer, and J. J. Schäffer: *Square roots of operators.*

The main purpose of this paper is to answer (negatively) the question: does every (bounded) invertible operator on a (complex) Hilbert space have a square root? Let H be the Hilbert space of all square-integrable analytic functions on a bounded domain D, and let A be the operator on H defined by $(Ax)(t)=tx(t)$ for all t in D. It is shown that the spectrum of A is the closure of D and, more particularly, that if $\lambda\in D$, then λ is not an approximate proper value of A but $\bar{\lambda}$ is a simple proper value of A^*. From this in turn it is deduced that A has a square root if and only if D surrounds the origin, in any one of several, rigorously definable, senses of that phrase. It follows that if, for instance, D is an annulus with center at the origin, then A is invertible, but A has no square root. The same methods solve the problem of nth roots for all $n\geqq 2$, and it turns out, in fact, that A has an nth root if and only if it has a square root. (Received April 9, 1952.)

627. J. M. Hammersley: *Nelder's conjecture.*

Let $F(x)$ be a nondecreasing function satisfying $0\leqq F(x)\leqq 1$. Then Nelder conjectured that $\int_0^\infty\int_0^\infty(x-y)^2e^{-(x+y)}dF(x)dF(y)\leqq 2/e^2$, and that equality is attained only in the case $F(x)=0$, $1/2$, or 1 according as $x<0$, $0\leqq x<2$, or $x\geqq 2$. The first part of this conjecture is proved by approximating the integral by a function of the saltuses of F, in case F is a step function, and obtaining a contradiction on the magnitude of any saltus at zero if F does not have the special form given above. The question of uniqueness remains open, although it is proved that the above F is unique among the class of F which have saltuses of commensurable magnitudes. Nelder's conjecture is an inequality in $[1, 1]$ space, and has application to the theory of moment generating functions. (Received July 21, 1952.)

628. Fritz Herzog and George Piranian: *Sets of radial continuity of analytic functions.*

A point set E on the unit circle $|z|=1$ is defined to be a *set of radial continuity*, provided there exists a function $f(z)$, regular in $|z|<1$, such that $\lim_{r\to 1}f(re^{i\theta})$ exists if and only if $e^{i\theta}$ is a point of E. It is proved that every set of type F_σ is a set of radial continuity. The result is not the best possible. (Received July 1, 1952.)

629. Fritz Herzog and George Piranian: *Sets of convergence of Taylor series.* II.

The authors describe two types of constructions of Taylor series. The first type, essentially geometric in nature, is used to prove the following: (1) every denumerable set on the unit circle C is the set of divergence of the Taylor series of a function which is schlicht and bounded in $|z|<1$; (2) there exists a function, schlicht and bounded in $|z|\leqq 1$, whose Taylor series converges everywhere on C, but not uniformly on any arc of C; (3) if the set M on C is closed and nowhere dense, there exists a Taylor series which converges everywhere on $|z|\leqq 1$ to a function which is schlicht in $|z|\leqq 1$, regular on every arc of $C-M$, and unbounded on every open arc of C that meets the set M. The second type of construction, of more arithmetic character, yields analogues to results (1) and (2), but with the condition of univalence replaced by continuity in $|z|\leqq 1$. (Received July 1, 1952.)

630*t*. Edwin Hewitt: *Linear functionals on almost periodic functions.*

Bochner's theorem on positive definite functions is generalized in the following way. Let $p(t)$ be an arbitrary complex-valued function defined for all real t ($t\in R$) such that $\sum_{j,k=1}^{N}p(t_j-t_k)\xi_j\bar{\xi}_k\geqq 0$ for all $t_1,\cdots,t_N\in R$ and all complex $\xi_1,\cdots,\xi_N$. Continuity or measurability of $p(t)$ is not assumed. For a bounded measurable set $T\subset R$, and any $\alpha\geqq\sup T-\inf T$, let $T_\alpha=\bigcup_{n=-\infty}^{+\infty}\{T+n\alpha\}$. Let $\mathfrak{S}$ be the smallest algebra of sets containing all T_α. Then there exists a finitely additive, non-negative, bounded measure ϕ defined for all sets in $\mathfrak{S}$ such that $p(t)=\int_{-\infty}^{+\infty}e^{itx}d\phi(x)$ for all $t\in R$. Subject to certain regularity conditions, ϕ is uniquely determined by this representation. The proof is obtained by considering the general form of a bounded linear functional L on the space $\mathfrak{A}$ of continuous almost periodic functions on R, regarding $\mathfrak{A}$ as the space of all complex continuous functions on a certain compact group bR, applying the generalized Bochner theorem to bR, and finally representing L as a finitely additive integral on R. As a corollary, it is proved that every positive definite function p can be uniquely written as p_c+p_d, where p_c and p_d are positive definite, p_c is continuous, and p_d cannot be written as q_1+q_2 with q_1, q_2 positive definite and q_1 continuous and $\neq 0$. The relations between measures on R and bR are also studied. (Received April 9, 1952.)

631. R. D. James: *Generalised nth primitives.*

In a previous paper (Canadian Journal of Mathematics vol. 2 (1950) pp. 297–306) the author considered some of the properties of a Perron second integral (P^2-integral), which, starting with a function, goes directly to its second primitive. The purpose of this paper is to define a P^{2m}-integral for $m>1$. The definition is made directly and not in terms of P^{2r}-integrals with $r<m$. It is shown that the P^{2m+2}-integral includes and is more general than the P^{2m}-integral. In a similar way a P^{2m+1}-integral is defined with the same property of consistency as the P^{2m}-integral. Moreover, the two scales fit together. The P^{r+1}-integral includes and is more general than the P^r-integral for $r>3$. It is also shown that the P^{r+1}-integral includes and is more general than the C_rP-integral defined by Burkill (Proc. London Math. Soc. (2) vol. 39 (1935) pp. 541–552). (Received July 18, 1952.)

632t. Fritz John: *Elementary expressions for the derivatives of weak solutions of elliptic differential equations.*

Let L be a linear elliptic differential operator of order $2m$ with continuous coefficients defined in a domain D in E^n. Let $\bar{L}$ be the adjoint operator of L. A function u is said to be a weak solution of $\bar{L}(u)=B(x)$ if $\int[uL(v)-vB(x)]dx=0$ for every v of class C^∞ vanishing outside a compact subset of D. An explicit expression for the derivatives of a function agreeing almost everywhere with u is given, using the method of spherical means (see Proceedings of the Symposium on Spectral Theory, Oklahoma, 1951, pp. 162 et seq.). This expression involves weighted integrals of u over solid spheres with kernels formed from B and the coefficients of L by elementary operations. [Existence theorems for derivatives of weak solutions have been proved by F. Browder by a different method (Bull. Amer. Math. Soc. Abstract 58-2-162).] The construction given here yields estimates for the derivatives of u in terms of u, which are related to but weaker than those given by K. O. Friedrichs. (Received June 26, 1952.)

633t. W. E. Johnson: *Asymptotic solutions of a linear second order differential equation with two turning points.*

The solutions of the differential equation $d^2y/dx^2-[\lambda^2\phi^2(x)-\chi(x)]y=0$ have been shown to be expressible asymptotically with regard to the parameter λ in terms of Bessel functions throughout an interval containing at most one zero of $\phi^2(x)$. If a representation with elementary functions is to be used, the Stokes' phenomenon is encountered. Connection formulas must then be employed. When the interval considered contains two distinct zeros of $\phi^2(x)$, Bessel functions are no longer adequate for the representation of a solution by means of a single functional form. If Bessel functions are to be used, connection formulas must again be employed. In this paper it is shown however that with the use of Weber functions, a representation can be achieved for any solution of the differential equation over a real finite interval containing two simple zeros of $\phi^2(x)$. The forms are asymptotic with respect to the real positive parameter λ. (Received July 14, 1952.)

634*t*. Edward Kasner and John DeCicco: *The Newtonian potential of a sphere of a Euclidean space E_N embedded in a Euclidean universe E_n.*

The two forces of attraction F_S and F_V exerted by a homogeneous material $(N-1)$-dimensional surface S and N-dimensional volume V of a sphere of radius r and mass M of a Euclidean space E_N upon a particle P of mass m in E_N at a distance a from the center of the sphere, such that E_N is embedded in a Euclidean universe E_n of n dimensions where $2 \leqq N \leqq n$, are studied. Either one of these two forces is directed along the line joining the position of the particle P to the center of the sphere. If $N=n$, the results are extensions of the corresponding well known formulas for $n=2$ and 3. If $2 \leqq N < n$, both F_S and F_V can be expressed in terms of hypergeometric functions. The corresponding fields of force are conservative and solenoidal. The Newtonian potential functions U_S and U_V are expressed in terms of hypergeometric functions. As a corollary, certain integrals of hypergeometric functions are evaluated. Finally some theorems of Gauss concerning the potential function due to the attraction of a circumference are extended. (Received June 19, 1952.)

635. W. S. Kimball: *The uniqueness requirement of Jacobi's condition in the calculus of variations.*

The integral $J=(1/2)\int y'^2dx/a \cos ax$ whose extremals are $y=c \sin ax$ and $y=b$, between $0 \leqq ax \leqq \pi$ for a fixed $c>0$, has for its first family of extremals above an envelope, $y=c$, under variations in the family parameter a, where ca is seen as the slope at the origin of the sinusoidal extremal arches. The two extremals $y=c \sin a'x$, and $y=c \sin a''x$ that cross at each point (x, y) under the envelope must have arguments that are supplementary angles $a'x=(\pi/2-r)$ and $a''x=(\pi/2+r)$, where $a''=$ constant gives the steeper sinusoidal extremal that touches the envelope $y=c$ before it reaches the end point (x, y) where it meets the extremal $a'=$ constant, which has not yet touched the envelope because $a'x=\pi/2-r$ is less than $\pi/2$ of the argument $ax_c=\pi/2$, that determines the conjugate points x_c of contact with the envelope. For both extrema, $J_{\min}$ (contacting envelope) $=(c^2/2) \sin a''x=J_{\min}$ (not touching envelope) $=(c^2/2) \sin a'x=cy/2$. Thus $J_{\min}$ (contacting envelope) is not $J_{\min}$, *unique* although equal to the smallest values of J. But still Jacobi's condition is satisfied provided it is taken to prescribe that *unique* minima of J cannot be provided along extremals having contact points with the envelope. (Received August 6, 1952.)

636*t*. W. J. Klimczak: *Differential operators of infinite order.*

Let $G(w)=\sum_{k=0}^{\infty} g_k w^k$ be an entire function of order less than or equal to σ and type at most β if the order equals σ. Let $D_z=\sum_{j=0}^{n} P_j(z)d^{n-j}/dz^{n-j}$. Then a necessary and sufficient condition that the operator $G(D_z)$ preserve holomorphism in any domain where the $P_j(z)$ $(j=0, 1, \cdots, n)$ are holomorphic is that $\sigma=1/n$ and $\beta=0$. In a certain sense this result is the best possible. If the $P_j(z)$ $(j=0, 1, \cdots, n)$ are polynomials and if the degree of $P_0(z)$ is at most $n-1$, then $G(D_z)f(z)$ is an entire function provided $f(z)$ is an entire function and the order of $f(z)$ and σ satisfy a certain inequality determined by an order relation depending upon the degrees of the polynomial coefficients. The emphasis is on the case $n=2$ and the extension to larger values of n is indicated. The results generalize previous results of H. Muggli, E. Hille, and M. K. Peabody and are sharp at least for $n=2$. (Received July 25, 1952.)

637. E. G. Kundert: *A relation between poles and zeros of a simple meromorphic differential form.*

It is well known that on a Riemann surface the degree of the divisor of a differential is always $=-X$ (Euler's characteristic). For a simple differential form α in a complex manifold M^n the following theorem holds, denoting by $C(\alpha)$ the zero-cohomology-class and by $S(\alpha)$ the polar-cohomology-class: $C(\alpha)$ can be expressed as a polynomial of degree n of $S(\alpha)$ in the cohomology ring of M^n: $C(\alpha)=S^n(\alpha)+\sum_{k=1}^{n}\Gamma_{n-k+1}\cdot S^{n-k}(\alpha)$; where the Γ_k are the covariant Chern classes of M^n. The proof uses an association of a field of complex line elements to α and then one can use the theory developed in the author's paper in Ann. of Math. vol. 54 (1951). (Received August 11, 1952.)

638*t*. E. G. Kundert: *A simple method to calculate Chern's classes of an algebraic variety.*

For various purposes it is important to have a method to calculate Chern's classes. The following formulas in the cohomology ring of an algebraic variety V^n serve this purpose: $\Gamma_1=\Pi_1^n+\sum_{k=2}^{n}\Pi_{n-k+1}\Pi_1^{n-k}$; $\Gamma_k=(1/(k-1)!)\partial^{(k-1)}\Gamma_1(\Pi_n)/\partial\Pi_n$, Γ_k being the contravariant Chern classes, Π_k denoting the cohomology classes dual to the intersection cycle of a $(k+m-n-1)$-dimensional plane in R^m with V^n, where P^m is the lowest-dimensional space in which V^n can be imbedded without singularities. The proof of this formula consists in constructing a field of line elements on V^n having Π_k as secondary obstructions and using it as reference field for the Chern field in formula (19) and (20) in the Author's paper in Ann. of Math. vol. 54. (Received August 11, 1952.)

639*t*. A. E. Livingston: *A note on products of summability methods.*

Let A and B be regular summability methods for sequences, and define the method AB by the A transforms of the B transforms of a given sequence. If, then, $s_n\to s$ (A) implies that $s_n\to s$ (AB), this will be indicated symbolically as $[A, B]$ is true. Let A, B, (C, α), (E, γ), and (H, μ) denote, respectively, the Abel, Borel, Cesaro (with parameter $\alpha>0$), Euler (with parameter γ, $0<\gamma\leqq 1$), and Hausdorff (with kernel $\mu(x)$) summability methods. Otto Szász (*On products of summability methods*, Proceedings of the American Mathematical Society vol. 3 (1952) pp. 257–263) has shown that $[A, (C, \alpha)]$, $[B, (C, \alpha)]$, and $[B, (E, \gamma)]$ are true. The author of this note shows that $[A, (H, \mu)]$ and $[B, (H, \mu)]$ are true, and these results include the results quoted above. Since $(H, \mu)(H, \sigma)=(H, \sigma)(H, \mu)$, it is trivial that $[(H, \mu)(H, \sigma)]$ is true. (Received July 24, 1952.)

640*t*. A. E. Livingston: *The space H^p, $0<p<1$, is not normable.*

It has been conjectured that the space H^p, $0<p<1$, is not normable (S. S. Walters, *Remarks on the space H^p*, Pacific Journal of Mathematics vol. 1 (1951) pp. 455–471). The author of this note shows that this conjecture is correct by proving the equivalent result: The space H^p, $0<p<1$, contains no bounded convex neighborhood of the origin. (Received July 24, 1952.)

641*t*. Dorothy Maharam: *The structure of abstract integrals.*

In this paper an "abstract integral" is a linear, countably additive, order-preserving mapping ϕ of one function space F in another, F'. One example of such an "integral" is as follows. Let X, Y be spaces in each of which a Borel field of "measurable" subsets and a σ-ideal of "null" sets are given, and suppose that in X these are in fact given by a measure, m. Let F be a space of (suitably restricted) "measurable" functions on $X\times Y$, and F' be the space of "measurable" functions on Y (functions differing only on "null" sets being identified throughout); these functions need not be finite. Take $\phi(f)=g$ where $g(y)=\int f(x, y)dm(x)$. It is proved (under mild assumptions, mostly of countability, and with no use of norms) that every "abstract integral" ϕ is "isomorphic" to one of this type. Conditions are given under which F will consist of (essentially) *all* "measurable" functions on some subset of $X\times Y$. (Received May 28, 1952.)

642. Hazleton Mirkil: *Translation-invariant function algebras over compact groups.* I.

All the major theorems of G. E. Šilov's paper *Homogeneous rings of functions* (see Mathematical Reviews vol. 13 (1952) p. 139) are here extended to arbitrary compact groups; Šilov did only the compact abelian case. Let $A=\{x\}$ be a commutative semi-simple symmetric banach algebra whose maximal-ideal-space is homeomorphic to a compact group $G=\{s\}$. (Multiplication is pointwise.) Call A "left-homogeneous over G" if A contains with every x all the left-translates sx, in which case the left-translations $\{x\to sx\}$ are automatically continuous $A\to A$. Write $Q(G)$ for the characteristic algebra of G, $Q(G)=$ linear combinations of matrix coefficients of irreducible left representations of G. Theorem: a necessary and sufficient condition that all the mappings $\{s\to sx\}$ be continuous $G\to A$ is that A contain $Q(G)$ and that $Q(G)$ be everywhere dense in A. The classical Peter-Weyl approximation theorem for continuous functions, clearly included as a special case, is used in the proof. Since the right-representations of G define exactly the same $Q(G)$ as the left-representations, it follows that for a left- and right-homogeneous A the continuity of all $\{s\to sx\}$ implies the continuity of all $\{s\to xs\}$. (Received July 21, 1952.)

643*t*. L. J. Mordell: *The minima of some nonhomogeneous functions of two variables.*

This paper contains a proof and some applications of the following: *Theorem.* Let x_0, y_0 be any real numbers. Then for certain functions, $f(x, y)$, there exist numbers x, y such that $x\equiv x_0 \pmod 1$, $y\equiv y_0 \pmod 1$, and $|f(x, y)|\leq k \max (|f(1, 0)|, |f(0, 1)|, f(1, \pm 1))$, where k is a number independent of x_0, y_0, depending only upon $f(x, y)$, and is really a constant defined by the existence of the inequality $|f(x, y)| \leq k|f(2x, 2y)|$ for all real x, y. The function $f(x, y)$ is supposed to be such that *inter alia*, the region defined by $|f(x, y)|\leq 1$ is in general appearance similar to that defined by $|x^3+y^3|\leq 1$ The paper has been accepted for publication in the Duke Mathematical Journal. (Received July 15, 1952.)

644*t*. Zeev Nehari: *Some inequalities in the theory of functions.*

Many of the inequalities of function theory and potential theory may be reduced to statements regarding the properties of harmonic domain functions with vanishing or constant boundary values. For the derivation of these inequalities a large number of different techniques and procedures have been used. It is the aim of this paper to show that many of the known inequalities of this type, and also others which are new, can be obtained as simple consequences of the classical minimum property of the Dirichlet integral. In addition to the resulting simplification, this method has the further advantage of being capable of generalization to a wide class of partial differential equations of elliptic type in two or more variables. While the ideal of using the positive-definite character of an integral as the source of function-theoretic inequalities is certainly not new, the present paper attempts to give a more or less systematic survey of the type of inequality obtainable in this way. (Received August 4, 1952.)

645*t*. W. V. Parker and B. Evans Mitchell: *Elementary divisors of certain matrices.*

If $f(x)$ and $g(x)$ are polynomials and P and Q are matrices such that $(P-Q)f(P) = (P-Q)g(Q) = 0$, then P and Q have identical elementary divisors except for those associated with characteristic roots which are roots of either $f(x) = 0$ or $g(x) = 0$. It is shown that two theorems of Flanders (Proceedings of the American Mathematical Society vol. 2 (1951) pp. 871–874) follow as a consequence of this. (Received July 14, 1952.)

646*t*. A. M. Peiser: *An addition formula for spherical harmonics.*

Let $\mathbf{r}$ denote a three-dimensional vector having magnitude r, polar angle θ, and azimuth angle ϕ. A spherical harmonic of $\mathbf{r}$ is a function of the form $F_n^m(\mathbf{r}) = r^n P_n^m(\cos\theta)\exp(im\phi)$, $|m| \leq n$. It is shown that if $\mathbf{r}_1 = \mathbf{r}_2 + \mathbf{r}_3$, then $F_n^m(\mathbf{r}_1) = \sum_{k=0}^{n}\sum_{j=-k}^{k} C_{n+m,j+k} F_k^j(\mathbf{r}_2) F_{n-k}^{m-j}(\mathbf{r}_3)$. Addition formulas of this type are useful in the evaluation of integrals involving products of spherical harmonics of different vectors. Such integrals have been encountered by the author (with S. Golden) in the quantum theory of chemical kinetics. (Received May 16, 1952.)

647*t*. L. L. Pennisi: *A class of linear integro-differential boundary value problems.*

This paper concerns a system of linear integro-differential equations of the first order (1) $\mathcal{L}_i(y) = y_i'(x) - \sum_{j=1}^{n} A_{ij}(x)y_j(x) - \int_a^b \sum_{j=1}^{n} C_{ij}(x, t)y_j(t)dt = 0$, together with linear homogeneous boundary conditions of the form (2) $s_i(y) = \sum_{j=1}^{n} [M_{ij}y_j(a) + N_{ij}y_j(b)] = 0$ $(i, j = 1, \cdots, n)$. We shall give a new construction of the Green's matrix for the incompatible system (1), (2) which is an improvement over the method of Jonah (Bull. Amer. Math. Soc. Abstract 36-3-54) in that it is not necessary to introduce an arbitrary matrix. We also study a special class of integro-differential systems which is linear in a characteristic parameter, and termed definitely self-conjugate adjoint. For this system all the characteristic values are real, and the characteristic solutions may be chosen normed and orthogonal. Also, certain expansion theorems of this system, in terms of the characteristic solutions, are established by showing that our system is equivalent to an integral equation of the type to which the results of Reid (Trans. Amer. Math. Soc. vol. 36 (1931) pp. 475–485) are readily extensible. (Received July 2, 1952.)

648. L. L. Pennisi: *Construction of the Green's matrix for an incompatible integro-differential system.*

This paper is concerned with the construction of the Green's matrix for an incompatible system of linear integro-differential equations of the first order (1) $\mathcal{L}_i(y) = y_i'(x) - \sum_{j=1}^{n} A_{ij}(x)y_j(x) - \int_a^b \sum_{j=1}^{n} C_{ij}(x, t)y_j(t)dt = 0$, together with linear homogeneous boundary conditions of the form (2) $s_i(y) = \sum_{j=1}^{n} [M_{ij}y_j(a) + N_{ij}y_j(b)] = 0$ $(i, j = 1, \cdots, n)$. The system (1), (2) is called incompatible when it admits only the trivial solution $y(x) \equiv 0$. In an unpublished dissertation Jonah (Bull. Amer. Math. Soc. Abstract 36-3-54) has proved the existence of a Green's matrix for a boundary value problem associated with a system of integro-differential equations of the first order. We shall give a new construction of the Green's matrix for the incompatible system (1), (2) which is an improvement over the method given by Jonah in the sense that it is not necessary to introduce a matrix $B(x)$ which is to a large extent arbitrary and on which the construction depends. Our new construction is based upon the following result. Let D be a matrix whose first row is $V_{\alpha\beta}$, $V_{\alpha\beta}$ and whose second row is $Z_{\alpha p}$, $W_{\alpha\beta}$, p, $q=1, \cdots, r$; $\alpha=1, \cdots, r$. The homogeneous system (1), (2) is compatible if and only if D is singular, where the elements of D are given by well-defined integral functions. In fact, the number of linear independent solutions of (1) and (2) is precisely $r+R-\rho$, where ρ is the rank of D. (Received July 11, 1952.)

649. Pasquale Porcelli: *Concerning uniform completeness of sums of reciprocals of linear functions.*

Suppose $F[a, b]$, $0 \leqq a < b$, is the normed linear space of continuous (complex-valued) functions on $[a, b]$ with $\|f\| = \max_{x \in [a, b]} |f(x)|$, $\{k_p\}_{p=1}^{\infty}$ numbers distinct from one another and from 0, and, if $S = \{(1+k_p x)^{-1}\}_{p=1}^{\infty}$ and $S \subset F[a, b]$, $M[a, b]$ the closed linear manifold in $F[a, b]$ generated by S. If $k_p \notin [-\infty, -1]$, then each of the conditions (1) $1 \in M[0, 1]$ and (2) $\sum \{1 - |k_p^{-1}[2+k_p-2(1+k_p)^{1/2}]|\} = \infty$ is necessary and sufficient for $M[0, 1] = F[0, 1]$ (van Herk, Compositio Mathematica vol. 9 (1951) pp. 1–79, proved (2) sufficient if $\{k_p\}_{p=1}^{\infty}$ is an unbounded increasing sequence of positive numbers). If $R(k_p) \geqq 0$, then (i) $a > 0$ implies $M[a, b] = F[a, b]$, (ii) $\epsilon > 0$ and $M[0, \epsilon] = F[0, \epsilon]$ imply $M[0, b] = F[0, b]$ for $b > 0$, and (iii) $\epsilon > 0$ and $M[0, 1] = F[0, 1]$ imply there exist $\{t_s^{(n)}\}_{s=0}^{n}$ in $[0, \epsilon]$ and $f_n(x) = \sum_{p=0}^{n} c_p^{(n)}(1+k_p x)^{-1}$, such that $f_n(t_s^{(n)}) = 1$, $\|f_n(x) - 1\| \to 0$ as $n \to \infty$, in $[0, b]$, $b > 0$. If $f \in F[a, b]$, f real, and $\epsilon > 0$, then there exist Stieltjes continued fractions F, G with all poles in $[-\infty, -1]$ such that $\|f - (F - G)\| < \epsilon$. (Received July 9, 1952.)

650*t*. M. H. Protter: *An existence theorem for the Tricomi problem.*

Consider the equation $K(y)u_{xx} + u_{yy} = 0$ where $K(y)$ is a monotone increasing function of y with a continuous third derivative. Suppose $K(0) = 0$ and $K'(0) \neq 0$. Let Γ be an arc in the upper half-plane with end points on the x-axis, and let γ_1 and γ_2 be the characteristics emanating from these end points which intersect; let D denote the domain bounded by Γ, γ_1, and γ_2. The boundary value problem in which values are assigned on Γ and γ_1 is solved for the domain D with certain restrictions on the nature of Γ. The method of proof makes use of the fundamental solution of the equation $yu_{xx} + u_{yy} = 0$ and a modified form of the results of Gellerstedt. A uniqueness theorem previously established by the author is essential. Since the Chaplygin equation is of the above form this answers in part a question raised by Frankl concerning the transonic flow out of a jet. (Received May 16, 1952.)

651. Gustave Rabson: *A note on the convergence of fourier series on compact groups.*

Theorem: On any compact group not of dimension zero there is a continuous function with a non-absolutely convergent fourier series. It is a consequence of a formula of André Weil (*L'intégration dans les groupes topologiques et ses applications*, 1940, p. 87) that if a continuous central function F on a compact group G has a fourier series which converges to it absolutely at the identity, then the restriction of F to any closed subgroup will also have this property. On any compact Lie group, not of dimension zero, there is a circular subgroup on which may be chosen a continuous even function whose fourier series diverges at the identity. This function may then be extended to a function on the group with the desired properties. If G is not a Lie group it may be assumed to be connected and one may consider any nontrivial unitary representation with kernel K. Then G/K is a Lie group of dimension at least one. We may then construct a function with the required properties on G/K. This induces a function on G, in the natural way, which also has the required properties. (Received June 9, 1952.)

652. H. J. Reiter: On a property of Fourier transforms.

Let G be a locally compact abelian group. The following theorem is proved (for the notation, cf. Bull. Amer. Math. Soc. Abstracts 58-3-139, 140, 141): if $\hat{\mathfrak{Z}}_I$ is denumerable and the elements of $\hat{\mathfrak{Z}}_I$ are independent, then the quotient-algebra L^1/I is isomorphic and isometric with the Banach algebra of all complex-valued functions $c(\hat{\mathfrak{x}})$ defined and continuous on $\hat{\mathfrak{Z}}_I$ and vanishing at infinity in case $\hat{\mathfrak{Z}}_I$ is not compact, with norm $\|c(\hat{\mathfrak{x}})\| = \max |c(\hat{\mathfrak{x}})|$, $\hat{\mathfrak{x}} \in \hat{\mathfrak{Z}}_I$. In particular: for any such $c(\hat{\mathfrak{x}})$ there is an $f(x) \in L^1$ such that the Fourier transform $\hat{f}(\hat{\mathfrak{x}}) = c(\hat{\mathfrak{x}})$ for all $\hat{\mathfrak{x}} \in \hat{\mathfrak{Z}}_I$. The proof is based on the results announced in the abstracts mentioned above, and on the following lemma: under the conditions of the theorem, if ϕ is orthogonal to I, then ϕ has an absolutely convergent Fourier series. (Received July 21, 1951.)

653. P. C. Rosenbloom: *Linear partial differential equations of parabolic type with constant coefficients.*

Equations of the form $u_t = Lu + f$ are studied, where L is a linear differential operator of elliptic type of arbitrary even order with constant coefficients. By the use of the method of Ladyzhenskaya, Mat. Sbornik (1950), for estimating the fundamental solution, her uniqueness theorem is sharpened, and various properties of the Poisson integral representation of the solution are obtained. The approach to the solution of the equation $Lu + f = 0$ as $t \to +\infty$ is also studied, and some uniqueness theorems for such equations of elliptic type are obtained. (Received June 9, 1952.)

654t. H. L. Royden: *Some counterexamples in the classification of open Riemann surfaces.*

Recently Ahlfors and the author constructed a Riemann surface of hyperbolic type which possessed no nonconstant harmonic functions with a finite Dirichlet integral. In this paper we explore some of the consequences of this example and construct a Riemann surface on which the spaces HD and HBD have dimension n. A bounded Riemann surface is given which has no HD functions on it which vanish on the relative boundary while it has a nonconstant HD function on it whose normal derivative vanishes on the relative boundary. Finally, a Riemann surface is constructed which admits of a nonconstant bounded harmonic function, but no nonconstant harmonic functions with a finite Dirichlet integral. This shows that the classes O_{HD} and O_{HD} are distinct. (Received June 27, 1952.)

655. Walter Rudin: *Analyticity, and the maximum modulus principle.*

The author investigates the extent to which the maximum modulus principle characterizes the class of analytic functions. Let K be a compact subset of the plane, bounded by a simple closed curve C, whose interior is D. An algebra A of complex-valued functions continuous on K is said to be a maximum modulus algebra on K if for every $f \in A$ there is a point $z_0 \in C$ such that $|f(z)| \leq |f(z_0)|$ for every $z \in K$. The principal result of the paper is the following theorem: Suppose (1) A is a maximum modulus algebra on K, (2) A contains a nonconstant function f which is analytic in D, and (3) A contains a function g which is schlicht on K (g may or may not coincide with f); then every member of A is analytic in D. The question whether (3) may be omitted from the hypotheses is not settled. However, if (1) is replaced by the stronger requirement that every $f \in A$ satisfy the maximum modulus principle with respect to every subdomain of D, then (3) may be omitted. (Received July 9, 1952.)

656. James Sanders: *The behaviour, near the boundary, of the Green's function of second-order partial differential equations.* Preliminary report.

Let $P(x, y)$ be a point in a domain D bounded by a contour C with outer normal n. Let $G(P, Q)$ be the Green's function for $\Delta_p U = U_{xx} + U_{yy} = 0$, in D, with singularity $-\log r$ at $P = Q$. r is the distance from P to Q. Then, when $P = p$ on C, (1) $\partial G(p, Q)/\partial n_p = -2\partial \log r/\partial n_p + F_1(p, Q)$. P. Lévy's result (Acta Math. vol. 42 (1920)) that $F_1(p, Q)$ is analytic for all $p \in C$, $Q \in C+D$, provided C is analytic, is proved by a new direct method. It is further shown that if C has continuous derivatives of order m, $F_1(p, Q)$ has continuous derivatives of order $(m-4)$ with respect to p and of order $(m-3)$ with respect to Q for $p \in C$, $Q \in D+C$. Next let $G(P, Q)$ be the Green's function, in D, of $U_{xx} + U_{yy} + a(x, y)U_x + b(x, y)U_y + c(x, y)U = 0$ which has a fundamental solution $L(P, Q) = A(x, y) \log r + B(p, Q)$: then, (2) $\partial G(p, Q)/\partial n_p = -2\partial L(p, Q)/\partial n_p - (\alpha \cdot n)_p L(p, Q) + F_2(p, Q)$ where α is the vector with components (a, b). If C is three times continuously differentiable, $F_2(p, Q)$ is continuous for $p \in C$, $Q \in E+C$. The coefficients a, b, c have been assumed analytic in $D+C$, for the sake of simplicity. (Received July 21, 1952.)

657. J. A. Schatz: *Representation of algebras with an involution.*

A *-algebra A is a complex Banach algebra with an identity element and a norm-preserving conjugate linear anti-automorphism $a \rightarrow a^*$ of period two. A BIP-space is a complex Banach space with an inner product satisfying $(x+\lambda y, z) = (x, z) + \lambda(y, z)$, $(x, y) = \overline{(y, x)}$, $(x, y) = 0$ for all y only if $x = 0$, and $|(x, y)| \leq \|x\| \|y\|$. A real functional on A is a linear functional satisfying $f(a^*) = \overline{f(a)}$ for all $a \in A$. Using methods similar to those of Gelfand and Neumark for C^*-algebras [Matematičeskiĭ Sbornik N.S. vol. 12 (1943) pp. 197–217], the following results can be obtained. Let S be a set of self-adjoint elements of A such that for each $s \in S$ there is a real functional f_s with $\|f_s\| = 1$ and $f_s(s) = \|s\|$. Then there is a BIP-space X and a norm-reducing *-homomorphism $a \rightarrow T_a$ of A onto an algebra of bounded operators on X such that $\|T_s\| = \|s\|$ for each $s \in S$. By choosing S as the set of all self-adjoint elements of A, we obtain a *-isomorphism. If A is a B^*-algebra, it is an immediate consequence that $a \rightarrow T_a$ is norm-preserving. The Gelfand-Neumark result can be obtained by choosing S appropriately and using results of Lax and Kaplansky. (Received June 26, 1952.)

658. Augusta L. Schurrer: *On the location of the zeros of the derivative of rational functions of distance polynomials.*

Let $R(x, y, z) = \prod_{k=1}^{q} F_k(x, y, z) / \prod_{k=q+1}^{p} F_k(x, y, z)$, $0 \leq q \leq p$, represent a rational function of the distance polynomials $F_k(x, y, z) = c_k \prod_{j=1}^{r} [(x-x_{jk})^2 + (y-y_{jk})^2 + (z-z_{jk})^2]^{m_{jk}}$, $m_{jk} > 0$, where $\sum_{j=1}^{r} m_{jk} = n_k$, the "degree" of F_k. Let $R'(x, y, z) = |\nabla R|^2/4R$ be the "derivative" of R in the sense of Nagy [Bull. Amer. Math. Soc.

vol. 55 (1949) pp. 329–342]. It is shown that the following theorem due to Marden [Bull. Amer. Math. Soc. vol. 42 (1936) pp. 400–405] can be extended to the class of distance polynomials: If $F_j(x, y, z)$ has all of its zeros in the spherical region $\sigma_j S_j(v) \leq 0$ where $\sigma_j^2 = 1$, $S_j(v) = (v - c_j)^2 - r_j^2$, $r_j > 0$, and v is the position vector of the point $P(x, y, z)$, it follows that every finite zero of R' either lies in one of the given spherical regions or in the region $(E(v)/\prod_{j=0}^{p} S_j(v)) = \sum_{j=0}^{p} NN_j/S_j(v) - \sum_{j=0,k=j+1}^{p} N_jN_kT_{jk}/S_j(v)S_k(v) \leq 0$ bounded by the p-spherical $2p$-ic surface $E(v) = 0$. Using the preceding theorem it is possible to extend to the class of distance polynomials some well known results, due to Bôcher and Walsh, concerning the location of the zeros of the derivative of a rational function. A Gauss-Lucas, a Laguerre, and a Walsh cross-ratio type theorem are also developed. (Received June 20, 1952.)

659*t*. Harold Shniad: *Convexity properties of integral means of analytic functions.*

For the class of functions analytic in an open circle $|z| < R$, the convexity properties of the circumferential means of order t over circles $|z| = r < R$ are studied as functions of r. It is known that these means of order t are convex for every analytic function providing t is zero or of the form $2/k$, where k is a positive integer. The following results are obtained. The mean of order four of all analytic functions is convex. If t is a number greater than 5.66, then there is an analytic function whose mean of order t is not convex. (Received July 21, 1952.)

660. Edward Silverman: *An intrinsic inequality for Lebesgue area.*

The measure of a parallelogram is not less than the product of the distances between the two pairs of opposite sides. A corresponding inequality for Lebesgue area permits the generalization to area of certain properties of arc length. In particular if one considers only functions of finite Lebesgue area, then (i) Lebesgue area is lower semi-continuous with respect to point-wise convergence, and (ii) if one considers quasi-harmonic functions in a square (continuous and harmonic on each of a finite number of Jordan regions into which the square is decomposed) to be generalizations of quasilinear functions on an interval, then the connection between arc length and inscribed polygons carries through to Lebesgue area and inscribed quasi-harmonic functions. The basic inequality, suggested by Choquet, is another approach to a problem considered by Cesari and Fullerton. (Received July 21, 1952.)

661*t*. C. T. Taam: *On linear differential equations with small perturbations.*

Let $y(x)$ be any solution of (1) $(p(x)y')' + q(x)y = 0$, where $p(x)$ and $q(x)$ are complex-valued, defined for $x \geqq 1$, and $p(x)^{-1}$ and $q(x)$ belong to $L(1, R)$ for every large R. Let $g(x)$ be real-valued having a positive lower bound in $1 \leqq x < \infty$. If $\int_1^x |p(t)|^{-1}dt = O(g(x))$ and if $g(x)q(x)$ belongs to $L(1, \infty)$, then it is proved in this paper that $y(x) = O(g(x))$ as $x \to \infty$. From this we prove that for some constants A and B, $y(x) - Ax - B$ tends to zero as $x \to \infty$ provided there exists a constant a such that $(p(x)^{-1} - a)$ and $x^2q(x)$ belong to $L(1, \infty)$. Given the weaker conditions $(p(x)^{-1} - a)$ and $xq(x)$ belong to $L(1, \infty)$ for some a, then for any constant C there exists a solution $y^*(x)$ of (1) which tends to a limit C as $x \to \infty$. On the other hand if $p(x)^{-1}$ and $q(x)$ belong to $L(1, \infty)$, then $y(x)$ tends to a limit as $x \to \infty$. These results are then extended to the general linear differential equations of the second order, of the third order, and with the help of the Phragmén-Lindelöf Theorems, to the complex domain. The main tool we use is an inequality due to R. Bellman (Duke Math. J. vol. 10 (1943) pp. 643–647). (Received July 10, 1952.)

662. W. J. Thron: *A class of meromorphic functions having the unit circle as a natural boundary.*

In the continued fraction (1) $1+K(d_n z^{\alpha_n}/1)$, where the α_n are positive integers, let (2) $\lim |d_n|^{1/\alpha_n}=1$ and $\lim \alpha_n=\infty$. Then (1) converges to a meromorphic function $f(z)$ for all $|z|<1$. Set $k_n=\sum_{k=1}^{n+1}\alpha_k$. It is shown in this paper that if in addition to conditions (2) the continued fraction (1) satisfies $\limsup k_n/k_{n-1}=\infty$, then the function $f(z)$ to which (1) converges for $|z|<1$ has the unit circle as a natural boundary. The methods employed in the proof of this result are elaborations of procedures used by the author in a recent paper (Proc. Nat. Acad. Sci. U.S.A. vol. 36 (1950) pp. 51–54). (Received July 18, 1952.)

663. E. W. Titt, Roger Osborn, A. J. Kainen, W. C. Long, L. G. Worthington, and W. S. McCulley: *On a theory of the linear second order partial differential equation with n independent variables.*

In contrast to the methods of Hadamard and M. Riesz, the method developed here is more closely allied to that of Volterra. This paper treats the normal hyperbolic interior, although the method has been applied to other problems in lower dimensional cases. Whereas Volterra integrates a potential along a line, here a potential is integrated over a hypersurface. For $n\geqq 4$ the initial potential is not the Green potential but an extension of a quantity used by Volterra outside the cone in $n=3$. In $n=2, 3$, the initial potential is sometimes of the latter type and sometimes Green's. The $(n-1)$-tuple integral when reduced to a single integral is in the nature of a transform of the initial potential, the kernel of which apparently varies only with the dimension. It turns out that this kernel is related to Legendre and Tschebyscheff polynomials in the even and odd dimensions, respectively. This paper concludes with a discussion of the integrating factor which is obtained by applying a certain number of differential operators to the above mentioned transform. With further differential operations to be performed after the Green's theorem stage, the final formula may involve a potential distinct from the initial one. (Received July 21, 1952.)

664. C. J. Titus and G. S. Young: *An extension theorem.* I.

Let $P(x)$ and $Q(x)$ be real polynomials of degree n and $n-1$ respectively where (a) the first coefficients are positive, (b) the roots are real and simple, and (c) the roots of Q separate the roots of P. Let $f(t)$ be an $n+1$ times differentiable function of period 2π such that $(f')^2+(f)^2>0$. Corresponding to the polynomials P and Q are polynomic differential operators. With these operators define $u=P[f]$, $v=Q[f]$. There exists then a mapping $I: D\rightarrow E^2$, where D is the closed unit disc, where I is a light interior on Int D, where Bdy $D=\{(x, y)|x=\cos t, y=\sin t\}$ and where $I(x(t), y(t))=(u, v)$, $0\leqq t\leqq 2\pi$; i.e., every curve of the form (u, v) is the point-wise image of the boundary of a light interior mapping of the unit disc. (Received July 21, 1952.)

665*t*. C. J. Titus: An extension theorem. II.

A class of matric differential operators is defined which yields an extension theorem similar to that in the previous abstract. The domain of definition in this case is not simply connected. (Received July 21, 1952.)

666*t*. Leonard Tornheim: *Derivatives of functions convex with respect to n-parameter families.*

Let F be an n-parameter family of functions on the interval $a\leqq x\leqq b$; i.e., through every n points, no two of which have the same abscissa, there is a unique member of F. Let g be a function convex with respect to F; i.e., each member of F intersects g in at most n points. If all functions of F have $n-1$ continuous derivatives on $a<x<b$, then g has $n-2$ derivatives. This generalizes an earlier result for $n=3$. (Received July 21, 1952.)

667*t*. J. L. Ullman: *The eigenvalues of a Toeplitz form.*

Let $f(\theta)$ be a Lebesgue integrable function satisfying $0<m\leqq f(\theta)\leqq M$, and having Fourier series $\sum_{-\infty}^{\infty} c_n z^n$, $z=\exp(i\theta)$, $c_n=(1/2\pi)\int_0^{2\pi} z^{-n}f(\theta)d\theta$. Let $\lambda_0^n, \lambda_1^n, \cdots, \lambda_n^n$ be the eigenvalues arranged in ascending order of magnitude associated with the Toeplitz matrix $T_n(f)=\left|c_{q-p}\right|_0^n$ whose determinant is $D_n(f)$. Carathéodory (Rendiconti del Circolo Matematico di Palermo vol. 32 (1911)) proved that $\lim_{n\to\infty}\lambda_n^n=M$ and $\lim_{n\to\infty}\lambda_0^n=m$. His result was essentially broadened by Szegö (Math. Zeit. vol. 76 (1920)) who showed that the eigenvalues had the same distribution as the equally spaced ordinates of $f(\theta)$. This result was obtained by first determining the limit of $(D_n(f))^{1/n}$. The purpose of this paper is to obtain the distribution directly. Formal manipulations with infinite Toeplitz matrices $T(f)$ yield $(T(f))^k=T(f^k)$. It is shown that the following exact relationship holds, namely $\lim_{n\to\infty}\mathrm{Tr.}\,(T_n(f))^k/(n+1)=\lim_{n\to\infty}\mathrm{Tr.}\,(T_n(f^k))/(n+1)=(1/2\pi)\int_0^{2\pi}f(\theta)^k d\theta$. Since $\mathrm{Tr.}\,(T_n(f))^k=\sum_0^n(\lambda_p^n)^k$, this implies that for any continuous $F(\lambda)$, $\lim_{n\to\infty}\sum_0^n(F(\lambda_i^n))/(n+1)=1/2\pi\int_0^{2\pi}F(f(\theta))d\theta$, which is the mathematical statement of Szegö's result on distributions quoted above. (Received July 14, 1952.)

668*t*. H. S. Wall: *Concerning continuous continued fractions and certain systems of Stieltjes integral equations.*

The idea of continuous continued fraction (P. Adam, Revista Matematica vol. 11 (1951) pp. 180–190) is developed and use to prove (1) if each of f and g is a continuous fraction in an interval $[a, b]$ and g of bounded variation in $[a, b]$, ${}_0I_s^t=\int_s^t f\,dg$, ${}_nI_s^t=\int_s^t {}_{n-1}I_s^u\,dg(u)$, $n=1, 2, 3, \cdots$, then the series $\sum_{p=0}^{\infty}{}_pI_s^t$ converges absolutely and uniformly for $a\leqq t, x\leqq b$, (2) if each of $f_1, \cdots, f_n$ is a function which is continuous and of bounded variation in $[a, b]$, then there exists only one $n\times n$ matrix A of functions such that $A_{ij}(x, t)=\delta_{ij}+\int_x^t A_{i+1,j}(s, t)df_i(s)$, $i=1, 2, \cdots, n$, $a\leqq x, t\leqq b$, where $A_{n+1,j}=A_{1,j}$. The matrix A has the properties: $A(x, x)=I$, $A(x, t)=A(x, r)A(r, t)$. If n is 2, $[A_{11}(x, t)z+A_{12}(x, t)]/[A_{21}(x, t)z+A_{22}(x, t)]$ is the value of the continuous continued fraction $\phi_x^t(f_1, f_2, z)=\lim_{\delta\to 0}[b_0+K_{p=1}^{2m+2}1/b_p]$, where δ is the norm of an ordered subdivision $\{x_i\}_{i=0}^{m+1}$ of $[x, t]$, $b_{2p}=f_1(x_{p+1})-f_1(x_p)$, $b_{2p+1}=f_2(x_{p+1})-f_2(x_p)$, $p=0, 1, \cdots, m$, $b_{2m+2}=z$. (Received June 10, 1952.)

669*t*. Alexander Weinstein: *A new solution of Cauchy's problem for the wave equation and the equation of Poisson.*

Let k be a real parameter, $k\neq -1, -3, -5, \cdots$, and let x denote a system of m variables $x_1, \cdots, x_m$. Let $u^k(x, t)$ satisfy the equation (1) $\Delta u^k=u_{tt}+kt^{-1}u_t^k$. For $k\geqq m-1$, the Cauchy problem for (1) with given $u^k(x, 0)$ and $u_t^k(x, 0)=0$ is solved by Poisson's method of mean values. For $k<m-1$, the Cauchy problem is reduced to the previous case by a repeated application of the recursive formulas $u_t^k=tu^{k+2}$, $(k+1)u^{k+2}(x, 0)=\Delta u^k(x, 0)$, $u_t^{k+2}(x\cdot 0)=0$. The case $k=0$ corresponds to the wave

equation which is solved by the new and simple formula $u^k(x, t) = 2^{1-n}\Gamma^{-1}(n) \cdot \int_0^t (t^2-r^2)^{n-1} r u^{kn}(x, r)dr + R_{n-1}$, where the integer n satisfies the inequality $2n \geq m-1$. u^{kn} is the Poisson mean value of $\Delta^n u(x, 0)$ divided by $1\cdot 3\cdot 5 \cdots (2n-1)$. The term R_{n-1} is given by the formula $R_{n-1} = \Gamma^{-1}(2n-1)t^{2n-2}\Delta^{n-1}u^k(x, 0) + \cdots + u^k(x, 0)$. In the exceptional cases $k = -1, -3, -5, \cdots$, Cauchy's problem can be solved if and only if $u^k(x, 0)$ is a polyharmonic function. (Received June 3, 1952.)

670. John Wermer: *On ideals in a class of commutative Banach algebras.*

It is known that the group algebra L of a locally compact abelian group has the following properties: (1) L is regular, (2) the elements of L which lie in all maximal ideals except for a compact set are dense, (3) there exists a set of linear isometric operators T on L commuting with multiplication such that any hyperplane of L invariant under all T in the set is an ideal, (4) given $f \in L$ and $\epsilon > 0$ there is some g with $\|fg-f\| < \epsilon$. The author shows that in any commutative semi-simple Banach algebra satisfying (1), (2), (3), and (4), a closed ideal the boundary of whose hull has no nonempty perfect subsets equals the intersection of maximal ideals containing it. (The hull of an ideal is the set of maximal ideals containing it.) For group algebras similar results are known. (Received July 21, 1952.)

671. L. C. Young: *Discontinuous parametric surfaces and their BL representations.*

Dirichlet integral methods [Young, Trans. Amer. Math. Soc. vol. 64 (1948) pp. 317–335] lead to admitting BL representations [Nikodým, Fund. Math. vol. 21 (1933) pp. 129–150]. Closure as well as compactness are then attained. An essential tool is again harmonic interpolation (which does not increase the Dirichlet integral of a BL function: this is little known). There is a corresponding representation theorem for generalized surfaces [Young, Bull. Soc. Math. France vol. 79 (1951) pp. 59–84]. (Received July 21, 1952.)

APPLIED MATHEMATICS

672t. H. W. Becker: *On Richardson's compensation theorem.*

In the classic compensation theorem, the compensating EMF is $-I\Delta v$, where I is the current originally in the variable branch v, acting in the circuit of present impedance. Expressing this abstractly, in the case of one source, if $\Delta_v Y_{sr}$ is the change in the circuit transfer admittance from source s to receiver r while the branch v impedance increases by Δv, f_{sv} and f_{vr} are the transfer factors from s to v and v to r, and θ and θ' are the network determinants before and after the change, then $\Delta_v Y_{sr} = \mp \Delta v \cdot f_{sv} f_{vr}/\theta\theta' = \mp \Delta v \cdot Y_{sv} Y_{v'r} = \mp \Delta v \cdot Y_{sv'} Y_{vr}$, where the sign is $-$ or $+$ according as v is effectively in series with s and r or in shunt with them. This shows that the theorem could also be phrased in terms of present current flowing in original circuit. Why not work with currents and admittances both taken at the same epoch? This was done, by D. E. Richardson, *Electrical network calculations*, New York, 1946, p. 150, in his "Equivalent EMF of Resistance Change Theorem." Succinctly, this is $\Delta_v Y_{sr} = \mp \{Z_v \Delta v/(Z_v + \Delta v)\} f_{sv} f_{vr}/\theta^2 = \mp (Z_v \| \Delta v) \cdot Y_{sv} Y_{vr}$, where Z_v is the circuit impedance as originally seen from a point in v. Analogously, referring all data to the present epoch, $\Delta_v Y_{sr} = \mp \{Z_{v'} \Delta v/(Z_{v'} - \Delta v)\} f_{sv} f_{vr}/\theta'^2 = \mp (Z_{v'} \|' \Delta v) Y_{sv'} Y_{v'r}$. Thus the price paid for simultaneity is the additional factor $(1 \pm \Delta v/Z_{v \text{ or } v'})^{-1}$. (Received July 21, 1952.)

673t. John DeCicco: *Magnetic fields of force in Euclidean space of n dimensions.*

To extend the concept of magnetic fields of force to Euclidean space E_n of n dimensions, the following form of Coulomb's law is adopted. The force between two magnetic poles is directly proportional to their strengths and inversely proportional to the $(n-1)$ power of the distance between them. If a magnetized body is considered to be a Riemannian manifold V_N of N dimensions where $1 \leqq N \leqq n$ embedded in the Euclidean space E_n together with an intensity of magnetization vector field defined in V_N, then the magnetic attraction produced by this body defines a conservative and solenoidal field of force. The corresponding potential function $U(x)$ is harmonic. The potential $U(x)$ due to a normally magnetized shell consisting of a V_{n-1} together with an intensity of magnetization vector field normal to V_{n-1} is proportional to the mean value of the intensity of magnetization relative to the solid angle subtended by V_{n-1} at the given point x. Also considered are the relations between the potentials $U(x)$ and $V(x)$ due to a uniformly magnetized body V_N and to a uniform charge of electricity throughout the region occupied by V_N. Finally applications to magnetized spheres in E_n are given. (Received June 18, 1952.)

674. R. J. Duffin and D. H. Shaffer: *On the modes of vibration of a ring-shaped plate.*

As is well known nodal lines cannot occur in the gravest mode of vibration of a membrane. The question has been raised by Weinstein whether the same statement is true for the vibrations of a thin, flat plate clamped on its boundary. A counterexample is considered here consisting of a plate clamped on two concentric circles whose radii are in the ratio K. For K sufficiently large it is found that the mode of vibration having a single nodal diameter and no nodal circles has the lowest frequency. The critical value for K is approximately 715 where it is found that the frequencies of the one nodal line mode and the mode with no nodal lines are equal. In the limit as K becomes large the frequency of the mode with no nodal lines approaches a value which is approximately 7% greater than that of the limiting fundamental frequency. Received July 18, 1952.)

675t. F. D. Faulkner: *The problem of Oberth.*

The "Problem of Oberth" is the problem of choosing a thrust function in order to attain maximum altitude with a rocket. It was shown by B. Hamel (*Zeitschrift für Angewandte Mathematik und Mechanik* vol. 7 (1927) pp. 451–452) that the problem has a solution. By changing the independent variables from the time to the distance along the path, and by a second elementary substitution, the proof of the existence of a solution is made simpler. It is shown that a single function is maximized at the upper end of the curve by the extremals. Also that extrema of point functions of distance along the path, velocity, and fuel consumption at the upper end of the curve can generally be obtained as the consequence of the maximization of that one function. (Received July 21, 1952.)

676. Bernard Friedman: *Characteristic roots of compound matrices.*

The use of direct products enables us to find the characteristic roots of certain matrices whose elements are also matrices. The following result is used: Let $A_1, A_2, \cdots, A_k$ be $n\times n$ matrices, $B_1, B_2, \cdots, B_k$ be $m\times m$ matrices, and let $C=A_1\times B_1+\cdots+A_k\times B_k$, $A_1\times B_1$ representing the direct product of A_1 and B_1. If

$A_1, \cdots, A_k$ have a common eigenvector x such that $A_1x=\lambda_1x, \cdots, A_kx=\lambda_kx$, then every eigenvalue of $\lambda_1B_1+\cdots+\lambda_kB_k$ is an eigenvalue of C. We discuss also the characteristic roots of the $kn\times kn$ matrix which has the matrices $A_1, A_2, \cdots, A_k$ as the elements of its first row while in the jth row the same matrices are repeated in cyclic order but shifted jr places to the right, r being a fixed integer. (Received July 18, 1952.)

677. R. M. Hayes: Iterative methods of solving linear problems on Hilbert space.

Many iterative procedures for solving linear problems, such as the Rayleigh-Ritz method, are allied to the method of "expanding subspaces." Thus, given a Hilbert space A and a linear equation on that space, the equation is successively solved on each of a sequence $\{A_k\}$ of subspaces of A satisfying the conditions (1) A_k is contained in A_{k+1} and (2) $A=\sum A_k$. This paper is concerned with the proof of the convergence of such iterative procedures for operators expressible as the sum of a positive definite operator and a completely continuous operator. In particular, one such procedure is the so-called "conjugate-gradient method," wherein the subspaces are successively determined by a sequence of conjugate vectors. Pertinent properties of the successive approximations in this method are considered; estimates on rates of convergence are given; and applications are shown. (Received July 15, 1952.)

678*t*. Stanley Katz and A. M. Peiser: *A system of nonlinear partial differential equations arising in heat transfer.*

Consider the differential equations $f(u)\partial u/\partial x=u-v=g(v)\partial v/\partial y$ in $x>0$, $y>0$, with boundary conditions $u(0+, y)=\phi(y)$, $v(x, 0+)=\psi(x)$, where ϕ, ψ, f, g are continuous and $|f|\geqq a>0$, $|g|\geqq a>0$. It is shown that there exist unique solutions, continuous in $x\geqq 0$, $y\geqq 0$. For the corresponding difference equations $f(U_{m,n})(U_{m+1,n}-U_{m,n})/h=U_{m,n}-V_{m,n}=g(V_{m,n})\cdot(V_{m,n+1}-V_{m,n})/k$ with boundary conditions $U_{0,n}=\phi(nk)$, $V_{m,0}=\psi(mh)$, it is proved that $U_{m,n}$ and $V_{m,n}$ converge to $u(x, y)$ and $v(x, y)$ respectively as the interval lengths h, k approach zero. In the linear case, $-f=g=1$, $\phi=1$, $\psi=0$, error estimates are obtained for $u(x, y)-U_{m,n}$ and for $v(x, y)-V_{m,n}$. (Received May 16, 1952.)

679. Franco Modigliani and F. E. Hohn: *Production planning over time: An example of nonlinear programming.*

This paper considers the problem of scheduling production of a given commodity over a sequence of equal, successive periods of time in such a way as to meet initially known demands in these periods while incurring the lowest possible total cost. The problem is solved for the case in which marginal cost in each period is the same, nonnegative, strictly increasing, differentiable function of production in that period only. Economic implications of the results are discussed. (Received July 21, 1952.)

680*t*. T. S. Motzkin and I. J. Schoenberg: *On the relaxation method for linear inequalities.*

Let H_i: $\sum_{j=1}^{n} a_{ij}x_j+b_i\geqq 0$ $(i=1, \cdots, m)$ be a set of closed half-spaces of E_n such that $A=\bigcap H_i$ is nonvoid. Let $p_0\notin A$, and let H_j be a half-space such that dist $(p_0, H_j)=\max_i$ dist (p_0, H_i). Let $|p_0-q|=$ dist (p_0, H_j), $q\in H_j$. Let $0<\lambda\leqq 2$ and let $p_1=p_0+\lambda(q-p_0)$. If $p_1\notin A$ we can repeat the process on p_1, with the same λ, obtaining the point p_2. If $p_2\notin A$ we obtain p_3 and so forth. S. Agmon has shown in a paper to appear in Ann. of Math. that if $0<\lambda<2$, then this process either terminates with a $p_k \in A$, or else it results in an infinite sequence $[p_n]$ which converges to a point on the boundary of A. A new geometric approach to this problem is given which also includes the case of $\lambda=2$. In particular it is shown that if $\lambda=2$ and A is n-dimensional, then the process always terminates. (Received July 21, 1952.)

681*t*. L. E. Payne: *Axially symmetric problems in elasticity. I. The punch problem.*

The usual procedure for solving the axially-symmetric punch problem amounts to finding a biharmonic function which satisfies a pair of dual integral equations. By formulating the problem in a manner analogous to that introduced by A. E. Green [*On Boussinesq's problem and penny-shaped cracks*, Proc. Cambridge Philos. Soc. vol. 45 (1947) pp. 252–257] one can reduce the problem to a determination of an axially-symmetric potential function satisfying certain boundary conditions. By formulating the problem in this way rather than in terms of a biharmonic function, curvilinear coordinates may be more conveniently introduced and consequently the problem is in many cases greatly simplified. It is found that in most cases the necessity of solving a pair of dual integral equations can be eliminated. In particular, the solution to the spherical punch problem, which was approximated by F. Chong [*Indentation of a semi-infinite medium by an axially symmetric rigid punch*, Iowa State College Journal of Science vol. 26 (1952) pp. 1–15], can now be obtained in closed form with little difficulty. (Received July 21, 1952.)

682. Mary H. Payne: *A generalization of Laplace's equation using spinors.*

The solutions of Laplace's equation in spherical coordinates are compared with the solutions of its analogue in spinor coordinates. All solutions of Laplace's equation are solutions of the spinor equation. There are, however, solutions of the spinor equation which are not solutions of Laplace's equation. Among these additional solutions are found functions which might be called "harmonic functions of half integral order," as well as some extra "harmonic" functions of integral order. The spinor coordinates are $S_1=(2r)^{1/2}\cos(\theta/2)e^{i(\phi+\psi)/2}$, $S_2=(2r)^{1/2}\sin(\theta/2)e^{i(\phi-\psi)/2}$, and their complex conjugates. (Received July 18, 1952.)

683. Charles Saltzer: *The second fundamental theorem of electrical networks.*

If a lumped-parameter linear network is represented by branches consisting of admittances in parallel with constant current sources, the mesh method can be replaced by the nodal method. The matrix equations of Kron are established rigorously by topological or graph-theoretical considerations, and conditions for the existence and uniqueness of the solution of the nodal equations are given. (Received July 23, 1952.)

684*t*. P. A. Samuelson: *The exact order of convergence of an accelerated iteration.*

For any sufficiently differentiable iteration, $X_{n+1}=g(X_n)$, with an isolated stationary point $A=g(A)$, the newly-defined accelerated iteration, $X_{n+1}=F(X_n)$, where $F(X)=[Xg(g(X))-g(X)^2]/[X+g(g(X))-2g(X)]$ has the same stationary point A and has local derivatives there with the following properties: (a) for $g'(A)\neq 1$, the first $r-1$ derivatives vanish and $F^{(r)}(A)=g^{(r)}(A)g'(A)[g'(A)^{r-1}-1]/[g'(A)-1]^2$; (b) for $g'(A)=1$, $F'(A)=1-r^{-1}$; and (c) for $g'(A)=0$, the first $2r-2$ derivatives of F vanish and $F^{(2r-1)}(A)=-C_{2r,r}g^{(r)}(A)^2/2r$—where in every case r represents the first nonvanishing higher derivative of g at A. Hence, it follows that F is always a locally stable iteration and if, as in case (b), it does not already converge more rapidly than any locally linear iteration, the accelerated iteration, $F_1(X_n)=[X_nF(F(X_n))-F(X_n)^2]/[X_n+F(F(X_n))-2F(X_n)]$, will surely do so. Unless $g'(A)=1$, it will usually not pay to compute F_2, just as it will usually not pay to compute F in the case where $g'(A)=0$, since in this case g already has convergence of higher than the first order and since $g(g(X_n))$, representing two whirls of g, will have more vanishing derivatives than will F, their number being $r^2-1>2r-2$. (Received May 16, 1952.)

685. Domina E. Spencer: *A Riemannian space for astronomy.*

The interpretation of astronomical data depends on the geometry that is postulated for cosmic space. All the data are obtained at essentially one point in the universe and give no sure indication of relationships at stellar distances. The practical interpretation of the data has ordinarily been made on the assumption of a euclidean metric, though general relativity employs a Riemannian metric with R of the order of 10^{10} lightyears. A study is made of the possibilities of a non-euclidean world, limited by the assumption that space is locally euclidean and isotropic. Masses, densities, and luminous outputs of stars remain unchanged. An investigation of data on binary stars and cepheids shows that if the space constant R of a Riemannian universe is 5 lightyears, Einstein's relativity is unnecessary. Time and space may be regarded as separate and distinct, a universal time scale can be established, velocity of light is constant with respect to the source, and the "expanding" universe ceases to expand. (Received July 9, 1952.)

686t. M. L. Stein: *A Rayleigh-Ritz-like procedure for minimising integrals*. Preliminary report.

In this paper a procedure is presented for constructing the minimum of an integral $I(y)=\int_a^b f(x, y, y')dx$ among the arcs $y=(y_1, \cdots, y_n)$ such that each $y(x)$ $(i=1, 2, \cdots, n)$ is of class c^1 on $a\leqq x\leqq b$ and satisfies certain fixed end conditions. In the algorithm it is assumed that the ith estimate of the minimum y^i has been found. The integral I is then replaced by a suitable quadratic integral $I^*(y^i)$ and the Rayleigh-Ritz method is applied to I^* in order to obtain a variation η^i. The succeeding approximation is defined to be $y^{i+1}=y^i+\eta^i$ and the process is repeated. Conditions are given under which the sequence $[y^i]$ converges uniformly to a strong relative minimum. The minimum is expressed in the form $y_0+\sum_i a_i s^i$ where y^0 is an initial estimate satisfying the end conditions, $[s^i]$ is a set of basic functions in a suitable Hilbert space, and a_i are scalars. (Received June 9, 1952.)

GEOMETRY

687t. Eugenio Calabi and M. A. Rosenlicht: *Complex analytic manifolds without countable base.*

Any Riemann surface is known to have a countable base for its open sets. In this paper we verify a conjecture of Bochner that there exist complex manifolds of complex dimension >1 that are non second-countable by explicitly constructing a large class of such manifolds. Some basic topological properties of these manifolds are developed. (Received June 23, 1952.)

688*t*. R. M. Conkling and D. O. Ellis: *Metric Δ-lattices.*

A lattice L with a third operation which commutes and distributes over join and meet is called a *Δ-lattice*. If L is normed, and Δ-translations are isometries, L is called a *metric Δ-lattice*. It is shown in this paper that a normed lattice admits a third operation to form an associative metric Δ-lattice if and only if it is isometrically lattice isomorphic to the metric lattice of a certain type of lattice of isometric endomorphisms on itself. It is also shown that the class of reduced Gaussian semigroups is algebraically equivalent to a class of suitably restricted associative metric Δ-lattices. (Received May 26, 1952.)

689*t*. David Gale: *On inscribing n-dimensional sets in a regular n-simplex.*

It is proved that every set of diameter 1 in n-space can be inscribed in a regular n-simplex of side $s<(n(n+1)/2)^{1/2}$ but in no smaller simplex. Among the consequences of this result is the fact that every plane set of diameter 1 is the union of 3 sets each of diameter not greater than $3^{1/2}/2$. (Received June 13, 1952.)

690*t*. C. C. Hsiung: *Some curves in Riemannian space.*

In a Riemannian space R_n of n dimensions, let $\overline{C}$ be a neighboring curve corresponding to a given curve C such that the corresponding point $\overline{P}$ on $\overline{C}$ of any point P on C is on the unit 2-normal vector $\xi_{(2)}$ of C at P with an infinitesimal distance ϵ from P. Let $\bar{\xi}_{(2)}'$ be the vector obtained by displacing the unit 2-normal vector $\bar{\xi}_{(2)}$ of $\overline{C}$ at $\overline{P}$ parallelly back to P by the infinitesimal displacement along $\xi_{(2)}$. $n-2$ necessary and sufficient conditions for $\bar{\xi}_{(2)}'$ to coincide with $\xi_{(2)}$ in the first order of ϵ are obtained in terms of the n curvatures of C at P and some other invariants associated with C and R_n at P. This result can be considered as an extension of the following one: In a Euclidean three-dimensional space if there exists a curve $\overline{C}$ such that its binormals are also the binormals of a given curve C, then C is a plane curve. (Received July 22, 1952.)

691. W. R. Hutcherson and S. T. Gormsen: *Maps of certain algebraic curves invariant under cyclic involutions of periods three, five, and seven.*

The authors contrast the mapping of general plane algebraic curves of orders 3, 5, 7 [Mlle J. Dessart, *Sur les surfaces representant l'involution engendree par une homographie de periode cinq du plan*, Mem. de Liege, vol. 16, no. 2] invariant under transformation T, $x_1':x_2':x_3'=x_1:\epsilon x_2:\epsilon^2x_3$ into spaces 3, 4, 5 getting surfaces of orders 3, 5, 7, where $p=3, 5, 7$ respectively, and $\epsilon^p=1$. Certain general invariant space curves in S_3 and supported by given algebraic surfaces of order 7, 3, 3 are mapped into spaces of orders 7, 11, 17. The involution of period three on the seventh degree surface has two isolated points of coincidence as well as a line of invariant points. The isolated points are found to be perfect. The imperfect point P_3 of I_3 contained on the cubic surface [W. R. Hutcherson, *Maps of certain cyclic involutions on two dimensional carriers*, Bull. Amer. Math. Soc. vol. 37 (1931) pp. 759–765] is more accurately treated and found to have a perfect point in the second order neighborhood instead of the third. The neighborhood of each of the three branch points (images of points of coincidence in the plane containing I_7) on the surface in S_4 are investigated. (Received July 11, 1952.)

692. W. R. Hutcherson and J. C. Morelock: *Invariants with respect to special projective transformations.*

A method for finding homogeneous polynomials invariant under an involutory transformation [W. R. Hutcherson, *Fifth order neighborhood of an involution of period thirteen*, Bull. Amer. Math. Soc. vol. 57 (1951) p. 484] is discovered and exhibited. The total number of terms in the general polynomial is synthetically built up by using the p (prime number) classifications as component parts [J. C. Morelock, *An algebraic formula*, Amer. Math. Monthly vol. 57 (1950) p. 703]. The I.B.M. machine was used to obtain data for classifying perfect points in the infinitesimal neighborhoods of an imperfect point, and along the two invariant directions from this point, which is located on a certain fourth degree algebraic surface. (Received July 11, 1952.)

693t. Edward Kasner and John DeCicco: *The Fourier heat equation in Riemannian space.*

If a body which occupies an n-dimensional region of Riemannian space V_n and which is capable of absorbing heat is heated by conduction in any manner, the resulting situation may be visualized as a flow of heat. It is deduced by the divergence theorem in V_n that the temperature U at a position x within the body and at a time t obeys the Fourier heat equation: div $(k\partial u/\partial x)=c\rho\partial u/\partial t$, where k is the conductivity, c the specific heat, and ρ the density of the body. If k, c, ρ are all constant so that the diffusivity $a^2=k/c\rho$ is constant, the Fourier heat equation is: $g^{-1}\,\partial(g^{1/2}g^{ij}\partial U/\partial x^j)/\partial x^i=(1/a^2)\partial U/\partial t$. If $v(x;\,t)$ obeys this latter equation, the heat hypersurfaces are defined by $U=v(x;\,t)$, where U and t are constants. By means of homothetic representations of Riemannian manifolds, it is found that the only heat families which consist of exactly ∞^1 hypersurfaces are given by $f(x)=\text{const.}$, where f is a solution of either the Laplace equation, or a special form of the Poisson equation, or the Helmholtz-Pockels equation. (Received June 19, 1952.)

694. L. M. Kelly and Leo Lapidus: *The geometry of L-metrized spaces.*

If with each two elements of a set S is associated an element $\rho(a,\,b)$ of a lattice L with an O such that (1) $\rho(a,\,b)=\rho(b,\,a)$; (2) $\rho(a,\,b)=0$ iff $a=b$; (3) $\rho(a,\,b)\cup\rho(b,\,c)\supset\rho(a,\,c)$, then $\rho(a,\,b)$ is called the distance from a to b and S an L-metrized space. When $S=L$, $\rho(a,\,b)$ is written $a\dagger b$, $\dagger$ is called a metric operation and L autometrized. The case in which $S=L=$a Boolean algebra with $a\dagger b=ab'\cup a'b$ has been called a Boolean geometry and studied by Ellis, Blumenthal, and Elliott. This paper is a study of the more general situation described above together with the special case of autometrized Brouwerian algebras (logics) with $a\dagger b=(a-b)\cup(b-a)$. Typical theorems: (1) A Brouwerian algebra is a Boolean algebra if and only if it admits a metric group operation; (2) A finite distributive lattice is a Boolean algebra iff it admits a metric operation such that the resulting space contains no isosceles triangles. (3) The congruence order of a Brouwerian chain with respect to the class of L-metrized spaces is 4. (4) The congruence order of a Brouwerian space with respect to the class of L-metrized spaces is three iff it is a Boolean geometry. (5) If $\dagger$ is a metric group operation in a lattice L and metric and lattice betweenness coincide, then L is a Boolean geometry. (Received July 21, 1952.)

695. T. Minagawa and Tibor Radó: *Infinitesimal rigidity of surfaces.*

Let S denote a regular piece of surface whose boundary consists of a finite number of simple closed curves. Let b be a portion of the boundary, where b may be a subarc of a boundary curve or it may consist of one or more of the boundary curves. Then S is termed IRb (infinitesimally rigid for b kept fixed point-wise) if the identity is the only infinitesimal deformation of S which leaves b fixed point-wise. The purpose of this study is to establish the IRb property for various categories of surfaces whose Gauss curvature has constant sign (positive, negative, or zero), including cases where the surface has corners. This study is being carried out under a research grant from the Office of Ordnance Research, U. S. Army. (Received July 17, 1952.)

696. Ernest Mitchell: *On Kasner's Neo-Pythagorean triangles.*

Kasner has called Neo-Pythagorean that class of triangles whose defining property is that the sum of the squares of the sides shall be zero. Such a triangle must of necessity lie in the complex plane, being in toto or at least partim imaginary. The object of this paper is to give this class of triangles a real representation, carrying along with it for the sake of analogy the Pythagorean class immersed in the complex plane. The method of representation is that of Laguerre. The two classes of triangles assume a quasi-conjugate relationship as exhibited by their "pictures" (Study). (Received July 10, 1952.)

697. Don Mittleman: *Trajectories of trihedral elements.*

Starting with the newtonian equations of translation of the center of mass and rotation about the center of mass of a rigid body of finite dimensions in a positional field of force, approximations to the equations are obtained when the maximum diameter of the body is small. These equations are called the equations of motion of a microscopic body. Also, limiting forms of the original equations are obtained when the maximum diameter approaches zero. These equations are called the equations of motion of a trihedral-particle. This latter result is dependent on the limits of the ratios of the principal moments of inertia. A trihedral element consists of a point (the center of mass of the body) and the three mutually orthogonal directions through the point (the principal axes of inertia). A trajectorial series is ∞^1 trihedral elements associated with the motion of a body for a given set of initial conditions. The differential equations for the trihedral series associated with all motions of a rigid body, all motions of a microscopic body, and all motions of a trihedral-particle are obtained. (Received June 12, 1952.)

598. T. S. Motzkin and I. J. Schoenberg: *On Fejér sets in linear and spherical spaces.*

Let M be a metric space, $A \subset M$. We denote by $F(A)$, called the Fejér set of A, the set of those points p with the property that there is no point p' such that $px > p'x$ for all $x \in A$. If M is a real, inner-product space it is shown that $F(A) = K(A)$, where $K(A)$ is the intersection of all closed half-spaces containing A. In particular for three points p_1, p_2, p_3 we have (1) $F(p_1, p_2, p_3) = \Delta(p_1, p_2, p_3)$, where $\Delta(p_1, p_2, p_3)$ is the triangle of vertices p_1, p_2, p_3. Let now M be a real Banach space B which may also be finite-dimensional. It is shown: 1. If B is 2-dimensional, then (1) holds for any three points if and only if its metric is strictly convex. 2. If the dimension of B

is ≥ 3, then (1) holds if and only if B is an inner product space. Also Fejér sets in spherical spaces are determined. (Received July 21, 1952.)

699t. T. K. Pan: *Relative first curvature and relative parallelism.*

Let a Riemannian space V_n with positive definite first fundamental form $g_{ij}dx^idx^j$ be immersed in a Riemannian space V_{n+1}. Let λ be unit vectors in V_{n+1}, which are not in V_n except in its asymptotic directions and whose contravariant components λ^i at a point P in V_n are analytic functions of x^i and dx^i at P. The totality of these vectors is called a λ-congruence. Let t be the unit tangent vector at a point P of a curve C in V_n. Let N be a unit vector at P satisfying the conditions: (1) linearly dependent on t and λ at P, (2) orthogonal to t. When the first curvature vector at P of C in V_n is decomposed along N and a direction in V_n, the magnitude of the tangential component is called the first curvature at P of C relative to the λ-congruence. A curve, at each point of which the relative first curvature is zero, is called a pseudo-geodesic in V_n. Pseudo-geodesics in V_n are relative auto-parallel curves in V_n according to a definition of relative parallelism, which is a generalization of the parallelism of Levi-Civita. Geometric interpretations of the relative curvature of a curve and properties of relative parallel displacement of vectors with respect to a curve in V_n are studied. (Received July 21, 1952.)

LOGIC AND FOUNDATIONS

700. R. L. Davis: *Structures of finite relations.*

Carnap pointed out recently that the number of structures (isomorphism classes) of a dyadic relation on a finite set was not known, adding that such information would have important applications in deductive and inductive logic and certain branches of science (*Logical foundations of probability*, Chicago, 1950, p. 124). If $\mathfrak{R}_n$ is the set of all relations of some particular kind defined on a set of n elements, there is a group of permutations of $\mathfrak{R}_n$ which is isomorphic to the symmetric group, $\mathfrak{S}_n$. The orbits (sets of transitivity) under this group are just the isomorphism classes in $\mathfrak{R}_n$. Hence, by a well known proposition of group theory, the number of structures (i.e., of orbits) is $(1/n!)\sum_{\pi\epsilon} f(t_\pi)$, where $f(t_\pi)$ is the number of relations fixed under the transformation t_π corresponding to an element π of $\mathfrak{S}_n$. This reduces the problem to that of counting the number of relations so fixed. Employing various combinatorial arguments, general formulas for the number of structures are derived in each of the following kinds of relations (all but (i) dyadic): (i) arbitrary m-adic; (ii) reflexive, or irreflexive; (iii) symmetric; (iv) irreflexive-symmetric; (v) asymmetric, and (vi) functions. (Received June 4, 1952.)

701. J. C. C. McKinsey and Patrick Suppes: *Transformations of systems of classical particle mechanics.*

This paper is a continuation of the investigation of the foundations of particle mechanics begun by McKinsey, Sugar, and Suppes (Bull. Amer. Math. Soc. Abstract 57-6-524). It is devoted to a determination of the set S of transformations which always carry systems of particle mechanics into systems of particle mechanics. It is shown that every Galilean transformation belongs to S, as does every change of units (of time, mass, distance, and force). Moreover, every member of S which satisfies the following weak hypothesis is expressible as a product of Galilean transformations and changes of units: the transformation of time is a one-to-one function mapping R (the set of real numbers) onto itself; the transformation of mass is a function mapping the set of positive real numbers into itself; the transformation of position is a function mapping $E_n \times R$ (where E_n is the set of all n-dimensional vectors) into E_n; and the transformation of force is a one-to-one function mapping E_n onto itself. In general the units of distance and force can be transformed differently along the different axes; this is not the case, however, if the system is *ultra-classical*: i.e., if the forces satisfy Newton's Third Law and are expressible as functions only of the distances between the particles. (Received July 14, 1952.)

STATISTICS AND PROBABILITY

702t. K. L. Chung: *On the renewal theorem in higher dimensions.*

Let $V_1, V_2, \cdots$ be independent r-dimensional random vectors with the common distribution $F(x_1, \cdots, x_r)$ which does not degenerate into a one-dimensional distribution. Let m_j, $j=1, \cdots, r$, be the mean of the jth component. Let $S_n = \sum_{k=1}^{n} V_k$ and let $E(n \mid S_n \in B)$ denote the mathematical expectation of the number of n's such that S_n assumes a value in the Borel set B in R^r. Let C be a compact set in R^r and if U is a vector in R^r let $C(+)U$ denote the set obtained by a translation of C through U. Let $|U|$ denote the length of U. Then, under the hypothesis that not all m_j are zero, we have $\lim E(n \mid S_n \in C(+)U) = 0$ as $|U| \to \infty$. (Received June 23, 1952.)

703. J. L. Hodges and Murray Rosenblatt: *Recurrence-time moments in a Markov chain.*

Consider an irreducible time-homogeneous Markov chain with discrete time. It is shown that if the recurrence time of one state has its first k moments finite, then the recurrence times of all the other states have their first k moments finite. The recurrence time moments of a class of random walks are examined. For $k=0, 1, 2, \cdots$ a random walk is exhibited whose first $k-1$ recurrence time moments exist and whose higher moments are infinite. A comparison theorem then permits the moment properties of other random walks to be determined. (Received April 10, 1952.)

704t. Miriam A. Lipschutz: *Strong laws for sums of independent random variables with stable distributions.*

Consider a sequence of identically distributed independent positive random variables such that $P(X \geqq x) = h(x)/x^{\alpha}$ $(0<\alpha<2)$ and $\lim_{(x\to\infty)} h(cx)/h(x) = 1$, for every positive constant c. Let $S_n = \sum_{k=1}^{n} X_k$. It is known that with b_n properly chosen $P(S_n \leqq b_n x) \to G_\alpha(x)$ and $P(S_n - np \leqq b_n x) \to G_\alpha(x)$ for $0<\alpha<1$ and $1<\alpha<2$ respectively, where p denotes the first moment, and $G_\alpha(x)$ is the stable distribution function of index $(\alpha_1 - 1)$. Under those conditions where a satisfactory estimate of the error, in the above limit theorem, has been obtained, one gets the following strong lower bounds for S_n. Let $\psi(n) \uparrow \infty$, $\psi(n) \leqq 1g n^{1/|1-\alpha|}$, then $P(S_n \leqq b_n/\psi(n)^{1/\alpha}\text{i.o.})$ and $P(S_n - np \leqq -b_n\psi(n)^{1/\alpha}\text{i.o.})$ equal 0 or 1 as $\int[\psi(n)^{(1/\alpha)(1/(1-\alpha))}/n] \exp(-k_\alpha\psi(n)^{1/(1-\alpha)})dn$ and $\int\psi(n)^{(1/\alpha)(1/(\alpha-1))}(\exp(-k_{1/\alpha}\psi(n)^{1/(\alpha-1)}))$ converge or diverge. k_α is a positive constant less than one. A recent lemma of Chung and Erdös (Trans. Amer. Math. Soc. vol. 72 (1952) p. 179) was used in the proof, which also allows one to obtain in a simple way the strong upper bounds previously discovered by Feller. (Trans. Amer. Math. Soc. vol. 67 (1949) p. 115). (Received May 22, 1952.)

705. Shu-Teh C. Moy: *Characterisations of conditional expectation as a transformation on function spaces.*

Let T be a transformation on the space S of all real non-negative measurable functions on a probability space (Ω, F, μ) satisfying the following conditions: T1. $T(\alpha x+\beta y)=\alpha Tx+\beta Ty$, T2. If x is bounded then Tx is bounded, T3. $T(xTy)=(Tx)(Ty)$, T4. If $x_n \uparrow x$ then $Tx_n \to Tx$. It is proved that the general form of such transformation is to take x to be the conditional expectation of xg with respect to a σ-algebra of subsets $F_T \subset F$ where F_T and g are determined by T. With slight modification of the argument the same formulation is obtained for a continuous linear transformation T on L_p into L_p satisfying conditions that T takes bounded functions to bounded functions and that $T(xTy)=(Tx)(Ty)$ for bounded x, y. In the case of $p=1$, if T satisfies the further conditions of $T1=1$ and $\|x\|_1=\|Tx\|_1$, where $\| \ \|_1$ denotes the L_1 norm, then it is proved that Tx is the conditional expectation of x with respect to F_T. (Received July 14, 1952.)

Topology

706. R. D. Anderson: *Monotone mappings and monotone interior mappings of manifolds.* Preliminary report.

The author shows that if f is a monotone mapping of a compact n-manifold S either with or without boundary and with $n>2$, then there exists a monotone interior mapping of S onto $f(S)$. (Received July 21, 1952.)

707t. H. W. Becker: *Nonconnective topology.* Preliminary report.

A nonplanar dual has indeterminate connectivity, but has the same branch symmetry partition as the nonplanar itself. Represent such a topological imaginary by its real dual drawn in a color different from that used for reals. Assemble a graphoid (akin to the matroids of Hassler Whitney, Amer. J. Math. vol. 57 (1935) p. 509) from m nonplanar circuits or their imaginary duals alike or different in any series-parallel-bridge pattern. Then if K_m is the number of combinations of m objects of given specification, the possibilities are one real graph, one pure imaginary, and K_m-2 imaginary graphoids having imaginary duals, "dually imaginary graphoids," d.i.g. If the m are all different, $K_m=2^m$; if all alike, $K_m=m+1$. Thus the reals are ultimately a drop in the bucket, among the nonconnectives. Using symmetry diversity methods previously applied to the enumeration of real graphs and passive circuits, for $16 \leqq n \leqq 19$ the d.i.g. $\langle a\rangle_n=1$, 18, 238, 2730, and the imbedded passive circoids $\langle b\rangle_{n-1}=4$, 110, 1912, 26586; $\langle a\rangle_{18}$ is self-dual. Tentatively, $\langle b\rangle_n \sim 8\cdot 15^{n-15}$. The natural habitat of Starr's theorem $\partial_s Z(s, \cdots)=\square$ is then $s_n+|b|_n+2(b)_n+\langle b\rangle_n$, the planar passive circuits, the nonplanar and their imaginary duals, and the d.i.p.c. (for which last, the Kuratowski's theorem analogue is that each contains one or more of the three elementary n.p.c. and one or more of the three duals of same). These are issues raised by R. M. Cohn. (Received July 21, 1952.)

708. Arthur Bernhart: *A triangle inequality in coloring rings.*

The occurrence of a polygon adjacent to three regions of an n-ring yields inequalities among three associated frequencies in its 4-color spectrum. These requirements permit arguing from the algebraic spectrum to the geometrical structure. This triangle rule is illustrated on various 6-ring configurations. (Received July 21, 1952.)

709. C. E. Burgess: *Continua which are the sum of a finite number of indecomposable continua.*

A continuum M is said to be the *finished sum* of the continua of a collection G if

M is the sum of the continua of G and no continuum of G is a subset of the sum of the others. A continuum M is said to be *indecomposable under index* n if M is the finished sum of n continua and is not the finished sum of $n+1$ continua. These two definitions were given by Swingle (*Generalized indecomposable continua*, Amer. J. Math. vol. 52 (1930) pp. 647–658). The following theorems are proved. If $n>1$ and M is a compact continuum in a space satisfying R. L. Moore's Axioms 0 and 1, then in order that M be indecomposable under index n, it is necessary and sufficient that M should be the finished sum of n indecomposable continua and be irreducible about some n points. If the compact continuum M in the plane is the finished sum of two indecomposable continua H and K such that some composant of H does not intersect K, then uncountably many composants of K lie in $M-H$. (Received July 16, 1952.)

710. D. O. Ellis: *Lattice theory problem 77.*

Problem 77 of G. Birkhoff's *Lattice theory* is answered in the affirmative. One first shows by metric methods that it is valid for 2^B where B is a countable set, by algebraic methods from validity in 2^B that it is valid in 2^S where S is any set, again by algebraic methods that it is valid in $2^S/J$ where J is a σ-ideal of 2^S, and an application of the Theorem of Loomis then yields the result for any σ-complete Boolean algebra. (Received June 30, 1952.)

711. Mary E. Estill: *Problems on indecomposable connexes and related sets in the plane.*

The purpose of this paper is to answer the questions raised by P. M. Swingle in a paper entitled *The closure of types of connected sets* which appeared in the April, 1951 issue of the Proceedings of the American Mathematical Society. The principal results are the following. If I is an indecomposable connexe in the plane, there does not exist a point p such that $I+p$ is decomposable. If W is a widely connected set in the plane, there always exists a point q such that $W+q$ is widely connected. There is a biconnected set without a dispersion point which is not widely connected and is therefore not contained in a widely connected set. (Received July 7, 1952.)

712t. M. K. Fort: *A theorem concerning various types of weak continuity.*

Let T_1 be a separable metric topology for a set X. A second topology T_2 for X is α-related to T_1 if and only if: there exists a countable collection of ordered pairs (U_n, K_n) of subsets of X such that $U_n \in T_1$ for each n, if $p \in U \in T_1$ then there exists n such that $p \in U_n \subset K_n \subset U$, if $q \in U_n$ for some n then there exists $V \in T_2$ such that $q \in V$ and $V - K_n \in T_2$. (In most examples $K_n = \overline{U}_n$.) It is proved that a function on a topological space into X which is T_2 continuous is also T_1 continuous except at points of a first category subset of the domain. The following examples are typical: the point-open topology is α-related to the compact-open topology for the function space of continuous functions on a locally compact separable metric space into a separable metric space, the "weak neighborhood topology" in the calculus of variations sense is α-related to the "strong neighborhood topology," the weak topology for a separable Banach space is α-related to the norm topology. The theory developed has application in various fields, including: topological groups, transformation groups, measure theory, and semi-continuity. (Received April 9, 1952.)

713. S. I. Goldberg: *Extensions of Lie algebras and the third cohomology group.* I.

Let $\{W, Q\}$ be a representation module of the Lie algebra L and U an L-module extending the L-module W by the L-module V. Hence the extension R_x $(x\in L)$ of Q_x to U is a linear endomorphism of U and $\{U, R\}$ is a representation module of L. The quotient U/W implies $\{V, P\}$. Choose linear representatives $\mu_v\in U$ $(v\in V)$ from the residue class corresponding to v by $U/W\cong V$. Then $R_x\mu_v=\mu_{P_xv}+\beta(x, v)$, $\beta(x, v)\in W$. The factor system $\{\beta\}$ completely determines the structure of U, and so we write $U=(L, V, W, \beta)$. Given the extension (L, V, W, β) there exists an exact relative cohomology sequence of homomorphisms, $\cdots\rightarrow H^q(L, V)\xrightarrow{\Lambda}H^{q+1}(L, W)\xrightarrow{\mu}H^{q+1}(L, U)\xrightarrow{\nu}H^{q+1}(L, V)\rightarrow H^{q+2}(L, W)\rightarrow\cdots$. Let $\psi: U\rightarrow V$ be the given homomorphism of U upon its quotient, and let g be any cocycle in $C^q(L, V)$. Pick representatives $\bar{g}(x_1, \cdots, x_q)$ at random so that $\bar{g}$ is multilinear and $\psi\bar{g}(x_1, \cdots, x_q)=g(x_1, \cdots, x_q)$. The map Λ is obtained by sending the cohomology class of g in $H^q(L, V)$ into that of $\delta\bar{g}$ in $H^{q+1}(L, W)$. The following is established: Given the representation module $\{W, Q\}$ then, for any given extensions $L^*=(L, V, g)$ and $U=(L, V, W, \beta)$, a necessary and sufficient condition for the existence of another extension L^{**} of L by U with $L^{**}/W\cong L^*$ is that $\Lambda(g)$ be a coboundary. (Received July 9, 1952.)

714t. S. I. Goldberg: *Extensions of Lie algebras and the third cohomology group.* II.

Let L be a Lie algebra over an arbitrary field and $\{X, S\}$ a representation module of L. Then for any given extension $U^*=(L, U, X, \gamma)$, there exist extensions $W^*=(L, W, X, \gamma)$ and $U=(L, V, W, \beta)$ (W^* is a submodule of U^* and $U/W\cong V$). Let $L^*=(L, V, g)$ and $L^{**}=(L, U, \bar{g}+\alpha)$ satisfying $L^{**}/W\cong L^*$. Select linear representatives $\sigma_x, \sigma_y, \cdots\in L^{**}$. Hence $[\sigma_x, \sigma_y]=\sigma_{[x,y]}+\bar{g}(x, y)+\alpha(x, y)$, $\alpha\in C^2(L, W)$. A necessary and sufficient condition for the existence of the extension $L^{***}=(L, U^*)$ with $L^{***}/X\cong L^{**}$ is that the 3-S-cocycle $\bar{\Lambda}(\bar{g}+\alpha)$ is a coboundary ($\bar{\Lambda}$ is the invariant coboundary, $\bar{\Lambda}: H^q(L, U)\rightarrow H^{q+1}(L, X)$). (Received July 9, 1952.)

715t. Mary-Elizabeth Hamstrom: *Concerning webs in the plane.*

The terms "web" and "simple web" have been defined by R. L. Moore. (Cf. R. L. Moore, *Concerning continua which have dendratomic subsets*, Proc. Nat. Acad. Sci. U. S. A. vol. 29 (1943) pp. 384–389, and *A characterisation of a simple plane web,* Proc. Nat. Acad. Sci. U. S. A. vol. 32 (1946) pp. 311–316.) In the present paper the notion of a W_n set is introduced. If $n>1$, a W_n set is a compact continuum M for which there exists a family F of n elements such that (1) each element o of F is an upper semicontinuous collection of mutually exclusive continua which fills up M and is an arc with respect to its elements and (2) if G is a collection of continua each belonging to some, but no two to the same, collection of the family F, then the continua of the collection G have a point in common and their common part is totally disconnected. It is shown that there exists a W_2 set which is not a W_3 set and that for each positive integer n a simple closed curve plus its interior is a W_n set. (Received June 23, 1952.)

716t. Mary-Elizabeth Hamstrom: *Concerning a certain type of web.*

A W_n' set is a W_n set satisfying the conditions obtained by replacing, in the definition of a W_n set (see abstract 58-6-715) the phrase "upper semicontinuous collection of mutually exclusive continua" by the phrase "continuous collection of mutually exclusive continuous curves." If M is a compact continuum in the plane, $B(M)$ will denote its boundary and $S(M)$ will denote the collection consisting of all boundaries of complementary domains of M, all points of $B(M)$ that are not on the boundary of any complementary domain of M, and all points of $B(M)$ that are common to the boundaries of two or more complementary domains of M. It is shown that the compact continuum M in the plane is a W_2' set if and only if (1) every boundary of a complementary domain of M is a simple closed curve and (2) the collection $S(M)$ is an upper semicontinuous collection of type 2 (cf. R. L. Moore, *Fundamental theorems concerning point sets*, The Rice Institute Pamphlet, vol. 23, 1936, pp. 42, 43, and 56) and there exists an upper semicontinuous collection, S', of type 2 such that $S(M)$ is a subcollection of S', each element of $S'-S(M)$ is a point of $M-B(M)$, S' is an arc with respect to its elements (cf. R. L. Moore, loc. cit. p. 2) and if J is the boundary of a complementary domain of M there are only two points P of J such that P is a limit point of the sum of the elements of S' different from J. If n is an integer greater than 2 the theorem remains true if "W_2'" is replaced by "W_n'." (Received June 23, 1952.)

717. Frank Harary: *The number of graphs and directed graphs.*

The powerful and elegant methods of Pólya, Acta Math. (1937), yield formulas for the number of graphs, directed graphs, and several generalizations of graphs. The generalizations considered are graphs of type t, graphs of strength s, and multiply rooted graphs. A graph of strength s is one in which there are at most s lines between any pair of points. A graph of type t has t different kinds of lines and at most t lines (each of a different kind) joining any pair of points. A multiply rooted graph is one in which the set of points is partitioned into distinguished classes. The formula for the number of graphs is due to Pólya; graphs of strength s were counted by R. Z. Norman. To find the number of graphs of n points and k lines, a polynomial $G_n(x)$ of degree $n(n-1)/2$ is determined in which the coefficient of x^k is this number. Directed graphs, graphs of type t, and of strength s are counted by suitable changes in the "figure power series" (Pólya's terminology). Multiply rooted graphs of n points are found by using as the appropriate permutation groups the direct product of symmetric groups having n as the sum of their degrees. (Received June 13, 1952.)

718t. S. T. Hu: *Cohomology relations in spaces with a topological transformation group.*

Let Q be a topological transformation group operating on the left of a topological space X. Let us denote by B the orbit space and $p: X\rightarrow B$ the projection. p is a continuous and open map of X onto B. For any abelian coefficient group G, the continuous map p induces homomorphisms $p^*: H^n(B, G)\rightarrow H^n(X, G)$ of the Alexander-Wallace cohomology groups. These induced homorphisms are, in general, not onto isomorphisms. They depend on the manner in which the topological transformation group Q operates on X. To measure the deviation of these induced homomorphisms p^* from the onto isomorphisms, the author introduces, in the present paper, the weakly residual cohomology groups $H^n_w(X, G)$ ($n\geq 0$). They are invariants depending on X, Q, G and the operations of Q on X. By means of these groups, the author establishes an exact sequence $H^0(B, G)\xrightarrow{p^*}\cdots\rightarrow H^n(B, G)\xrightarrow{p^*}H^n(X, G)\rightarrow H^n_w(X, G)\rightarrow H^{n+1}(B, G)\xrightarrow{p^*}\cdots$. This indicates that the weakly residual cohomology groups $H^n_w(X, G)$ might play an important role in the further studies of the cohomology structures of the orbit space. For each point x of X, there is a canonical homomorphism

k_x^* of $H_w^n(X, G)$ into $H^n(Q, G)$ for each $n \geq 0$. It is proved that if Q is compact and if x and y are two points contained in a compact connected subset of X then $k_x^* = k_y^*$. (Received July 9, 1952.)

719*t*. E. G. Kundert: *A generalisation of the fundamental group.*

The following generalization of the fundamental group to the second dimension seems to be a natural one: let (T_k, f) be the continuous mapping f of the carrier $(T_k; p)$ into the topological space $(Y; y_0)$ such that $p \to y_0$, T_k being an orientable, ordered (handles ordered) 2-dimensional manifold with k handles. By certain continuity preserving, reversible processes $(T_k, f) \leftrightarrow (T_{k'}, f')$ and the usual homotopy relation, one is able to collect the (T_k, f) in equivalence classes, which form a group $\Pi(Y)$. Some properties: $\Pi = \Pi_1 \times \Pi_2$ such that $\Pi_1 \approx \pi_1$ (fundamental group). π_2 (homotopy group) is subgroup of $\Pi_2 \cdot \Pi_2 \xrightarrow{\Gamma} H_2$; $\gamma = {}^{\Gamma}\cap \pi_2$; $\pi_2 \xrightarrow{\gamma} S_2$, $\Pi_2/\Gamma \cdot \pi_2 \approx G_1^*$. If $\pi_1 \approx 0$: $\Pi_2 \approx \pi_2$. (For the definitions of S_2 and G_1^* see H. Hopf, *Fundamental Gruppe und zweite bettische Gruppe*, Comment. Math. Helv. vol. 14.) (Received August 11, 1952.)

720*t*. Deane Montgomery and Leo Zippin: *Small subgroups of finite-dimensional groups.*

The authors prove the following "reduction-theorem": Theorem A. If G is a separable metric, locally compact, finite-dimensional, connected, and locally connected topological group and if all of the proper subgroups of G are generalized Lie groups (in the sense of Gleason, or of Iwasawa) then G contains an invariant generalized Lie group H such that the factor-group G/H is finite-dimensional and has no small subgroups. This means that there is a neighborhood of the identity of G/H which contains no subgroup of G/H except the trivial one consisting of the identity alone. The subgroup H is determined in a natural way so that the group G/H shall have no center but the identity. A key device in the proof is that of finding a local Lie subgroup R of G/H such that R is equal, in the small, to the identity component of its own normalizor and such that any conjugate subgroup R' to R either coincides with R, locally, or else intersects R in the identity alone. Such a subgroup, under conjugation by the elements of a neighborhood of G/H, sweeps out some open subset of G/H. This gives one control of the topology of G/H and also of the topological space which results when G/H is fibered by R. (Received March 27, 1952.)

721*t*. J. C. Moore: *On tetrads and triads.*

The triad $(X; A, B)$ is said to be excisive if and only if $X = A \cup B$, A, B, and $A \cap B$ are arcwise connected, and $H_q(X, A) = H_q(B, A \cap B)$ for all q, the homology theory considered being singular homology. It is proved that if $(X; A, B, C)$ is a tetrad such that the triads $(X; A, B)$, $(X; A, C)$, $(X; B, C)$, and $(A; A \cap B, A \cap C)$ are excisive, X is 1-connected, (X, A), $(A, A \cap B)$, and $(A, A \cap C)$ are 2-connected, R is a principal ideal domain, $H_q(X, A; R) = 0$ for $q \leqq r$, $H_q(X, B; R) = 0$ for $q \leqq m$, and $H_q(X, C; R) = 0$ for $q \leqq n$, then $\pi_q(X; A, B, C) \otimes R = 0$ for $q \leqq m + n + r$. The preceding theorem is applied to prove that if $(X; A, B)$ is an excisive triad, X is 1-connected, C^* is the join of $A \cap B$ and a point, $A^* = A \cup C^*$, $B^* = B \cup C^*$, and $X^* = A^* \cup B^*$, (X, A), (A, C), and (B, C) are 2-connected, R is a principal ideal domain, $H_q(C; R) = 0$ for $0 < q < r$, $H_q(X, A; R) = 0$ for $q \leqq m$, and $H_q(X, B; R) = 0$ for $q \leqq n$, then $\pi_q(X; A, B) \otimes R = \pi_q(X^*; A^*, B^*) \otimes R$ for $q < m + n + r$, and $\pi_{m+n+r}(X; A, B) \otimes R \to \pi_{m+n+r}(X^*; A^*, B^*) \otimes R \to 0$ is exact, where the homomorphisms are those induced by the inclusion map. Since $C^* = A^* \cap B^*$ is contractible, $\vee\, \pi_q(X^*; A^*, B^*) = \pi_{q+1}(A^* \times B^*, A^* \vee B^*)$ for all q. (Received June 6, 1952.)

722. J. C. Moore: *On the homotopy groups of odd-dimensional spheres.*

For G an abelian group and p a prime let $C(G, p)$ denote the p-primary component of G. It is then proved that for p an odd prime, $C(\pi_q(S^3), p)$ is a cyclic group of order p for $q=4p-3$, $4p-2$, and is trivial for $4p-2<q<6p-6$. Let F_2^n denote the space of two cells in S^n. Then in a standard fashion S^n may be imbedded in F_2^{n+2} so that the inclusion map is equivalent to the double suspension. Let X_p^r denote the space obtained by adjoining an $r+1$ cell to the r sphere by a map of degree p. It is proved that if n is odd and greater than or equal to 3, then $C(\pi_q(F_2^{n+2}, S^n), p)=\pi_q(X_p^{n+p-2})$ for q less than $(p+1)(n+1)-4$. Moreover $C(\pi_q(F_2^5 S^3))$ is a cyclic group of order p for $q=4p-2$, and is trivial otherwise for q less than $6p-5$. These facts imply that if n is odd and greater than 3, then $C(\pi_q(S^n), p)$ is nontrivial for $q=n+4p-5$, and is trivial for $n+4p-5<q<n+6p-9$. (Received June 26, 1952.)

723. P. S. Mostert: *On the existence of local cross sections in compact groups.*

Let G be a second countable topological group, H a subgroup of G. Theorem. If G has a discrete normal subgroup N such that G/N is compact and of finite dimension, then H has a local cross section. Moreover, if G is 0-dimensional compact, then H has a full cross section. G (compact) is the projective limit of Lie groups G_k. Local cross sections are constructed for $H_k\subset G_k$, where H_k is the projection of H on G_k, which "agree" under the projections. (Received July 21, 1952.)

724. Everett Pitcher: *Homology theory of critical levels for a pair of spaces.* Preliminary report.

A homology theory of critical levels of a real-valued function f on a space X relative to a subspace Y is constructed. The theory is formulated in terms of group homomorphisms. It connects the Morse critical level theories for f on X and for $f|Y$ through interlocking exact sequences. (Received July 21, 1952.)

725. L. E. Pursell: *Rings associated with topological spaces.* Preliminary report.

A ring R is "weakly associated" with a T_1-space X if, for every r in R, there is a closed set $Z(r)$ in X such that: (i) $r=0$ if and only if $Z(r)=X$, (ii) $rs=0$ if and only if $Z(r)\cup Z(s)=X$, and (iii) if the point x is not in a closed set F, there is an r such that $F\subset Z(r)$ but $x\notin Z(r)$. The "support" $S(r)$ is the set $\mathrm{Cl}(X-Z(r))$. In this ring $rs=0$ implies $sr=0$. The "annihilator" $A(r)$ is the ideal of all s such that $rs=sr=0$. Elements r and r' in R are "disjunct" if, for every s and s' in R, $\exists t\in R$ such that $(t-s)\in A(r)$ and $(t-s')\in A(r')$. A ring R weakly associated with X is "associated with X" if (iv) r and r' in R are disjunct if and only if $S(r)$ and $S(r')$ are disjoint and (v) $x\notin$ closed set F implies $\exists r\in R$ such that $F\subset S(r)$ and $x\notin S(r)$. If condition (iv) is satisfied, if every $S(r)$ is compact, and if (v) is satisfied for F compact, then R is "compact associated" with X. A ring associated with a compact Hausdorff space determines its topology. A ring compact associated with a regular locally-compact Hausdorf space determines its topology. The ring of all continuous functions on a normal T_1-space is associated with the space. The ring of such functions with compact supports is compact associated with the space. Rings can be constructed of tensors on differentiable manifolds which are associated or compact associated with the manifold. (Received July 18, 1952.)

726. M. E. Shanks and L. E. Pursell: *The Lie algebra of a smooth manifold.*

Let M be an infinitely differentiable manifold and D the algebra of real infinitely differentiable functions on M. A tangent vector T is an abstract derivative on D and $T\cdot f = r^i\partial f/\partial x^i$, where r^i are infinitely differentiable. The set $\mathcal{L}$ of all tangent vectors is a Lie algebra over the real field under the composition $[T_1, T_2]\cdot f = T_1\cdot(T_2\cdot f) - T_2\cdot(T_1\cdot f)$. Let $\mathcal{L}_0$ be the subalgebra of $\mathcal{L}$ with compact supports. It is shown that two manifolds with isomorphic Lie algebras $\mathcal{L}_0$ are homeomorphic. Furthermore if G is the Grassmann algebra of differential forms over M and G_p is the subalgebra of G consisting of forms of degree p and higher, $p<n$, then the Lie algebra of M is determined by the algebraic structure of G_p. (Received July 21, 1952.)

727t. Henry Sharp: *Strongly topological imbedding of F-sigma subsets of E_n.*

Denote by E_n Euclidean space of n dimensions, by M_n^k the set of points in E_n having at most k rational coordinates, by L_n^k the set of points in E_n having at least k rational coordinates. It is an open question whether M_n^k $(n<2k+1)$ contains a topological image of every k-dimensional subset of E_n. For the special cases $k=n$ and $k=n-1$ the answer to this question is in the affirmative; in fact, for these values of k the set M_n^k contains a strongly topological image (image under a homeomorphism of the space onto itself) of every k-dimensional subset of E_n. The present paper is concerned with a related problem: does the set M_n^k contain a strongly topological image of every j-dimensional F_σ-subset of E_n. S. W. Hahn has given the answer to this problem for certain special cases, namely: affirmative for $n=2$, $j=k=0$, and $n=3$, $j=0$, $k=1$; negative for $n=3$, $j=k=0$. The solution of this problem is completed in the present paper in which it is proved that the answer is affirmative for $n\geqq 4$, $j\leqq k=n-2$, and negative in all other (previously unsolved) cases. (Received June 4, 1952.)

728t. A. H. Stone: *A note on infinitely multicoherent spaces.*

Eilenberg (Fund. Math. vol. 27 (1936) pp. 153–190) has defined the "degree of multicoherence," $r(S)$, of a connected topological space S to be sup $b_0(A\cap B)$, the supremum being taken over all pairs of closed connected sets A, B such that $A\cup B=S$. Here $b_0(X)$ denotes the number of components of X less than one (or ∞ if this number is infinite). Assume that S is a Peano space and that $r(S)=\infty$; need A, B exist as above with $b_0(A\cap B)=\infty$? It is shown that the answer is affirmative if S is 1-dimensional, but negative for some 2-dimensional Peano spaces. (Received May 28, 1952.)

729t. A. H. Stone: *On coverings of 2-dimensional spaces.*

The "density" of a normal space S is the least integer n such that S has arbitrarily fine finite open (or, equivalently, closed) coverings in which no set meets more than n others. Boltyanskii (Doklady Akad. Nauk SSSR vol. 75 (1950) pp. 605–608) has shown that if S is a 2-dimensional compactum its density is at least 6. The present paper shows that if S is 2-dimensional its density is at most 7, and that the value 7 does occurs for suitable S. (Received May 28, 1952.)

730t. D. D. Strebe: *Certain applications of irreducibility to connected Hausdorff spaces.*

A study is made of Hausdorff spaces which are irreducibly connected or irreducibly closed and connected about arbitrary subsets. The nature of spaces which are irreducibly connected about minimal subsets (basic sets B) is investigated. If a nondegenerate connected space M has a basic set B consisting of n points, where n is a positive integer, it is proved that M can be expressed as the sum of $n-1$ or less subsets, each of which is irreducibly connected about 2 points of B, and such that each point of B is a point of irreducibility of at least one of these subsets. With regard to minimal subsets (basic sets B^*) about which a space M is irreducibly closed and connected, it is proved that B^*, when it exists, is an isolated, nondense subset of M, consisting entirely of non-cut points. The existence of B^* implies the existence of a basic set B. If a locally compact metric space M has a basic set B^*, a necessary and sufficient condition that B^* be the unique basic set is that M be locally connected at each point of B^*. (Received July 14, 1952.)

731. Harold Tinnappel: *On the topological index*. Preliminary report.

Utilizing a theorem of T. Radó (*Length and area*, Amer. Math. Soc. Colloquium Publications, vol. 30, IV.1.71) and certain auxiliary transformations the following theorem is proved. If T: $z=f(w)$ is a continuous transformation from a bounded, simply-connected Jordan region R in the w-plane into the z-plane and if for a point z_0 we have the topological index equal to $k\neq 0$, then there exists a $\delta>0$ such that there are at least $|k|$ maximal model continua lying in D for each z for which $0<|z-z_0|<\delta$, where D is the component of $R-f^{-1}(z_0)$ which contains the boundary of R. An analogous theorem is proved for the case where R is a bounded, finitely connected Jordan region. (Received July 16, 1952.)

732*t*. A. D. Wallace: *An axiom for cohomology*.

In this note we consider the effect of replacing (in a cohomology theory) the homotopy postulate by the following postulate: If (Y, B) is an adequate pair and $h\in H^n(Y, B)$, then there is an open cover $\mathcal{U}$ of Y such that if $f, g\colon (X, A)\rightarrow(Y, B)$ are adequate maps satisfying $x\in X$ implies the existence of a $V\in\mathcal{U}$ with $f(x), g(x)\in V$, then $f^*(h)=g^*(h)$. This is known to hold for the Alexander-Kolmogoroff groups (Spanier, Ann. of Math. (1948)). There are various modifications obtainable by prefixing an adjective to "open cover." This adjective depends on the particular type of "nearness" of the functions f and g that seems suitable for the problem at hand. If the spaces in question are metric then the postulate takes the familiar form: If $h\in H^n(Y, B)$, then there is an $\epsilon(h)>0$ such that $\rho(f, g)<\epsilon(h)$ implies $f^*(h)=g^*(h)$. (Received August 11, 1952.)

733*t*. A. D. Wallace: *A note on mobs*. I.

A mob is such a map ($=$continuous function) m: $S\times S\rightarrow S$ that S is Hausdorff and m is associative. Write $xy=m(x, y)$ and say that S is a mob. If S has a unit and is compact, then the set I of inverses is a compact (topological) group. If S is also connected, then no element of I is a cutpoint of S. If I is compact it is a group, and if also S is connected no point of I cuts I in S. (Received July 31, 1952.)

734*t*. A. D. Wallace: *A note on mobs*. II.

Let S be a mob (preceding abstract), let E be its idempotents and K its (unique) minimal closed ideal if such exists. Each algebraic subgroup is contained in a maximal such and these are pairwise disjoint. If S is compact, each such is a compact (topological) group. If S is compact, then K exists. If $e \in E$, let $H(e)$ be the maximal subgroup containing e. Let S be compact. Then $H(e) = eSe$ if and only if $e \in K \cap E$ and $K = \bigcup \{H(e) \mid e \in K \cap E\}$. This topologizes algebraic results of Clifford-Miller, Green, Rees, Schwarz, and Suschkewitsch. As corollaries: (i) If S has a unit and no other idempotent, it is a group; (ii) if K has a unit, it is a group; (iii) If S is simple and connected, then S has no zero and $K = S$. If also S has a unit, then S is a group. (Received July 31, 1952.)

735*t*. G. T. Whyburn: *A unified space for mappings.*

In this paper it is shown that any mapping from one separable metric space to another is topologically equivalent to a partial mapping of a compact mapping (actually a retraction). This is accomplished by constructing a space unifying both the domain and the range spaces and showing that a compact mapping can be defined on this space which is topologically equivalent to the given one on the prototype of the original domain space. The unified space is separable and metrizable when the given spaces are locally compact, and it is in this setting that the new viewpoint and result seem to be most useful. (Received May 26, 1952.)

E. G. BEGLE,
Secretary

BOOK REVIEWS

Handbuch der Laplace-Transformation. Vol. 1. *Theorie der Laplace-Transformation.* By G. Doetsch. (Mathematische Reihe, vol. 14.) Basel, Birkhäuser, 1950. 15+581 pp.

The theory of the Laplace and of the Laplace-Stieltjes transforms is essentially a creation of the last thirty years though much of the theory may be said to be inspired by the somewhat older theory of Dirichlet series. The organization of the theory of Laplace-Stieltjes transforms some twenty odd years ago unified the theory of Dirichlet series with the much less developed theory of Laplace integrals and with the theory of analytic almost periodic functions. In the unified theory the Laplace transform

$$\mathcal{L}\{F\} = f(s) = \int_0^\infty e^{-st}F(t)dt$$

plays a preferred role owing to its importance for the applications. It enters in a natural manner in the theory of boundary value problems and much of the Heaviside Ars conjectandi can be rationalized by a judicious use of Laplace transforms. There are also close contacts with analytic function theory and with the theory of Fourier transforms. The Laplace transform would seem to be assured of a place in the sun for some time to come merely on the score of its wide range of applications.

Gustav Doetsch has devoted almost thirty years of his life to the theory of Laplace transforms and their applications to boundary value problems, especially for the heat equation. Since his connection with the Technological Institute of Stuttgart in 1924–31 he has always paid much attention to the applications and he seems to have kept close and mutually profitable contacts with engineers and physicists. He writes with an eye to this larger audience and he wants to be understood by people who have an immediate use for the results. Naturally, these are not always the results having significance, for instance, in the theory of locally compact groups, supposedly the standard of excellence in up-to-date analysis.

This is the author's fourth book on Laplace transforms. It was preceded by *Theorie und Anwendung der Laplace-Transformation,* Berlin, Springer, 1937, *Tabellen zur Laplace-Transformation und Anleitung zum Gebrauch,* Berlin and Göttingen, Springer, 1947, jointly with H. Kniess and D. Voelker, and *Die zweidimensionale Laplace-Transformation,* Basel, Birkhäuser, 1950 (volume 12 of the same series), jointly with D. Voelker. In the first book, the author's concern with the applications and with the qualifications of the students of applied mathematics led him to dismiss the Lebesgue integral. This attitude has now changed radically, the Lebesgue integral is taken as the basic concept and the author makes a rather eloquent plea for its acceptance and use by applied mathematicians. To help the less qualified readers there is an excellent appendix containing what is needed of auxiliary theorems in real and complex function theory. There are also occasionally alternate proofs provided based on the Riemann integral.

The present treatise, of which the first volume is now before us, is written with skill and painstaking care. It provides a clear, easily readable and detailed introduction to the theory. Within the limits set by the author, it carries the reader practically to the borders of present day knowledge. Naturally there are omissions and, this being a critical review, some omissions will be pointed out below. But the main structure is there in all its splendor and there is an abundance of special results, nowhere else to be found.

While the Laplace-Stieltjes transform as well as the Mellin transform are mentioned and discussed in some detail, the treatise is concerned chiefly with the Laplace transform proper, both the unilateral and the bilateral cases. It is quite natural that the author should treat the Laplace-Stieltjes transform somewhat briefly; this corresponds both to his own predominant interests and to the fact that the Laplace-Stieltjes transform has been discussed rather fully in D. V. Widder's treatise. The author very carefully points out the differences in function theoretical behavior between Laplace-Stieltjes transforms and proper Laplace transforms. Perhaps the reader is apt to get a somewhat exaggerated view of these differences: after all any Dirichlet series can be converted into a Laplace transform by division by a logarithmico-exponential function of arbitrarily slow growth.

The theory is divided into five parts: Basic analytical and function theoretical properties of the Laplace transformation. Inversion of the Fourier and Laplace transformations, Parseval's identity and related questions. A generalization of the Laplace transformation. The Laplace transforms of special classes of functions. Abelian and Tauberian theorems.

Part I starts out with some generalities concerning linear transformations and the basic concepts of functional analysis. Having defined the various forms of transforms to be studied in the treatise, the author goes very fully into the question of how operations on the elements of the domain of the Laplace transform are reflected in the range. This technique is highly useful in the applications for much of the power of the Laplace transformation is derived from the fact that it translates transcendental operations on the domain, such as differentiation, integration, convolution, and Hankel's transform, into algebraic operations on the range such as multiplication of the transform by a power or by another transform or a simple change of variables. This part ends with the discussion of the various half-planes of convergence, boundedness, and holomorphism. Here the notion of order of boundedness is a concept introduced by the author.

Part II is concerned chiefly with the inversion formulas: the complex integral formula, the Phragmén operator, and the operators of Post, Widder, and Boas, as well as inversion by series expansions. In connection with the latter, there are some results of W. B. Caton and the reviewer which escaped the author's attention owing to war conditions though the paper appears in the Bibliography. Chapter 5 contains new material; it is concerned with the representation of the integral over the interval $(0, T)$ by a contour integral, involving the transform, with applications to Dirichlet series. Chapter 6, on Parseval's identity, contains a number of interesting results. The following strikes the reviewer as rather peculiar (Theorem 3, p. 257): If $f_n(s)=\mathfrak{L}[F_n]$ and $e^{-x_0 t}F_n(t)\in L_1(0, \infty)\cap L_2(0, \infty)$, then $f_1(s)$ and $f_2(s)$ are orthogonal to each other over every vertical line $x=\xi$ with $\xi>x_0$.

Part III is comparatively brief, just one chapter devoted to the (C, k)-summability of Laplace integrals which is discussed in some detail and contains original material. One misses a reference to the papers of M. Riesz in Acta Litt. ac Scient., Szeged, volumes 1 and 2, 1923–1924, where general Laplace-Stieltjes transforms are considered. A proof of the convexity of the abscissa of summability as function of the index would also be relevant. There is no discussion of other methods of summability; the reviewer naturally misses a reference to the work of Hille and Tamarkin where the problem of summability is subsumed under that of representing a function as the quotient of two Laplace integrals.

Part IV is concerned with the Laplace transforms of (1) entire functions of exponential type, (2) functions analytic in a strip (bilateral transforms), or in a sector (Mellin transforms), and (3) functions in $L_2(0, \infty)$. These are among the most important special classes to which the Laplace transform applies. Since the book is so full of special results, it is really a pity that most of the results of

Hausdorff concerning logarithmico-exponential functions are omitted. They are both elegant and useful.

The final part V deals with Abelian and Tauberian theorems and ends with the Ikehara theorem. The Wiener theorem is mentioned but not proved in detail and most of the work is based on the more elementary methods of Hardy, Littlewood, and Karamata. This part also contains original material and is basic for the applications to asymptotic representations.

The book contains at least fifteen research problems inserted in the text, relating to open questions in the theory. Some are fairly easy to solve on the basis of available results, others will require a more substantial effort.

References occupy a section of twelve pages and there is an excellent bibliography. The press work is beautiful and misprints very rare.

EINAR HILLE

Leçons de logique algébrique. By H. B. Curry. (Collection de Logique Mathématique, Série A, Monographies réunies par Mme Destouches-Fevrier, no. II.) Paris, Gauthier-Villars, 1952. 183 pp. $4.86.

This expository monograph is the outcome of a course given at Louvain in 1950–51. Although the important theoretical results have appeared in the works of the author and others, it is a valuable contribution to the literature by reason of its lucid architectonic treatment of elementary mathematical logic from the viewpoint of modern mathematics.

Chapter 1 treats of the nature and methodology of formal systems as previously proposed [*A theory of formal deducibility*, Notre Dame Mathematical Lectures, no. 6, University of Notre Dame, Notre Dame, Ind., 1950]. Mathematics is defined as the study of formal systems and mathematical logic is concerned with those formal systems which have some connection with philosophical logic.

These lectures are restricted to the logical algebras as characterized in Chapter II. Algebras are formal systems with free but no bound variables and with a fundamental transitive relation. Where this relation is reflexive, the algebra is called relational or in its reduced form, logistic. Logical algebras are relational or logistic algebras with two binary operators (sum and product) and the idempotent laws (tautology). Those considered of interest are also (at least) general lattices. A variety of interpretations are given, some propositional.

Postulates for semi-lattices (group logics) and lattices are introduced in Chapter III. The properties of general, distributive and modular lattices are discussed and some basic theorems demonstrated.

An operator which corresponds to implication in the propositional interpretation is introduced in Chapter IV. Implicative lattices are those for which detachment and exportation hold. Four formulations are proposed, one relational (with detachment and exportation taken as postulates) and three logistic. The latter are *TA* (Gentzen's natural formulation), *HA* (the familiar calculus construction) and *LA* (Gentzen's *L*). Some theorems are proved about the implicative lattices and their duals (the subtractive lattices). *HA* and *TA* are shown to be implicative lattices equivalent to each other, to Bernays' positive logic, and to intuitionistic logic without negation.

Where, in the propositional interpretation, implication is construed as deducibility, the above systems (called the positive systems) are applicable. For a truth functional definition of implication, they must be strengthened to accommodate some form of Pierce's law. An implicative lattice so strengthened is a classic implicative lattice. Appropriate modifications of *TA*, *HA*, and *LA* are made. *TC*, *HC*, and *LC* (the classic positive systems) are shown to be classic implicative lattices. Classic subtractive lattices (Boolean rings) are treated in some detail.

Four kinds of negation are considered in Chapter V. *M*, minimal (refutability); *N*, intuitionist (absurdity); *D*, strict (refutability with excluded third); *K*, classic (absurdity with excluded third). Algebras with each type of negation are discussed and some results established about their interrelation including an extension of Glivenko's theorem. The larger part of Chapter V is devoted to the classic (Boolean) algebra.

In conclusion, extensions beyond the classical are briefly considered, particularly in the direction of modalities.

Following each chapter are notes of a bibliographical, historical, and occasionally expository nature which add greatly to the value of the book. The preface by R. Feys is oriented toward the student of philosophy. (On page 99, line 23, add "Nous employerons pour cette espèce la lettre "*K*"." at the end of the line. On page 136, line 18, for "§7" read "§1.")

R. BARCAN MARCUS

Theory of perfectly plastic solids. By W. Prager and P. G. Hodge. New York, Wiley, 1951. 10+264 pp. $5.50.

This book is an important and valuable contribution to the literature of mechanics, because it presents in consecutive and easily readable form the views and the main results of a considerable body of investigators. Accordingly, the emphasis is on the solution of specific examples in the simplest rate theories. There is little discussion of the foundations or of the nature of plastic phenomena, and there is no development of the incremental or accumulative theories. There is compromise, however, between the Russo-English school of pure calculation of special cases and the American tendency to seek general properties of solutions. In an era of hasty and unsound books, the reader will welcome the dignified, easy exposition, even though at times he may find it a little tedious, but most of all the scholarly but unobtrusive references to the basic papers from which the material was taken. The problems for the student are moderate in number, interesting in content, and sometimes difficult.

The book is designed for engineers, not mathematicians. Not only are mathematical points either altogether disregarded, as may well be only proper in a field whose mechanical foundations are still insecure, or even skirted in mystifying terms (e.g. p. 66), but also the engineering method of dealing with particular examples concerning frameworks before formulating the general theory, and of sometimes reverting thereafter to "physical" methods rather than using the basic equations to solve special problems, will occasion some difficulty. For example, I could not follow the analysis of bending of a beam in §7, or the discussion of elastic-plastic torsion in §10. In my opinion a much stronger case can be made out for the mathematical method than the explanations offered by the authors on p. 1: one has only to cite rigid body mechanics, elasticity, and fluid dynamics, taking due note of the manner in which those theories were discovered and developed originally.

The entire book deals with the Prandtl-Reuss theory, which may be described as postulating that in an incompressible body subject to infinitesimal strain the stress is the sum of that given by classical elasticity plus that given by the classical theory of viscous fluids, except that the viscosity instead of being a material constant is a certain function of the stress deviator—the nature of this function being specified by the yield condition, which for an isotropic material is a single scalar equation connecting the stress invariants. The Tresca condition of constant maximum shear stress and the v. Mises condition of constant intensity of the stress deviator are the only ones used. Much attention is paid to the simpler theory of v. Mises, which results from that of Prandtl and Reuss when the elastic shear modulus is set equal to infinity. Not all readers will agree with the authors' distinction between the two theories on p. 30 and pp. 82–84, particularly when they insist that the Mises theory regards the material as rigid under stresses below the yield stress and when they imply that that theory cannot be used in treating elastic-plastic problems. A welcome reassurance comes at last on p. 121 when the authors explain that the v. Mises equations are applicable to deformations of any magnitude, provided what the authors have called strain rates be interpreted as the instantaneous rates of deformation. The student of rational mechanics would have been able to take a happier part in the developments of Chap. I if this remark, along with a proper discussion of strain, had been inserted there; but, as the authors notice on p. 120, there are difficulties in finite strain problems if a part of the work of deformation is recoverable even during plastic flow. The modification of the Prandtl-Reuss formulae which the authors suggest leads to tensorially illegitimate equations if their components (18.14) are used. Indeed, a strong advantage of the v. Mises equations lies in their meaningfulness and reasonableness in problems of large deformation, and in this respect they are a partial generalization, rather than a special case, of the Prandtl-Reuss equations.

The special problems treated in the book are without exception one-dimensional or two-dimensional. There is a chapter on torsion and four chapters on plane strain. I shall not attempt to detail their contents, which includes or refers to most of the particular solutions now known. A special gem is Sokolovsky's inverse method for the elastic-plastic torsion problem (§11). The mathematically inclined reader will find in Chap. 5 many ideas already familiar in gas dynamics. The authors make a conscientious attempt to deal with the problem of contained plastic flow, which is of course the major problem to be solved. After some cases in torsion and axially-symmetric plane strain, however, they frankly give up the attempt as hopeless (§19), dealing thenceforward almost entirely with unrestricted plastic flow. For this they are not to blame: it has been the natural tendency of the field to find problems which can be solved, rather than to solve problems which are important.

The student of other branches of mechanics, accustomed to complete solutions in special cases, may learn with surprise from the examples given here that in the theory of plasticity not so much is expected. Although there is a great deal of discussion of plastic flow, very few flow fields are exhibited, and a problem is apparently regarded as solved when stresses satisfying the equations have been found. From the practical side it can be said that indeed determination of the stresses is often the main objective. However, the reader may grow a little queasy about the whole matter if he asks himself in a particular case how a specimen suffering the stresses in question will deform. In other parts of continuum mechanics exact solutions obtained by inverse or semi-inverse methods are extremely valuable. In plasticity, too, I hazard the conjecture that such methods may soon yield some *complete* solutions which can illuminate the true nature of plastic flow as embodied in the governing differential equations. To add to the present unsettled nature of the subject, there appear to be no existence or uniqueness theorems or even any plausible arguments in this connection (cf. pp. 169–182). These remarks may explain why plasticity has so far been avoided by many students of other branches of mechanics, being cultivated mainly by specialists.

An almost completely new body of material is contained in Chapter 7 on "limit analysis," a field to a great extent created by the senior author and his school. The principal result is contained in two beautiful theorems of Drucker, Greenberg, and Prager, for which the printing of the work was held up for a year. Suppose all surface loads be written with a multiplicative factor which is allowed to increase from 0. The ratio of surface traction at the instant of impending plastic flow to a given value is called the *safety factor*. A number is a *statically admissible multiplier* if there exists for this multiple of the loads a stress field satisfying the equations of equilibrium, the boundary conditions, and the yield inequality; a *kinematically admissible multiplier* is defined in a somewhat more complicated way. Then the two new theorems are:

THEOREM 1. *The safety factor is the largest statically admissible multiplier.*

THEOREM 2. *The safety factor is the smallest kinematically admissible multiplier.*

The proofs require considerable manipulation. These theorems are applied by the authors to justify use of the St. Venant-Mises theory for unrestricted plastic flow, despite its rejection in the preceding period of contained plastic flow.

The book closes with a chapter on the extremum principles of Sadowsky-Hill-Markov and Greenberg, and their application to prove the safety factor theorems in three dimensions.

This book will be read and used intensively in the immediate future. For a definitive treatment of the mathematical theory of plasticity we must wait, however, for substantial improvements in the theory itself.

C. TRUESDELL

Methods of algebraic geometry. Vol. 2. By W. V. D. Hodge and D. Pedoe. Cambridge University Press, 1952. 9+394 pp. $7.50.

Having carefully laid the algebraic foundations in volume I (see Bull. Amer. Math. Soc. vol. 55 (1949) pp. 315–316), the authors first define an algebraic *variety* to be the aggregate of points determined by the vanishing of a set of homogeneous polynomials over a given field K. The ground field K is quite general except that it is not allowed to have a finite characteristic. In elementary treatments of real analytic geometry, one often finds it convenient to make temporary use of complex coordinates. Here, similarly, algebraic extensions of K are often introduced for special purposes. Great care is taken in defining a *generic* point (van der Waerden's "allgemeine Nullpunkt") of an algebraic variety, and a generic member of a system of k-spaces S_k. It is proved that a generic S_{n-d} meets an irreducible d-dimensional variety V_d in a finite number of points, each of which is a generic point of V_d. The number of points is the *order* of V_d. Much use is made of the so-called *Cayley form*, which is the "zugeordnete Form" of van der Waerden and Chow (Math. Ann. vol. 113 (1937) pp. 692–704).

Chapter XI is a thorough treatment of algebraic correspondences, following van der Waerden and Weil. The general principles are illustrated by application to two classical problems: finding the transversals of four skew lines in S_3, and reducing the general ternary cubic form to $X_0^3+X_1^3+X_2^3+6\lambda X_0X_1X_2$.

In Chapter XII, the theory of intersection leads naturally to the theory of equivalence, which is defined as follows. Two varieties V_a and V'_a on a V_n (in projective space of more than n dimensions) are said to be "equivalent in the narrow sense" if they belong to the same continuous system. Two such varieties are said to be "equivalent in the wide sense" ($V_a \equiv V'_a$) if there exists another variety V''_a such that $V_a+V''_a$ and $V'_a+V''_a$ are equivalent in the narrow sense. Equivalences $U_a \equiv U'_a$ and $V_a \equiv V'_a$ imply $U_a+V_a \equiv U'_a+V'_a$, but there does not necessarily exist a variety X_a such that $U_a+X_a \equiv V_a$. The desirable group property is achieved by inventing a *virtual* variety V_a-U_a and writing

$$V_a - U_a \sim V'_a - U'_a$$

when $V_a+U'_a \equiv U_a+V'_a$. The authors prove that, when the fundamental variety V_n is a flat n-space, any variety V_a of order g is equivalent to gS_a, where S_a is a flat a-space. They then mention the generalization "which has exercised the minds of many geometers:"

For any absolutely irreducible variety V_n, and for each value of a $(0 \leqq a \leqq n)$, there exists a *base* $V_a^{(1)}, \cdots, V_a^{(h)}$ such that any V_a on V_n satisfies an equivalence

$$\rho V_a \sim \rho_1 V_a^{(1)} + \cdots + \rho_h V_a^{(h)},$$

where $\rho, \rho_1, \cdots, \rho_h$ are integers $(\rho \neq 0)$. No attempt is made to prove this "theorem of the base" in its full generality, but various methods are suggested while proving it for quadrics (Chapter XIII) and for Grassmann varieties (Chapter XIV). These methods apply to varieties which admit transitive groups of automorphisms.

The chapter on quadrics is particularly welcome for its clarity and completeness. Here it is shown, for instance, that the S_d's on a non-singular quadric Q_{n-1} form a single irreducible system of dimension $(d+1)(2n-2-3d)/2$. The classical theory of "eight associated points" (the complete intersection of three ordinary quadrics Q_2) is generalized as follows: If $P_1 = 0, \cdots, P_r = 0$ are the envelope equations of r points $(r \geqq 2n+2)$ such that any quadric Q_{n-1} through all but one of the points passes through the remaining one, then there exists an identity of the form

$$\lambda_1 P_1^2 + \cdots + \lambda_r P_r^2 = 0.$$

The chapter on Grassmann varieties includes applications to enumerative geometry. For instance, the theory immediately yields the interesting result (p. 366) that two quadrics Q_3 in S_4 have, in general, sixteen common lines.

The book closes with a few pages of Bibliographical Notes, giving due credit to Cayley, Castelnuovo, Enriques, Severi, Macaulay, Lefschetz, van der Waerden, Chevalley, Zariski, Weil and others. The authors have skilfully blended the work of these illustrious men with many original ideas of their own. Their generality of outlook inevitably makes the book somewhat difficult to read, and there are no diagrams. However, an excellent index enables the reader to find much of interest without going through all the details. The Cambridge University Press must again be congratulated on a superb job of printing.

H. S. M. COXETER

BRIEF MENTION

Introduction to the theory of distribution. By Israel Halperin. Based on Lectures by Laurent Schwartz. (Canadian Mathematical Congress, Lecture Series, No. 1.) University of Toronto Press, 1952. 33 pp.

This brief pamphlet is a prolegomenon, though not exclusively so, to volume 1 of Schwartz' own two-volume book, and thus it deals only with the general operational aspect of distributions, and not with the more toilsome (and less innovatory) Fourier analysis in open Euclidean space.

Our one criticism of the exposition is this—that Halperin follows Schwartz too closely in claiming the "Dirac function" all-out for the theory of distributions, whereas in fact the Dirac function has been used by other theories of representation of linear functionals to illustrate their points at issue with equal fitness.

But what the pamphlet sets out doing it does very well indeed, and although written compactly, it is readable and informative.

S. BOCHNER

Linear computations. By P. S. Dwyer. New York, Wiley, 1951. 12+344 pp. $6.00.

The central theme of Professor Dwyer's book is the computational solution of simultaneous linear equations and various allied topics; considerable emphasis is given to effecting these solutions in a practical computational way on a desk calculator. The text contains a wealth of material on this topic and should provide a valuable tool to the young student of applied mathematics who needs to undertake numerous problems within the compass of hand calculation.

The book differs markedly from other texts on classical computational methods in the singleness of its purpose. It is virtually self-contained in that it pre-supposes essentially nothing beyond a knowledge of high school algebra and builds up such theorems on matrices as are needed. Happily the author has not felt a need to sacrifice rigor for clarity.

The text opens with a quite careful and indeed painstaking elementary discussion of computation with approximate numbers and does not leave this topic until the student has received a thorough grounding in the basic notions of his craft. Only then does it proceed to the main theme. Again near the end of the book the author re-emphasizes the approximate character of the subject by a careful discussion of the errors of linear computation.

The next two chapters are devoted to discussions of various exact methods for solution of a system of linear equations, i.e. all problems involving "round-off" are ignored; while the following chapter is concerned with some of the additional problems that arise when "round-off" is no longer ignored.

The principal linear problems discussed are these: solution of equations; inversion of matrices and consideration of the characteristic equation. In addition to these, as has been mentioned above, there are discussions of error analysis, and finally applications to statistics and non-linear problems.

The young student will find the very numerous and extremely detailed illustrations helpful and he should profit from the carefully selected problems and list of references to the literature.

H. H. GOLDSTINE

Moderna teoria delle funzioni di variabile reale. By G. Vitali and G. Sansone. Part II. *Sviluppi in serie di funzioni ortogonali.* By G. Sansone. (Consiglio Nazionale delle Ricerche. Monografie di Matematica Applicata.) 3d ed. Bologna, Zanichelli, 1952. 7+614 pp. 7000 lire.

The first edition of this treatise was reviewed in vol. 45 (1939) p. 218 and the second edition in vol. 58 (1947) p. 736 of this BULLETIN. The fact that the second edition was exhausted in less than three years is as good a proof of the excellence of the book as could be desired. The new edition is 103 pages longer than the second one. This is partly due to a change of the format and the addition of a number of figures.

Various additions have been made in chapter II: a discussion of the Gibbs phenomenon, bounds for the partial sums of Fourier series of functions of bounded variation, and applications of Fourier series to the stationary state of temperature in an infinite semi-strip and to the isoperimetric problem. A number of additions have been made in chapter IV: expansions in Tchebychef-Laguerre and Tchebychef-Hermite series, discussion of the zeros of Hermite and Laguerre polynomials, their asymptotic behavior for large n, and the applications of these estimates to the expansion problem. In connection with the developments on pp. 357–363 the author might have found some use for the reviewer's paper *Contributions to the theory of Hermitian series* [Duke Math. J. vol. 5 (1939) and Trans. Amer. Math. Soc. vol. 47 (1940)]. The reviewer is gratified to see that the misspelling of the name of Walsh has been corrected. The discussion of convergent sequences of characteristic functions in chapter VI on the Stieltjes integral has been revised. May the author soon have to prepare a fourth edition.

EINAR HILLE

Contributions to the founding of the theory of transfinite numbers. By G. Cantor. Translated, and provided with an introduction and notes, by P. E. B. Jourdain. New York, Dover, n.d. [1952]. 9+211 pp. Paper bound, $1.25; cloth bound, $2.75.

A philosophical essay on probabilities. By Pierre Simon, Marquis de Laplace. Translated from the sixth French edition by F. W. Truscott and F. L. Emory. With an introductory note by E. T. Bell. Dover, New York, n.d. [1952]. 8+196 pp. Paper bound. $1.25; cloth bound, $2.50.

These are reprints, apparently by a photographic process, of translations originally published in 1915 and 1902, respectively; the introductory note in the second is new. Although a form letter accompanying the paper-bound review copies suggests that "for bibliographical accuracy" a review might mention the existence of the cloth-bound edition, the publishers nowhere give the slightest indication that these are merely reprints of old translations. They invite comment on their venture in making these works available in inexpensive form. In view of the current high cost of publication, any economy in the production of technical books is welcome. However, the publishers have chosen to bring out low-priced editions not of new books but of old ones which are now of interest only to historians or to adherents of the "great books" theory of education. It is to be hoped that students will not be misled into buying ostensible bargains with money which would be better spent in making at least a down-payment on a good modern work on one of the subjects concerned.

R. P. Boas, Jr.

Funzioni ellittiche. By F. Tricomi. 2d ed. Bologna, Zanichelli, 1951. 12+343 pp. 4500 lire.

The first edition (1937) was reviewed in Bull. Amer. Math. Soc. vol. 44 (1938) p. 610. The preface states that this edition is closer to the first edition than to the German edition of 1948 (reviewed in Mathematical Reviews vol. 10 (1949) p. 532).

NOTES

The Department of Mathematics of the Massachusetts Institute of Technology wishes to announce the availability of C. L. E. Moore Instructorships in Mathematics for 1953–1954, open to young mathematicians with doctorates who show definite promise in research.

Applications should be filed not later than January 23, 1953, on forms obtained from the Department.

Dr. A. G. Walker of the University of Sheffield has been appointed to a professorship at The University, Liverpool, England.

Dr. Shmuel Agmon of New York University has been appointed lecturer at The Hebrew University.

Dr. J. W. Armstrong of Purdue University has accepted a position with the Aerophysics Section, Convair, Fort Worth, Texas.

Dr. R. B. Barrar of the University of Michigan has accepted a position as mathematician with the McMillan Laboratory, Inc., Ipswich, Massachusetts.

Professor F. S. Beckman of Pratt Institute has accepted a position as senior mathematician with the International Business Machine Corporation, New York City.

Professor H. A. Bernhard of the Evans Signal Corp Laboratory has accepted a position as senior engineer with the Curtiss-Wright Corporation.

Assistant Professor Evelyn Boyd of Fisk University has accepted a position as mathematician with the National Bureau of Standards, Washington, D. C.

Dr. Paul Brock of the Reeves Instrument Company, Inc. has accepted a position as mathematician with the Consolidated Engineering Corporation, Pasadena, California.

Dr. R. K. Butz of the University of Georgia has been appointed to an assistant professorship at Colorado Agricultural and Mechanical College.

Dr. Jean M. Calloway of the University of Pennsylvania has been appointed to an assistant professorship at Carleton College.

Dr. J. W. Carr of Massachusetts Institute of Technology has accepted a position as research mathematician with the Willow Run Research Center, University of Michigan, Ypsilanti, Michigan.

Dr. Peter Chiarulli of Carnegie Institute of Technology has been appointed to an assistant professorship at Brown University.

Dr. E. A. Coddington of Massachusetts Institute of Technology has been appointed to an assistant professorship at the University of California, Los Angeles.

Assistant Professor E. G. H. Comfort of Illinois Institute of Technology has been appointed to an associate professorship at Ripon College.

Dr. B. V. Dean of the University of Illinois has accepted a position as mathematician with the Department of Defense, Arlington, Virginia.

Dr. R. Y. Dean of California Institute of Technology has accepted a position as mathematician with the General Electric Company, Richland, Washington.

Visiting Professor A. H. Diamond of the University of Kansas has accepted a position as Chief of the Mathematics Branch, Office of Ordnance Research, Durham, North Carolina.

Dr. V. E. Dietrich of Alabama Polytechnic Institute has been appointed to a professorship of physics at North Central College.

Dr. Flora Dinkines of the University of Chicago has been appointed to an assistant professorship at the Chicago Branch of the University of Illinois.

Dr. Jim Douglas of Rice Institute has been appointed to an assistant professorship at the University of Alabama.

Dr. G. F. D. Duff of Massachusetts Institute of Technology has been appointed to an assistant professorship at the University of Toronto.

Professor W. L. Duren of Tulane University of Louisiana has been appointed acting program director for mathematics at the National Science Foundation.

Dr. Joanne Elliott of Swarthmore College has been appointed to an assistant professorship at Mount Holyoke College.

Dr. M. P. Emerson of the University of Illinois has been appointed to an assistant professorship at Harpur College, State University of New York.

Dr. M. H. M. Esser of Illinois Institute of Technology has been appointed to an assistant professorship at Georgia Institute of Technology.

Professor Chester Feldman of Antioch College has accepted a position as mathematical consultant with Aircraft Marine Products, Inc., Harrisburg, Pennsylvania.

Dr. Abraham Franck of Kansas State College of Agriculture and Applied Science has accepted a position as senior mathematician with Engineering Research Associates, St. Paul, Minnesota.

Mr. E. T. Frankel of the Health and Welfare Federation of Allegheny County has accepted a position as regional research analyst with Federal Security Agency, New York, New York.

Miss Joyce B. Friedman of the Logistics Research Project, George Washington University, has accepted a position as mathematician with the Department of Defense, Washington, D. C.

Dr. L. E. Fuller of the Goodyear Aircraft Corporation has been appointed to an assistant professorship at Kansas State College of Agriculture and Applied Science.

Dr. T. S. George has accepted a position as consultant with the Missile Test Center, Patrick Air Force Base, Florida.

Dr. Murray Gerstenhaber has been appointed a member of the Institute for Advanced Study.

Dr. Abolghassem Ghaffari of the Institute for Advanced Study has been appointed to a professorship at the University of Teheran.

Professor Casper Goffman of the University of Oklahoma has been appointed to an associate professorship at Wayne University.

Assistant Professor H. E. Goheen of the University of Pennsylvania has been appointed to an associate professorship at Iowa State College of Agriculture and Mechanic Arts.

Dr. T. N. E. Greville of the National Office of Vital Statistics, Public Health Service, has been appointed a technical adviser in Vital and Health Statistics at the Institute of Inter-American Affairs, Rio de Janeiro, Brazil.

Professor F. L. Griffin of Reed College has been appointed to a visiting professorship at Wesleyan University.

Assistant Professor John Gurland of the University of Chicago has been appointed to an associate professorship at Iowa State College of Agriculture and Mechanic Arts.

Dr. Carl Hammer of Walter Harvey Junior College has been appointed a research consultant at the Bureau of Applied Social Research, Columbia University.

Dr. Mary-Elizabeth Hamstrom of the University of Texas has been appointed to an assistant professorship at Goucher College.

Mr. L. S. Hart of Kenyon College has been appointed an analyst at the Department of Defense, Washington, D. C.

Professor H. L. Harter of Michigan State College has been appointed a mathematical statistician at Wright-Patterson Air Force Base.

Mr. R. M. Hayes of the Institute for Numerical Analysis, National Bureau of Standards, has accepted a position as mathematician with the Hughes Aircraft Company, Los Angeles, California.

Mr. G. B. Hedrick of Stanford University has accepted a position as training specialist with the Northrop Aircraft, Inc., Hawthorne, California.

Mr. Hugo Heermann of the University of Cincinnati has accepted a position with the National Advisory Committee for Aeronautics.

Dr. C. C. Hsiung of Harvard University has been appointed to an assistant professorship at Lehigh University.

Dr. R. V. Kadison of the Institute for Advanced Study has been appointed to an assistant professorship at Columbia University.

Miss Dora E. Kearney has been appointed to an assistant professorship at Westminster College.

Dr. J. L. Kelley of the University of California has been appointed to a visiting associate professorship at the University of Kansas.

Professor J. R. Kline of the University of Pennsylvania is on leave of absence and has been appointed to a visiting professorship at the University of Tubingen, Germany.

Mr. R. E. Krucklin of the American Viscose Corporation has accepted a position as senior engineer with A. H. Johnson and Company, Inc., New York, New York.

Dr. Harold W. Kuhn of the University of Paris has been appointed to an assistant professorship at Bryn Mawr College.

Dr. J. R. Larkin of the University of Kansas has accepted a position as mathematician with the Bureau of Ordnance, Navy Department, Washington, D. C.

Dr. G. E. Latta of Princeton University has been appointed to an assistant professorship at the University of British Columbia.

Professor Walter Leighton, who was a consultant to the Air Research and Development Command in Baltimore during the summer, has returned to Washington University.

Professor Hans Lewy has been appointed to a visiting professorship at Harvard University.

Mr. H. M. Lieberstein has accepted a position as mathematician with the U. S. Naval Proving Ground, Dahlgren, Virginia.

Mr. O. I. Litoff of the University of Chicago has been appointed to an assistant professorship at the University of Massachusetts.

Mr. D. B. Lowdenslager of the University of Virginia has accepted a position with the Institute for Cooperative Research, Johns Hopkins University.

Dr. H. R. Luxenberg of the Hughes Research and Development Laboratories has been appointed a lecturer at the University of California, Los Angeles.

Mr. J. J. Lynn has been appointed to the staff of the Oak Ridge National Laboratory.

Dr. Ella C. Marth of Harris Teachers College has been appointed to a professorship at Wilson Teachers College.

Dr. W. K. Moore of the University of Kansas has been appointed to an assistant professorship at Albion College.

Dr. J. C. Morelock of the University of Florida has been appointed to an assistant professorship at Alabama Polytechnic Institute.

Dr. G. D. Mostow of Syracuse University has been appointed to an assistant professorship at Johns Hopkins University.

Professor E. J. Moulton of Northwestern University has retired.

Mr. J. D. Neff of Purdue University has accepted a position as a member of the Technical Staff with the Bell Telephone Laboratories, New York, New York.

Mr. L. W. Neustadt of the Bell Aircraft Corporation has accepted a position as mathematician with the Reeves Instrument Company, New York, New York.

Mr. August Newlander of the University of Chicago has been appointed a staff research mathematician at the University of Denver.

Mr. E. M. Olson of Columbia University has accepted a position as aeronautical research scientist with the National Advisory Committee for Aeronautics, Hampton, Virginia.

Dr. Anne F. O'Neill of Smith College has been appointed to an assistant professorship at Wheaton College.

Dr. D. B. Owen of the University of Washington has been appointed to an assistant professorship at Purdue University.

Mr. S. T. Paine of the University of California has accepted a position as research mathematician with the Convair Aircraft Company, San Diego, California.

Dr. Howard Raiffa of the University of Michigan has been appointed to an assistant professorship at Columbia University.

Mr. A. G. Rawling of the Naval Ordnance Laboratory has accepted a position as D. I. C. staff member with the Digital Computer Laboratory, Massachusetts Institute of Technology.

Dr. W. C. Royster of the University of Kentucky has been appointed to an assistant professorship at Alabama Polytechnic Institute.

Dr. Walter Rudin of Massachusetts Institute of Technology has been appointed to an assistant professorship at the University of Rochester.

Assistant Professor Arthur Saastad of DePaul University has accepted a position as industrial engineer with United States Steel Company, Chicago, Illinois.

Mr. Yomei Sawanobori of Princeton University has accepted a position as junior mathematician with the Cornell Aeronautical Laboratory, Inc.

Dr. D. R. Scholz of Southwest Louisiana Institute has been appointed to an assistant professorship at Louisiana State University.

Dr. R. J. Seeger of the University of Maryland has been appointed physical science administrator at the National Science Foundation, Washington, D. C.

Dr. H. S. Shapiro of Massachusetts Institute of Technology has accepted a position as a member of the technical staff with the Bell Telephone Laboratories, Murray Hill, New Jersey.

Dr. R. W. Shephard of the Rand Corporation has accepted a position with the Sandia Corporation, Albuquerque, New Mexico.

Dr. I. M. Singer of Massachusetts Institute of Technology has been appointed to an assistant professorship at the University of California, Los Angeles.

Dr. K. T. Smith of the University of Wisconsin has been appointed to an assistant professorship at the University of Kansas.

Professor D. E. South of the University of Kentucky has been appointed to a professorship at the University of Florida.

Dr. W. L. Stamey of the University of Missouri has been appointed to an assistant professorship at the Atlanta Division, University of Georgia.

Dr. M. L. Stein of the University of California has accepted a position as mathematician with the International Business Machines Corporation, Los Angeles, California.

Dr. R. C. Taliaferro of the Portsmouth Priory School has been appointed to a visiting associate professorship at Notre Dame University.

Dr. D. L. Thomsen of the California Institute of Technology has been appointed to an assistant professorship at Pennsylvania State College.

Dr. Peter Thullen of the Social Security Department, Panama, has accepted a position as a member of the Social Security Division with the International Labour Office, Geneva, Switzerland.

Dr. C. B. Tompkins of George Washington University has been appointed a mathematician at the National Bureau of Standards, University of California.

Mr. C. H. Tross of the University of Illinois has accepted a position as mathematician with the Wright Air Development Center, Dayton, Ohio.

Mr. J. S. Vigder of the University of Saskatchewan has accepted a position as research officer with the Defense Research Board of Canada, Ottawa, Ontario, Canada.

Dr. A. H. Wallace of the University of Chicago is on leave of absence and has been appointed a lecturer in mathematics at the University College, Dundee, Scotland.

Dr. M. J. Walsh of the University of Illinois has been appointed to an assistant professorship at Florida State University.

Dr. S. S. Walters of the University of California has accepted a position as operations analyst with the Operations Research Office, Chevy Chase, Maryland.

Dr. H. C. Wang of Louisiana State University has been appointed to a research professorship at Alabama Polytechnic Institute.

Associate Professor L. E. Wear of California Institute of Technology has retired.

Professor Hassler Whitney of Harvard University has been appointed a permanent member of the Institute for Advanced Study.

Dr. N. Z. Wolfsohn of the University of Maryland has been appointed a member of the Institute for Advanced Study.

Professor C. H. Yeaton of Oberlin College has retired with the title professor emeritus.

Dr. D. M. Young of the Ballistic Research Laboratory, Aberdeen Proving Ground, has been appointed to an assistant professorship at the University of Maryland.

Dr. F. H. Young of the U. S. Naval Bureau of Ordnance has accepted a position as mathematician with the U. S. Naval Ordnance Test Station, China Lake, California.

The following promotions are announced:

Lars Gårding, Lund University, to a professorship.

Kurt Mahler, University of Manchester, to a professorship.

Hans Rådström, University of Stockholm, to an assistant professorship.

J. P. van Alstyne, Hamilton College, to an assistant professorship.

R. R. Bernard, Yale University, to an assistant professorship.

J. G. Bowker, Middlebury College, to a Baldwin professorship of Mathematics and Natural Philosophy.

G. H. Butcher, Howard University, to an assistant professorship.

A. J. Coleman, University of Toronto, to an assistant professorship.

W. J. Dixon, University of Oregon, to a professorship.

F. G. Dressel, Duke University, to a professorship.

J. C. Eaves, Alabama Polytechnic Institute, to a professorship.

D. O. Ellis, University of Florida, to an associate professorship.

Thorleif Fostvedt, University of Alberta, to an assistant professorship.

I. C. Gentry, Wake Forest College, to an associate professorship.

W. H. Gottschalk, University of Pennsylvania, to an associate professorship.

R. E. Greenwood, University of Texas, to an associate professorship.

L. A. Henkin, University of Southern California, to an associate professorship.

E. D. Jenkins, Kent State University, to a professorship.

William Karush, University of Chicago, to an associate professorship.

M. S. Kramer, New Mexico College of Agriculture and Mechanic Arts, to an associate professorship.

R. M. Lakness, University of California, to an assistant professorship.

W. G. Leavitt, University of Nebraska, to an associate professorship.

A. J. Lohwater, University of Michigan, to an assistant professorship.

G. G. Lorentz, University of Toronto, to an assistant professorship.

Leonard McFadden, Virginia Polytechnic Institute, to a professorship.

G. R. MacLane, The Rice Institute, to an associate professorship.

William Marcaccio, Xavier University, to an associate professorship.

K. O. May, Carleton College, to a professorship.

H. A. Meyer, University of Florida, to a professorship.

J. P. Nash, University of Illinois, to a research professorship.

D. A. Norton, University of California, to an assistant professorship.

J. K. Peterson, Western Michigan College, to an associate professorship.

P. E. Pfeiffer, The Rice Institute, to an assistant professorship.

Adrienne S. Rayl, University of Alabama, to a professorship.

H. J. Ryser, Ohio State University, to an associate professorship.

Edith R. Schneckenburger, University of Buffalo, to an associate professorship.

Annette Sinclair, Southern Illinois University, to an assistant professorship.

Jerome C. Smith, Lafayette College, to an associate professorship.

J. S. Thale, Loyola University, to an assistant professorship.

C. J. Thorne, University of Utah, to a professorship.

A. E. Ventriglia, Manhattan College, to an assistant professorship.

L. F. Walton, Santa Barbara College, University of California, to an associate professorship.

Albert Wilansky, Lehigh University, to an associate professorship.

František Wolf, University of California, Berkeley, to a professorship.

The following appointments to instructorships are announced: Alfred University: Mr. N. O. Tiffany; Ball State Teachers College: Mr. D. E. Deal; Brooklyn College: Dr. Melvin Hausner; Polytechnic Institute of Brooklyn: Mr. S. P. Hoffman, Mr. J. L. Katz; University of California, Berkeley: Dr. Maurice Sion; Catholic University of America: Dr. R. J. Silverman; University of Colorado: Mr. Robert Osserman; University of Connecticut: Dr. A. G. Kostenbauder; University of Delaware, Mr. J. L. Howell; Duke University: Mr. J. R. Shoenfield; Harvard University: Dr. E. R. Fadell; Hofstra College: Miss Azelle Brown; Indiana University: Dr. D. M. Nead; Lafayette College: Mr. W. R. Ferrante; Lehigh University: Dr. H. H. Wicke; Loyola University: Mr. T. J. Pignani; University of Maryland: Dr. Elizabeth H. Cuthill, Mr. C. W. McArthur, Miss Jacqueline L. Penez; Massachusetts Institute of Technology: Mr. Edwin Weiss; University of Nebraska: Dr. T. A. Newton; University of New Hampshire: Mr. S. B. Hobbs, Mr. R. E. Hux; Oberlin College: Mr. W. R. Orton; Ohio State University: Dr. L. E. Pursell; University of Oregon: Dr. H. J. Reiter; Princeton University: Mr. L. W. Green, Dr. J. C. Moore; Purdue University: Mr. Leonard Gillman, Dr. Melvin Henriksen; Queens College: Dr. A. L. Fass; Rice Institute: Dr. Arlen H. Brown; University of Rochester: Mr. A. E. Danese; University of Rochester: Mr. R. A. Raimi; Rutgers University: Mr. Solomon Leader, Mr. B. H. McCandless, Mr. R. E. Montgomery, Dr. V. L. Shapiro, Mr. H. G. Tucker; Saint Peter's College: Mr. R. J. Beeber; University of Saskatchewan: Dr. G. H. M. Thomas; Saint Xavier College for Women: Mr. T. E. Hagensee; Texas Western College of the University of Texas: Mr. W. L. Shepherd; University of Washington: Dr. L. A. Kokoris; United States Military Academy: Captain A. M. Maish; University of Wisconsin, Milwaukee Extension Division: Mr. J. G. Renno; Yale University: Dr. R. G. Bartle, Dr. F. P. Pedersen.

Professor M. C. Foster of Wesleyan University died on April 9, 1952 at the age of fifty-eight years. He had been a member of the Society for thirty years.

Professor R. R. Shumway of the University of Minnesota died on April 19, 1952. He had been a member of the Society for forty-three years.

Professor Otto Szász of the University of Cincinnati died on September 19, 1952 at the age of sixty-seven years. He had been a member of the Society for twenty-six years.

The note about Mr. Shader in the July issue of this Bulletin, p. 522, should read: Mr. M. A. Shader of Syracuse University has accepted a position as aeronautical research scientist with the Ames laboratory, National Advisory Committee for Aeronautics, Moffett Field, California.

The note about Professor Rauch in the September issue of this Bulletin, p. 588, should read: Dr. H. E. Rauch of the Institute for Advanced Study has been appointed to an assistant professorship at the University of Pennsylvania.

NEW PUBLICATIONS

A guide to tables of the normal probability integral. (National Bureau of Standards Applied Mathematics Series, no. 21.) Washington, Government Printing Office, 1952. 4+16 pp. $.15.

Angot, A. *Compléments de mathématiques à l'usage des ingénieurs de l'électrotechnique et des télécommunications.* 2d ed. Paris, Editions de la Revue d'Optique, 1952. 8+688 pp.

Bari, N. K. See Luzin, N. N.

Bernšteĭn, S. N. *Sobranie sočinenii.* Vol. I. *Konstruktivnaya teoriya funkciĭ* [1905–1930]. Moscow, Izdatel'stvo Akademiĭ Nauk SSSR., 1952. 581 pp., 1 plate. 31.50 rubles.

Bieberbach, L. *Theorie der geometrischen Konstruktionen.* (Lehrbücher und Monographien aus dem Gebiete der exakten Wissenschaften. Mathematische Reihe, vol. 13.) Basel, Birkhäuser, 1952. 8+162 pp. 18.70 Swiss fr.

Black, M. See Frege, G.

Boivie, O. See Laadi, H.

Bückner, H. *Die praktische Behandlung von Integral-Gleichungen.* (Ergebnisse der angewandten Mathematik, vol. 1.) Berlin, Springer, 1952. 6+127 pp. 18.60 DM.

Chattelun, L. *Calcul vectoriel.* Vol. I. *Algèbre linéaire. Applications.* Paris, Gauthier-Villars, 1952. 8+605 pp. 5000 fr.

Cundy, H. M., and Rollett, A. P. *Mathematical models.* Oxford, Clarendon, 1952. 240 pp. $5.50.

La cybernétique. Théorie du signal et de l'information. Réunions d'études et de mises au point tenues sous la présidence de Louis de Broglie. Paris, Editions de la Revue d'Optique, 1951. 6+318 pp. 1600 fr.

Deimel, R. F. *Mechanics of the gyroscope. The dynamics of rotation.* New York, Dover, 1950. 9+192 pp. $3.00.

Description of a magnetic drum calculator. By the Staff of the Computation Laboratory. Cambridge, Harvard University Press, 1952. 12+318 pp., 1 plate. $8.00.

Dobrovol'skiĭ, V. V. *Teoriya mehanismov.* Moscow, Gosudarstvennoe Naučno-Tehniko Izdatel'stvo Mašinostroit. Literatury, 1951. 465 pp.

Eilenberg, S., and Steenrod, N. *Foundations of algebraic topology.* Princeton University Press, 1952. 15+328 pp. $7.50.

Frege, G. *Translations from the philosophical writings of Gottlob Frege.* Ed. by P. Geach and M. Black. New York, Philosophical Library, 1952. 10+244 pp. $5.75.

Freudenthal, H. *Inleiding tot het denken van Einstein.* Assen, Uitgeverij Born, N. V., 1952. 56 pp., 1 plate. 1.45 florins.

Geach, P. See Frege, G.

Gel'fond, A. O. *Isčislenie konečnyh raznosteĭ.* Moscow, Gosudarstvennoe Izdatel'stvo Tehniko-Teoretičeskoi Literatury, 1952. 479 pp. 10.20 rubles.

Gel'fond, A. O., *Rešenie uravneniĭ v celyh čislah.* Moscow, Gosudarstvennoe Izdatel'stvo Tehniko-Teoretičeskoi Literatury, 1952. 63 pp. .85 rubles.

Goodstein, R. L. *The foundations of mathematics.* Leicester, University College, 1951. 27 pp. 1s.

Hardy, G. H. *A course of pure mathematics.* 10th ed. Cambridge University Press, 1952. 12+509 pp. $4.75.

Herrmann, H. *Übungen zur projektiven Geometrie.* (Lehrbücher und Monographien aus dem Gebiete der exakten Wissenschaften. Mathematische Reihe, vol. 18.) Basel, Birkhäuser, 1952. 1+168 pp. 17.00 Swiss fr.

Kaprekar, D. R. *Cycles of recurring decimals (from N=3 to 161 and some other numbers).* Deolali, Privately printed, 1951. 6+55 pp. 5 rupees.

Kneebone, G. T., See Semple, J. G.

Kober, H. *Dictionary of conformal representations.* New York, Dover, 1952. 16+208 pp. $3.95.

Korovkin, P. P. *Neravenstva.* Moscow, Gosudarstvennoe Izdatel'stvo Tehniko-Teoretičeskoi Literatury, 1952. 56 pp. .95 rubles.

Laadi, H., and Boivie, O. *Lagged product sums of H. Wold's normal deviates.* Uppsala, University Institute of Statistics, 1952. 11+51 pp.

van Langendonck, T. *Funções ortogonais na resolução de problemas da teoria da elasticidade.* Vol. I. *Generalidades e torção.* São Paulo, Associação Brasileira de Cimento Portland, 1952. 8+69 pp.

Lavrent'ev, M. A., and Šabat, B. V. *Metody teorii funkcii kompleksnogo peremennogo.* Moscow, Gosudarstvennoe Izdatel'stvo Tehniko-Teoretičeskoi Literatury, 1951. 606+2 pp. 17.70 rubles.

Lietzmann, W. *Altes und Neues vom Kreis.* 2d. ed. Leipzig, Teubner, 1951. 55 pp. 2.10 DM.

Luzin, N. N. *Integral i trigonometričeskii ryad.* Ed. and commentary by N. K. Bari and D. E. Men'šov. Moscow, Gosudarstvennoe Izdatel'stvo Tehniko-Teoretičeskoi Literatury, 1951. 550 pp., 1 plate. 20.45 rubles.

Lyusternik, L. A., and Sobolev, V. I. *Elementy funkcional'nogo analiza.* Moscow, Gosudarstvennoe Izdatel'stvo Tehniko-Teoretičeskoi Literatury, 1951. 360 pp. 11.30 rubles.

Markuševič, A. I. *Očerki po istorii teorii analitičeskih funkcii.* Moscow, Gosudarstvennoe Izdatel'stvo Tehniko-Teoretičeskoi Literatury, 1951. 127 pp. 4.30 rubles.

Mason, M., and Weaver, W. *The electromagnetic field.* New York, Dover, 1952. 13+390 pp. $3.95.

Men'šov, D. E. See Luzin, N. N.

Mercalov, N. I. *Teoriya prostranstvennyh mehanizmov.* Moscow, Gosudarstvennoe Naučno-Tehniko Izdatel'stvo Mašinostroit. Literatury, 1951. 205 pp. 11.40 rubles.

Newton, Sir Isaac. *Opticks, or a treatise of the reflections, refractions, inflections and colours of light.* New York, Dover, 1952. 105+406 pp. $3.95.

Ostroumov, G. A. *Svobodnaya konvekciya v usloviyah vnutrennei zadači.* Moscow, Gosudarstvennoe Izdatel'stvo Tehniko-Teoretičeskoi Literatury, 1952. 256 pp., 27 plates. 10.90 rubles.

Picone, M., and Viola, T. *Lezioni sulla teoria moderna dell'integrazione.* Torino, Einaudi, 1952. 404 pp. 5000 lire.

Ramberg, E. G. See Sommerfeld, A.

Rao, C. R. *Advanced statistical methods in biometric research.* New York, Wiley, 1952. 17+390 pp. $7.50.

Rollett, A. P. See Cundy, H. M.

Šabat, B. V. See Lavrent'ev, M. A.

Segre, B. *Forme differenziali e loro integrali.* Vol. I. *Calcolo algebrico esterno e proprietà differenziali locali.* Rome, Docet, 1951. 520 pp. 3900 lire.

Semple, J. G., and Kneebone, G. T. *Algebraic projective geometry.* Oxford, Clarendon, 1952. 7+404 pp. $7.00.

Smirnov, V. I. *Kurs vysšei matematiki.* Vol. 4. 2d ed. Moscow, Gosudarstvennoe Izdatel'stvo Tehniko-Teoretičeskoi Literatury, 1951. 804 pp. 23.54 rubles.

Snow, C. *Hypergeometric and Legendre functions with applications to integral equations* *of potential theory.* (National Bureau of Standards Applied Mathematics Series, no. 19.) Washington, Government Printing Office, 1952. 11+427 pp. $3.25.

SOBOLEV, V. I. See LYUSTERNIK, L. A.

SOMMERFELD, A. *Electrodynamics.* (Lectures on theoretical physics, vol. 3.) Trans. by E. G. Ramberg. New York, Academic Press, 1952. 13+371 pp. $6.80.

STEENROD, N. See EILENBERG, S.

ŠUMYAGSKIĬ, B. M. *Tablicy dlya rešeniya kubičeskih uravneniĭ metodom osnov.* Moscow, Gosudarstvennoe Izdatel'stvo Tehniko-Teoretičeskoĭ Literatury, 1950. 135 pp. 3.90 rubles.

Tables for the analysis of beta spectra. (National Bureau of Standards Applied Mathematics Series, no. 13.) Washington, Government Printing Office, 1952. 3+61 pp. $.35.

TANNAKA, T. *Sôtai genri.* [The principle of duality.] Tokyo, Iwanami Shoten, 1951. 5+128 pp. 3.20 yen.

VIOLA, T. See PICONE, M.

VOGEL, A. *Klassische Grundlagen der Analysis.* Leipzig, Hirzel, 1952. 10+194 pp. 8.40 DM.

WEAVER, W. See MASON, M.

ZACHARIAS, M. *Einführung in die projektive Geometrie.* 4th ed. Leipzig, Teubner, 1951. 54 pp. 2 DM.

ZÜHLKE, P. *Konstruktionen in begrenzter Ebene.* 3d ed. Leipzig, Teubner, 1951. 42 pp. 2.10 DM.

ERRATA, VOLUME 55

Becker, H. W., *A new proof and application of Kirchoff's rules.* Bull. Amer. Math. Soc. Abstract 55-7-437.

P. 718, lines 5 and 6 of the abstract. For "modes" read "nodes," by which was meant cut-sets of branches meeting at the same nodes.

ERRATA, VOLUME 57

Bourgin, D. G. *Classes of transformations and bordering transformations.*

p. 223. Following equation (1) add: "More generally, with S on X to Z, T on X to Y, and R on Y to Z,

$$(S(x_1 \bigcirc x_2), R(Tx_1 \oplus Tx_2)) \subset \mathfrak{M}.\text{"} \tag{1'}$$

p. 223, line 6 from the bottom, equation (2), and line 4 from the bottom. For "$\mathfrak{N}'$" read "$\mathfrak{M}'$."

p. 224, line 23. For "S is the identity operator" read "Z is the real axis."

p. 224, line 25. For "(1) is" read "(1′) is (with $S = R = \|\ \ \|$)."

p. 225, line 12. For "$< \infty$" read "$= 0$."

p. 225, lines 21 and 22. For "studied $U(x)$ turns out to be" read "$U(x)$ is."

p. 233, line 15. For "neighborhood retract which is" read "neighborhood retract or LC* space which is."

Brahana, H. R., *George Abram Miller, 1863–1951.*

p. 382, line 5. For "1901" read "1909."

Stoker, J. J. Review of *Hydrodynamics, a study in logic, fact, and similitude,* by G. Birkhoff.

p. 497, line 6 from the bottom. For "treats" read "emphasizes."

p. 498, line 8. For "not mentioned" read "insufficiently stressed."

p. 498, line 19. After "differential equations" add "(as was mentioned earlier in the same chapter by the author)."

ERRATA, VOLUME 58

Becker, H. W., *Pythagorean tetrahedrons, hexahedrons, and graphs.* Bull. Amer. Math. Soc. Abstract 58-5-510.

p. 561. For the next to the last sentence of the abstract read "But a P.H. is *geometrically* a hexahedron, pentahedron, or tetrahedron (due to faces lying in the same plane) according as the common face contains the longest edge of one or both component tetrahedrons, or neither. For the latter case, see H. S. M. Coxeter, *Regular polytopes,* 1947, p. 71."

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Proceedings Editorial Committee	G. A. HEDLUND NATHAN JACOBSON* A. C. SCHAEFFER
Transactions Editorial Committee	L. V. AHLFORS J. L. DOOB SAUNDERS MACLANE G. T. WHYBURN
Colloquium Editorial Committee	A. A. ALBERT EINAR HILLE R. L. WILDER
Mathematical Reviews Editorial Committee	WILLIAM FELLER EINAR HILLE HASSLER WHITNEY
Mathematical Surveys Editorial Committee	W. T. MARTIN A. W. TUCKER LEO ZIPPIN
Representatives on Board of Editors of the American Journal of Mathematics	REINHOLD BAER SAMUEL EILENBERG

FORMER PRESIDENT

T. H. HILDEBRANDT

FORMER SECRETARY

J. R. KLINE

MEMBERS AT LARGE

To serve until December, 1951	*To serve until December, 1952*	*To serve until December, 1953*
EMIL ARTIN	WARREN AMBROSE	D. H. BLACKWELL
HERBERT BUSEMANN	P. R. HALMOS	H. F. BOHNENBLUST
M. H. HEINS	MARK KAC	H. S. M. COXETER
G. W. MACKEY	S. B. MYERS	IRVING KAPLANSKY
L. C. YOUNG	D. C. SPENCER	W. T. REID

BOARD OF TRUSTEES

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T. H. HILDEBRANDT
J. R. KLINE
P. A. SMITH
G. T. WHYBURN
JOHN VON NEUMANN, ex officio
A. E. MEDER, JR., ex officio

EXECUTIVE COMMITTEE OF THE COUNCIL

E. G. BEGLE
WILLIAM FELLER
M. H. HEINS
EINAR HILLE
JOHN VON NEUMANN
A. W. TUCKER
J. L. WALSH

---

* For the academic year 1951–52, Professors Green and Jacobson were on leave of absence. Their places on the Council were filled by Professors W. T. Puckett and G. P. Hochschild respectively.

1952
MEMBERS OF THE COUNCIL

Officers

President	John von Neumann
President Elect	G. T. Whyburn
Vice Presidents	Richard Brauer G. A. Hedlund Deane Montgomery
Secretary	E. G. Begle
Associate Secretaries	L. W. Cohen J. W. Green* W. M. Whyburn J. W. T. Youngs
Treasurer	A. E. Meder, Jr.
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Bulletin Editorial Committee	W. T. Martin G. B. Price
Proceedings Editorial Committee	G. A. Hedlund Nathan Jacobson* A. C. Schaeffer
Transactions Editorial Committee	L. V. Ahlfors J. L. Doob Saunders MacLane G. T. Whyburn
Colloquium Editorial Committee	A. A. Albert Einar Hille R. L. Wilder
Mathematical Reviews Editorial Committee	William Feller Einar Hille Hassler Whitney
Mathematical Surveys Editorial Committee	W. T. Martin R. J. Walker Leo Zippin
Representatives on Board of Editors of the American Journal of Mathematics	Reinhold Baer Samuel Eilenberg

Former President

T. H. Hildebrandt

Former Secretary

J. R. Kline

Members at Large

To serve until December, 1952	*To serve until December, 1953*	*To serve until December, 1954*
Warren Ambrose	D. H. Blackwell	L. V. Ahlfors
P. R. Halmos	H. F. Bohnenblust	C. B. Allendoerfer
Mark Kac	H. S. M. Coxeter	R. H. Bing
S. B. Myers	Irving Kaplansky	E. R. Lorch
D. C. Spencer	W. T. Reid	J. C. Oxtoby
M. H. Heins†		

BOARD OF TRUSTEES

B. P. Gill
T. H. Hildebrandt
Einar Hille
J. R. Kline
P. A. Smith
John von Neumann, ex officio
A. E. Meder, Jr., ex officio

EXECUTIVE COMMITTEE OF THE COUNCIL

E. G. Begle
William Feller
G. A. Hedlund
M. H. Heins
John von Neumann
G. T. Whyburn
R. L. Wilder

---

* For the academic year 1951–52, Professors Green and Jacobson were on leave of absence. Their places on the Council were filled by Professors W. T. Puckett and G. P. Hochschild respectively.

† Professor Heins was elected a member-at-large of the Council for the period 1949–51. In January 1951, he was elected to the Executive Committee of the Council. According to Article VII, Section 4, of the by-laws, he remained on the Council through 1952.

REPRESENTATIVES AND COMMITTEES OF THE SOCIETY

Representatives of the Society in the Division of Mathematics of the National Research Council:

1951–1952—G. C. Evans, Einar Hille, J. R. Kline, Marston Morse, D. C. Spencer, Oscar Zariski.
1952–1953—G. C. Evans, Einar Hille, J. R. Kline, E. J. McShane, D. C. Spencer, R. L. Wilder.
1953–1954—W. L. Duren, J. R. Kline, Norman Levinson, E. J. McShane, D. C. Spencer, R. L. Wilder.

Representatives on the Council of the American Association for the Advancement of Science:

1951—H. F. Bohnenblust, L. R. Ford.
1952–1953—S. S. Cairns, M. R. Hestenes.

Representatives on the Editorial Board of the Annals of Mathematics:

1951—J. L. Doob, Norman Levinson, R. L. Wilder.
1952–1954—Richard Arens, J. L. Doob, Charles Loewner.

Representatives on the Editorial Board of the Duke Mathematical Journal:

1951–1952—C. B. Allendoerfer, R. P. Boas, Jr.
1953–1954—R. H. Cameron, S. S. Chern.

Representative on the American Year Book:

Hans Rademacher.

Representative on Advisory Board of Applied Mechanics Reviews:

K. O. Friedrichs.

Representative on U. S. Committee on Theoretical and Applied Mechanics:

1951–1952—Eric Reissner.
1953–1954—M. H. Martin.

Liaison Officer in Connection with Editorial Management of Quarterly of Applied Mathematics:

J. B. Rosser.

Colloquium Lecturers:

1896—James Pierpont.
Maxime Bôcher.
1898—W. F. Osgood.
A. G. Webster.
1901—Oskar Bolza.
E. W. Brown.
1903—H. S. White.
F. S. Woods.
E. B. Van Vleck.
1906—E. H. Moore.
E. J. Wilczynski.
Max Mason.
1909—G. A. Bliss.
Edward Kasner.
1913—L. E. Dickson.
W. F. Osgood.
1916—G. C. Evans.
Oswald Veblen.
1920—G. D. Birkhoff.
F. R. Moulton.
1925—L. P. Eisenhart.
Dunham Jackson.
1927—E. T. Bell.
Anna Pell-Wheeler.
1928—A. B. Coble.
1929—R. L. Moore.
1930—Solomon Lefschetz.

1931—Marston Morse.
1932—J. F. Ritt.
1934—R. E. A. C. Paley.
Norbert Wiener.
1935—H. S. Vandiver.
1936—E. W. Chittenden.
1937—John von Neumann.
1939—A. A. Albert.
M. H. Stone.
1940—G. T. Whyburn.
1941—Oystein Ore.
1942—R. L. Wilder.
1943—E. J. McShane.
1944—Einar Hille.
1945—Tibor Radó.
1946—Hassler Whitney.
1947—Oscar Zariski.
1948—Richard Brauer.
1949—G. A. Hedlund.
1951—Deane Montgomery.
1952—Alfred Tarski.

Committee to Select Gibbs Lecturers for 1952 and 1953:

L. V. Ahlfors (Chairman), S. Bochner, N. Jacobson.

Gibbs Lecturers:

1923—M. I. Pupin.
1924—Robert Henderson.
1925—James Pierpont.
1926—H. B. Williams.
1927—E. W. Brown.
1928—G. H. Hardy.
1929—Irving Fisher.
1930—E. B. Wilson.
1931—P. W. Bridgman.
1932—R. C. Tolman.
1934—Albert Einstein.
1935—Vannevar Bush.
1936—H. N. Russell.
1937—C. A. Kraus.
1939—Theodore von Kármán.
1941—Sewall Wright.
1943—Harry Bateman.
1944—John von Neumann.
1945—J. C. Slater.
1946—Subrahmanyan Chandrasekhar.
1947—P. M. Morse.
1948—Hermann Weyl.
1949—Norbert Wiener.
1950—G. E. Uhlenbeck.
1951—Kurt Gödel.
1952—Marston Morse.

Committee on Society Visiting Lectureship:

R. P. Boas, Jr. (Chairman), Richard Courant, H. S. Wall.

Visiting Lecturers:

1927–1928—Constantin Carathéodory, University of Munich.
1928–1929—Hermann Weyl, Zurich Technical School.
1929–1930—Enrico Bompiani, University of Rome.
1930–1931—Wilhelm Blaschke, University of Hamburg.
1931–1932—R. L. Moore, University of Texas.
1936–1937—Thirukkannapuram Vijayaraghavan, University of Dacca.
1948–1949—Casimir Kuratowski, University of Warsaw.
1951–1952—Marshall H. Stone, University of Chicago.

Auditors for 1952:

B. P. Gill, C. H. W. Sedgewick.

Committee on Publication of Moore Manuscript:

R. W. Barnard, E. W. Chittenden, L. M. Graves, T. H. Hildebrandt, M. H. Ingraham.

Committees to Select Hour Speakers:

For Annual and Summer Meetings: E. G. Begle (Chairman), Deane Montgomery, Tibor Radó.

For Eastern Sectional Meetings: L. W. Cohen (Chairman), M. H. Heins, R. D Schafer.

For Western Sectional Meetings: J. W. T. Youngs (Chairman), P. R. Halmos, Marshall Hall.

For Far Western Sectional Meetings: J. W. Green (Chairman), A. P. Morse, R. P. Dilworth.

For Southeastern Sectional Meetings: W. M. Whyburn (Chairman), B. J. Pettis, Wallace Givens.

Committee on Places of Meetings:

J. M. Thomas (Chairman), Orrin Frink, M. F. Smiley.

Policy Committee for Mathematics:

Representatives of American Mathematical Society: S. S. Cairns, J. R. Kline (Chairman), W. T. Martin, Marston Morse.

Representative of Association for Symbolic Logic: Alfred Tarski.

Representative of Institute of Mathematical Statistics: Mina Rees.

Representative of Mathematical Association of America: J. S. Frame, H. M. Gehman, Saunders MacLane.

Representative of National Council of Teachers of Mathematics: E. H. C. Hildebrandt.

Secretary: E. G. Begle.

Committee on Applied Mathematics:

M. H. Martin (Chairman), R. V. Churchill, K. O. Friedrichs, F. J. Murray, John von Neumann, Eric Reissner.

Editorial Committee for Applied Mathematics Symposium Proceedings:

Eric Reissner (Chairman), R. V. Churchill, A. H. Taub.

Committee on Translation of Russian and Other Mathematical Articles:

J. V. Wehausen (Chairman), Samuel Eilenberg, Irving Kaplansky, D. H. Lehmer, William Prager, G. Y. Rainich.

FORMER PRESIDENTS

J. H. Van Amringe, 1889–1890.
Emory McClintock, 1891–1894.
G. W. Hill, 1895–1896.
Simon Newcomb, 1897–1898.
R. S. Woodward, 1899–1900.
E. H. Moore, 1901–1902.
T. S. Fiske, 1903–1904.
W. F. Osgood, 1905–1906.
H. S. White, 1907–1908.
Maxime Bôcher, 1909–1910.
H. B. Fine, 1911–1912.
E. B. Van Vleck, 1913–1914.
E. W. Brown, 1915–1916.
L. E. Dickson, 1917–1918.
Frank Morley, 1919–1920.
G. A. Bliss, 1921–1922.
Oswald Veblen, 1923–1924.
G. D. Birkhoff, 1925–1926.
Virgil Snyder, 1927–1928.
E. R. Hedrick, 1929–1930.
L. P. Eisenhart, 1931–1932.
A. B. Coble, 1933–1934.
Solomon Lefschetz, 1935–1936.
R. L. Moore, 1937–1938.
G. C. Evans, 1939–1940.
Marston Morse, 1941–1942.
M. H. Stone, 1943–1944.
T. H. Hildebrandt, 1945–1946.
Einar Hille, 1947–1948.
J. L. Walsh, 1949–1950

ENDOWMENT FUND

In 1923 an Endowment Fund was collected to meet the greater demands on the Society's publication program caused by the ever increasing number of important mathematical memoirs. Of this fund, which now amounts to some $94,000, a considerable proportion was contributed by members of the Society. Under the terms of the will of the late Robert Henderson, for many years a Trustee of the Society, the Society receives approximately $4,000 yearly. Upon the death of the other legatees, the Society will receive the entire principal of the estate for its Endowment Fund.

SPECIAL FUNDS

The Bôcher Memorial Prize

This prize was founded in memory of Professor Maxime Bôcher. It is awarded every five years for a notable research memoir in analysis which has appeared during the preceding five years in a recognized journal published in the United States or Canada; the recipient must be a member of the Society, and not more than fifty years old at the time of publication of his memoir.

First (Preliminary) Award, 1923: To G. D. Birkhoff, for his memoir *Dynamical systems with two degrees of freedom.*

Second Award, 1924: To E. T. Bell, for his memoir *Arithmetical paraphrases,* and to Solomon Lefschetz, for his memoir *On certain numerical invariants with applications to abelian varieties.*

Third Award, 1928: To J. W. Alexander, for his memoir *Combinatorial analysis situs.*

Fourth Award, 1933: To Marston Morse, for his memoir *The foundations of a theory of the calculus of variations in the large in m-space,* and to Norbert Wiener, for his memoir *Tauberian theorems.*

Fifth Award, 1938: To John von Neumann, for his memoir *Almost periodic functions and groups.*

Sixth Award, 1943: To Jesse Douglas, for his memoirs *Green's functions and the problem of Plateau, The most general form of the problem of Plateau,* and *Solution of the inverse problem of the calculus of variations.*

Seventh Award, 1948: To A. C. Schaeffer and D. C. Spencer for their memoir *Coefficients of schlicht functions.*

The Frank Nelson Cole Prize in Algebra.
The Frank Nelson Cole Prize in the Theory of Numbers.

These prizes were founded in honor of Professor Frank Nelson Cole, on the occasion of his retirement as Secretary of the American Mathematical Society and Editor of the Bulletin, after twenty-five years of service; the fund was later doubled by his son, Charles A. Cole. They are awarded at five-year intervals for contributions to algebra and the theory of numbers, respectively, under restrictions similar to those for the Bôcher prize.

First Award, 1928: To L. E. Dickson, for his book *Algebra und ihre Zahlentheorie*, Zurich, 1927.

Second Award, 1931: To H. S. Vandiver, for his several papers on Fermat's last theorem published in the Transactions of the American Mathematical Society and the Annals of Mathematics during the last five years, with special reference to a paper entitled *On Fermat's last theorem*, which appeared in volume 31 of the Transactions.

Third Award, 1939: to A. A. Albert, for his papers on the construction of Riemann matrices, published in volumes 35 and 36 of the Annals of Mathematics.

Fourth Award, 1941: To Claude Chevalley, for his paper entitled *La théorie du corps de classes*, published in volume 41 of the Annals of Mathematics.

Fifth Award, 1944: To Oscar Zariski, for four papers on algebraic varieties, published in volumes 61 and 62 of the American Journal of Mathematics and volumes 40 and 41 of the Annals of Mathematics.

Sixth Award, 1946: To H. B. Mann for his paper entitled *A proof of the fundamental theorem on the density of sums of sets of positive integers*, published in volume 43 of the Annals of Mathematics.

Seventh Award, 1949: To Richard Brauer, for his memoir *On Artin's L-series with general group characters*, published in volume 48 of the Annals of Mathematics.

Eighth Award, 1951: To Paul Erdös, for his many papers in the Theory of Numbers, and in particular for his paper, *On a new method in elementary number theory which leads to an elementary proof of the prime number theorem*, published in the Proceedings of the National Academy, vol. 35, pp. 374–385, July 1949.

The Eliakim Hastings Moore Fund.

This fund was founded in 1922 in honor of Professor Eliakim Hastings Moore, on the occasion of the twenty-fifth anniversary meeting of the Chicago Section of the American Mathematical Society. The income from the fund is to be used at the discretion of the Council of the Society, for the publication of important mathematical books or memoirs, or the award of prizes.

The Marion Reilly Fund.

Dean Marion Reilly willed to the American Mathematical Society a portion of her estate to be used for the advancement of research in pure mathematics. The principal of this fund is $23,500.

The Ernest William Brown Fund.

From the estate of Professor Ernest William Brown, a fund of $1,000 is available, the interest on which, at the discretion of the Council, can be used for the furtherance of such mathematical interests as (a) the publication of important mathematical books, memoirs, and periodicals, and (b) lectures to be delivered on special occasions by invited guests of the Society.

The Robert Henderson Fund.

Upon his retirement from the Board of Trustees on December 31, 1940, Dr. Robert Henderson presented to the Society an unrestricted gift of $1,000 which the Trustees have set aside as the Robert Henderson Fund.

The James K. Whittemore Fund.

From the estate of Professor James K. Whittemore, a fund of $1,000 is available for use in the interest of mathematical research.

The Generva B. Hutchinson Fund.

A fund from the estate of the late Professor Generva B. Hutchinson for use, at the discretion of the Council and Trustees, in the advancement of mathematical research.

The Helen A. Merrill Fund.

From the estate of the late Professor Helen A. Merrill, a fund of $650 is available for use in the interest of mathematical research.

Library Proceeds Fund.

A fund of $50,000 established by the Trustees in 1951 from the proceeds of the sale of the Society's Library.

BY-LAWS OF THE
AMERICAN MATHEMATICAL SOCIETY

ARTICLE I

OFFICERS

Section 1. There shall be a President, a President Elect (during the even-numbered years only), an Ex-president (during the odd-numbered years only), three Vice Presidents, a Secretary, four Associate Secretaries, and a Treasurer.

Section 2. It shall be a duty of the President to deliver an address before the Society at the close of his term of office or within one year thereafter.

ARTICLE II

BOARD OF TRUSTEES

Section 1. There shall be a Board of Trustees consisting of seven trustees, five trustees elected by the Society in accordance with Article VII, together with the President and Treasurer of the Society ex officio. The Board of Trustees shall designate its own presiding officer and secretary.

Section 2. The function of the Board of Trustees shall be to receive and administer the funds of the Society, to have full legal control of its investments and properties, to make contracts and, in general, to conduct all business affairs of the Society.

Section 3. The Board of Trustees shall have the power to appoint a manager and such assistants and agents as may be necessary or convenient to facilitate the conduct of the affairs of the Society, and to fix the terms and conditions of their employment. The Board may delegate to the officers of the Society duties and powers normally inhering in their respective corporative offices, subject to supervision by the Board. The Board of Trustees may appoint committees to facilitate the conduct of the financial business of the Society and delegate to such committees such powers as may be necessary or convenient for the proper exercise of those powers. Agents appointed, or members of committees designated, by the Board of Trustees need not be members of the Board.

Nothing herein contained shall be construed to empower the Board of Trustees to divest itself of responsibility for, or legal control of, the investments, properties and contracts of the Society.

ARTICLE III

PUBLICATIONS COMMITTEES

Section 1. There shall be eight Publications Committees, which shall be the seven Editorial Committees specified in Section 2 of this Article and the Committee on Printing and Publishing specified in Section 3 of this Article.

Section 2. There shall be seven Editorial Committees, as follows: a committee of two members for the *Bulletin;* committees of three members each of the *Proceedings*, the *Colloquium Publications*, *Mathematical Surveys* and *Mathematical Reviews;* a committee of four members jointly for the *Transactions* and the *Memoirs*, and one consisting of two representatives of the Society on the Board of Editors of the *American Journal of Mathematics*.

Section 3. There shall be a Committee on Printing and Publishing consisting of three members.

Section 4. At the first election of members of the Committee on Printing and Publishing, one member shall be elected for a term of one year, one for a term of two years, and one for a term of three years. This section shall be repealed without further action on January 1, 1954.

ARTICLE IV

COUNCIL

Section 1. The Council shall consist of fifteen members at large and the following ex officio members: the officers of the Society specified in Article I, the members of the Publications Committees specified in Article III, any former Secretary for a period of two years following his term of office, and members of the Executive Committee who remain on the Council by the operation of Article VII, Section 4.

Section 2. The Council shall formulate and adminster the scientific policies of the Society and shall act in an advisory capacity to the Board of Trustees.

Section 3. In the absence of the Secretary from any meeting of the Council, one of the Associate Secretaries present may be designated as Acting Secretary for the meeting, either by written authorization of the Secretary, or, failing that, by majority agreement among the Associate Secretaries present.

Section 4. All members of the Council shall be voting members. The method for settling matters before the Council at any meeting shall be by majority vote of the members present. If the result of a vote is challenged, it shall be the duty of the presiding officer to determine the true vote by a roll call. In a roll call vote, each Council member shall vote only once (although he may be a member of the Council in several capacities), and he shall state before the vote in which capacity he votes. The group consisting of the four Associate Secretaries shall have one vote, and it shall be divided equally among those who vote as Associate Secretaries. Each of the eight Publications Committees shall have one vote, and it shall be divided equally among those who vote as members of the respective Publications Committees. All other members of the Council shall have one vote each. Fractional votes shall be counted.

Section 5. Any group of members of the Council who have a total of five votes as defined in Section 4 of this Article shall constitute a quorum for the transaction of business at any meeting of the Council.

Section 6. Between meetings of the Council, business may be transacted by a mail vote. Votes shall be counted as in the case of a roll call vote as specified in Section 4 of this Article, "members present" being replaced by "members voting." An affirmative vote by mail on any proposal shall be declared if and only if (a) more than half of the total number of possible votes is received by the time announced for the closing of the polls, and (b) at least three-quarters of the votes received by then are affirmative. If members who have a total of five or more votes request postponement at the time of voting, action on the matter at issue shall be postponed until the next meeting of the Council, unless either (1) at the discretion of the Secretary, the question is made the subject of a second vote by mail, in connection with which brief statements of reasons, for and against, are circulated or (2) the Council places the matter at issue before the Executive Committee for action.

Section 7. The Council may delegate to the Executive Committee (Article V) certain of its duties and powers. Between meetings of the Council the Executive Committee shall act for the Council on such matters and in such ways as the Council may specify. Nothing herein contained shall be construed as empowering the Council to divest itself of responsibility for formulating and administering the scientific policies of the Society.

ARTICLE V

EXECUTIVE COMMITTEE

Section 1. There shall be an Executive Committee of the Council, consisting of four elected members and the following ex officio members: the President, the Secretary, the President Elect (during even-numbered years), and the Ex-president (during odd-numbered years).

Section 2. The Executive Committee of the Council shall be empowered to act for the Council on matters which have been delegated to the Executive Committee by the Council. If three members of the Executive Committee request that any matter be referred to the Council, the matter shall be so referred. The Executive Committee shall be responsible to the Council and shall report its actions to the Council. It may consider the agenda for meetings of the Council and may make recommendations to the Council.

Section 3. Each member of the Executive Committee shall have one vote. An affirmative vote on any proposal before the Executive Committee shall be declared if and only if at least four affirmative votes are cast for the proposal. A vote on any proposal may be determined at a meeting of the Executive Committee, but it shall not be necessary to hold a meeting to determine a vote.

ARTICLE VI

EXECUTIVE DIRECTOR

Section 1. There shall be an Executive Director who shall be a paid employee of the Society. He shall have charge of the central office of the Society, and he shall be responsible for the general administration of the affairs of the Society in accordance with the policies that are set by the Board of Trustees and by the Council.

Section 2. The Executive Director shall be elected by the Council for a specified term and approved by the Board of Trustees. The terms and conditions of his employment shall be fixed by the Board of Trustees.

Section 3. The Executive Director shall work under the immediate direction of the Board of Trustees and of the Council, and in cooperation with the Secretary. He shall attend meetings of the Board of Trustees, of the Council, and of the Executive Committee, but he shall not be a member of any of these bodies.

ARTICLE VII

ELECTION AND TERMS OF OFFICERS

Section 1. The term of office shall be one year in the case of the President Elect and the Ex-president; two years in the case of the Trustees, the President, the Vice Presidents, the Secretary, the Associate Secretaries, and the Treasurer; three years in the case of the members of the Publications Committees. The term of office for members at large of the Council shall be three years, five of the members at large retiring annually. The term of office for elected members of the Executive Committee shall be two years, two of the elected members retiring annually. In every case, however, the officials specified in Articles I, II, III, IV, and V, with the exception of the President Elect and the Ex-president, shall continue to serve until their successors have been duly elected and qualified.

Section 2. Election of officers specified in Article I (with the exception of the President and the Ex-president), Trustees, members of the Publications Committees, and members at large of the Council shall be by ballot at the Annual Meeting. An official ballot shall be sent to each member of the Society by the Secretary at least one month before the Annual Meeting, and such ballots, if returned to the Secretary in envelopes bearing the name of the voter and received prior to the closing of the polls, shall be counted at the Annual Meeting. Each ballot shall contain one or more names proposed by the Council for each office to be filled, with blank spaces in which the voter may substitute other names. A plurality of all votes cast, whether cast in person or by mail, shall be necessary for election. In case of failure to secure a plurality for any office, the members present at the Annual Meeting shall choose by ballot among the members having the highest number of votes.

Section 3. At the end of his term of office, the President Elect shall become the President. At the end of his term of office, the President shall become the Ex-president.

Section 4. Within fourteen days after the election of officers at the Annual Meeting the Secretary shall send to all members of the Council for a mail vote a ballot containing two names for each place to be filled on the Executive Committee. The nominees shall be chosen by a committee appointed by the President. Members of the Council may vote for persons not nominated. Any member of the Council who is not an ex officio member of the Executive Committee (see Article V, Section 1) shall be eligible for election to the Executive Committee. In case a member is elected to the Executive Committee for a term extending beyond his regular term on the Council, he shall automatically continue as a member of the Council during his term on the Executive Committee.

Section 5. The President and Vice Presidents shall not be eligible for immediate re-election to their respective offices. A member at large or an ex officio member of the Council shall not be eligible for immediate election (or re-election) as a member at large of the Council.

Section 6. If the President of the Society die or resign while a President Elect is in office, the President Elect shall serve as President for the remainder of the year and thereafter shall serve his regular two-year term. If the President of the Society die or resign when no President Elect is in office, the Council, with the approval of the Board of Trustees, shall designate one of the Vice Presidents to serve as President for the balance of the regular presidential term. If the President Elect of the Society die or resign before becoming President, his office shall remain vacant until the next regular election of a President Elect, and the Society shall, at the next Annual Meeting, elect a President for a two-year term. If the Ex-president die or resign before the expiration of his term of office, the Council, with the approval of the Board of Trustees, shall designate a former President of the Society to serve as Ex-president during the remainder of the regular term of the Ex-presidency. Such vacancies as may occur at any time in the group consisting of the Vice Presidents, the Secretary, the Associate Secretaries, the Treasurer, and the members of the Publications Committees shall be filled by the Council with the approval of the Board of Trustees. If a member of a Publications Committee takes temporary leave from his duties, that Committee shall nominate a substitute for consideration by the Council. The Council shall then elect a substitute. The Council shall fill from its own membership any vacancy in the elected membership of the Executive Committee.

Section 7. If any Trustee die or resign during his tenure of office, the vacancy thus created shall be filled for his unexpired term by the Board of Trustees.

Section 8. If any member at large of the Council die or resign more than one year before the expiration of his term, the vacancy for the unexpired term shall be filled by the Society at the next Annual Meeting.

ARTICLE VIII

MEMBERS AND THEIR ELECTION

Section 1. Election of members shall be by vote of the Council or of its Executive Committee.

Section 2. There shall be three classes of members, namely, ordinary, contributing and institutional.

Section 3. Application for admission to ordinary membership shall be made by the applicant on a blank provided by the Secretary, and shall be approved either by two ordinary members of the Society or by the institutional member whose nominee he is (see Article IX, Section 6). Such applications shall not be acted upon until at least thirty days after their presentation to the Council (at a meeting or by mail), except in the case of members of other societies entering under special action of the Council approved by the Board of Trustees.

Section 4. An ordinary member may become a contributing member by paying the dues for such membership. (See Article IX, Section 3.)

Section 5. A university or college, or a firm, corporation, or association interested in the support of mathematics may be elected an institutional member.

ARTICLE IX

DUES AND PRIVILEGES OF MEMBERS

Section 1. Any applicant shall be admitted to ordinary membership immediately upon election by the Council (Article VIII) and the discharge within sixty days of election of his first annual dues. Dues may be discharged by payment or by remission when the provision of Section 7 of this Article is applicable. The first annual dues shall apply to the year of election, except that if any applicant is elected after August 15 of any year he may elect to have his first annual dues apply to the following year.

Section 2. The annual dues of an ordinary member of the Society shall be fourteen dollars, except that the Council, with the approval of the Board of Trustees, may establish special rates in exceptional cases and for members of an organization with which the Society has a reciprocity agreement.

Section 3. The minimum dues for a contributing member shall be twenty dollars per year. Members may, upon their own initiative, pay larger dues.

Section 4. The minimum dues of an institutional member shall depend on the amount of published material credited to that member in certain journals during a specific period. The formula for computing these dues shall be established from time to time by the Council subject to approval by the Board of Trustees. Institutions may pay larger dues than the computed minimum.

Section 5. Any member whose annual dues are at least one hundred dollars shall be designated as a Sustainer of the Society; any member whose annual dues are at least five hundred dollars shall be designated as a Patron of the Society.

Section 6. The privileges of an institutional member shall depend on its dues in a manner to be determined by the Council, subject to approval by the Board of Trustees. These privileges shall be in terms of Society publications to be received by the institution and of the number of persons it may nominate for ordinary membership in the Society.

Section 7. the dues of an ordinary member of the Society shall be remitted for any years during which he is the nominee of an institutional member.

Section 8. After retirement from active service on account of age, any ordinary or contributing member who is not in arrears of dues and with membership extending over at least twenty years may, by giving proper notification to the Secretary, have his dues remitted, on the understanding that he will thereafter receive the programs of the meetings but not the *Bulletin* and *Proceedings*.

Section 9. An ordinary or contributing member shall receive the *Bulletin* and *Proceedings* as privileges of membership during each year for which his dues have been discharged.

Section 10. Nine dollars of the annual dues of those who receive the *Bulletin* and *Proceedings* under the provisions of this Article shall be allocated in payment therefor.

Section 11. The annual dues of ordinary and contributing members shall be payable after October first of the preceding year and due by January first of the year to which they apply. The Society shall submit bills for dues. If the annual dues of any member remain undischarged beyond what the Board of Trustees deem to be a reasonable time, his name shall, after due notice, be removed from the list of members. If a member wishes to discontinue his membership at any time, he shall submit his resignation in writing to the Society.

Section 12. Any member who became a life member before October 25, 1941, by the payment of a sum determined in accordance with actuarial principles shall have for life the status and privileges of an ordinary member without further payment of dues. No additional applications for life memberships will be accepted.

ARTICLE X

MEETINGS

Section 1. The Annual Meeting of the Society shall be held between the fifteenth of December and the fifteenth of January next following. Notice of the time and place of this meeting shall be mailed by the Secretary or an Associate Secretary to the last known post office address of each member of the Society. The times and places of the Annual and other meetings of the Society shall be designated by the Council. No matter of general business shall be considered at any meeting of the Society except the Annual Meeting, without the recommendation of the Council.

Section 2. Meetings of the Executive Committee may be called by the President; he shall call a meeting at any time upon the written request of two of its members.

Section 3. The Council shall meet at the Annual Meeting of the Society. Special meetings of the Council may be called by the President; he shall call a special meeting at any time upon the written request of five of its members. No special meeting of the Council shall be held unless written notice of it shall have been sent to all members of the Council at least ten days before the day set for the meeting.

Section 4. The Board of Trustees shall hold its regular Annual Meeting each year between December 15 and January 20, but not before the Annual Meeting of the Society. Special meetings of the Board of Trustees may be called by the Chairman of the Board upon three days' notice of such meeting mailed to the last known post office address of each Trustee. He should call a meeting upon the receipt of a written request of two of the Trustees. Meetings for the transaction of business may also be held by common consent of all the Trustees.

Section 5. Papers intended for presentation at any meeting of the Society shall be passed upon in advance by a program committee appointed by or under the authority of the Council; and only such papers shall be presented as shall have been approved by such committee. Papers in form unsuitable for publication, if accepted for presentation, shall be referred to on the program as preliminary communications or reports.

ARTICLE XI

PUBLICATIONS

Section 1. The Society shall publish an official organ called the *Bulletin of the American Mathematical Society*. It shall publish two journals, known as the *Transactions of the American Mathematical Society* and the *Proceedings of the American Mathematical Society*. It shall publish a series of mathematical papers known as the *Memoirs of the American Mathematical Society*. The object of the *Transactions*, *Proceedings*, and *Memoirs* is to make known important mathematical researches. It shall publish a periodical called *Mathematical Reviews*, containing abstracts or reviews of current mathematical literature. It shall publish a series of volumes called *Colloquium Publications* which shall embody in book form new mathematical developments. It shall publish a series of monographs called *Mathematical Surveys* which shall furnish expositions of the principal methods and results of particular fields of mathematical research. It shall also cooperate in the conduct of the *American Journal of Mathematics*.

Section 2. The editorial management of the publications of the Society listed in Section 1 of this article, and the participation of the Society in the editorial management of the *American Journal of Mathematics* shall be in charge of the respective Editorial Committees as provided in Article III, Section 2.

Section 3. The Committee on Printing and Publishing shall advise the Council and Board of Trustees on business and non-editorial matters concerning the publications of the Society, and shall perform such other functions and shall exercise such powers as are properly assigned or delegated to it by the Council or the Board of Trustees.

ARTICLE XII

AMENDMENTS

These By-Laws may be amended or suspended at any meeting of the Society on recommendation of the Council and by a two-thirds vote of the members present, provided notice of such proposed action and of its general nature shall have been given in the call for such meeting.

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